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Research Article

On the Symmetric Properties for the Generalized Twisted Bernoulli Polynomials

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We study the symmetry for the generalized twisted Bernoulli polynomials and numbers. We give some interesting identities of the power sums and the generalized twisted Bernoulli polynomials using the symmetric properties for the p-adic invariant integral.

1. Introduction

Let p be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x)dx = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \tag{1.1}$$

(see [1]). From (1.1), we note that

$$I(f_1) = I(f) + f'(0), \tag{1.2}$$

where $f'(0) = df(x)/dx|_{x=0}$ and $f_1(x) = f(x+1)$. For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. Then we can derive the following equation from (1.2):

$$I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i) \tag{1.3}$$

(see [1–7]).

Let d be a fixed positive integer. For $n \in \mathbb{N}$, let

$$\begin{gathered} X = X_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p, \\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\,\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \left(\operatorname{mod} dp^N \right) \right\}, \end{gathered} \tag{1.4}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. It is easy to see that

$$\int_X f(x)dx = \int_{\mathbb{Z}_p} f(x)dx, \quad \text{for } f \in UD(\mathbb{Z}_p). \tag{1.5}$$

The ordinary Bernoulli polynomials $B_n(x)$ are defined as

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.6}$$

and the Bernoulli numbers B_n are defined as $B_n = B_n(0)$ (see [1–19]).

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n \geq 1} \mathbb{C}_{p^n} = \lim_{n \to \infty} \mathbb{C}_{p^n}, \tag{1.7}$$

where $\mathbb{C}_{p^n} = \{\omega \mid \omega^{p^n} = 1\}$ is the cyclic group of order p^n. It is well known that the twisted Bernoulli polynomials are defined as

$$\frac{t}{\xi e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\xi}(x) \frac{t^n}{n!}, \quad \xi \in T_p, \tag{1.8}$$

and the twisted Bernoulli numbers $B_{n,\xi}$ are defined as $B_{n,\xi} = B_{n,\xi}(0)$ (see [15–18]).

Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$. Then the generalized twisted Bernoulli polynomials $B_{n,\chi,\xi}(x)$ attached to χ are defined as follows:

$$\sum_{a=0}^{d-1} \frac{\chi(a)\xi^a e^{at} t}{\xi^d e^{dt} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}(x) \frac{t^n}{n!}, \quad \xi \in T_p. \tag{1.9}$$

The generalized twisted Bernoulli numbers attached to χ, $B_{n,\chi,\xi}$, are defined as $B_{n,\chi,\xi} = B_{n,\chi,\xi}(0)$ (see [16]).

Recently, many authors have studied the symmetric properties of the p-adic invariant integrals on $\mathbb{Z}_p$, which gave some interesting identities for the Bernoulli and the Euler polynomials (cf. [3, 6, 7, 13, 14, 20–27]). The authors of this paper have established various identities by the symmetric properties of the p-adic invariant integrals and investigated interesting relationships between the power sums and the Bernoulli polynomials (see [2, 3, 6, 7, 13]).

The twisted Bernoulli polynomials and numbers and the twisted Euler polynomials and numbers are very important in several fields of mathematics and physics(cf. [15–18]). The second author has been interested in the twisted Euler numbers and polynomials and the twisted Bernoulli polynomials and studied the symmetry of power sum and twisted Bernoulli polynomials (see [11–13]).

The purpose of this paper is to study the symmetry for the generalized twisted Bernoulli polynomials and numbers attached to χ. In Section 2, we give interesting identities for the power sums and the generalized twisted Bernoulli polynomials using the symmetric properties for the p-adic invariant integral.

2. Symmetry for the Generalized Twisted Bernoulli Polynomials

Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$. For $\xi \in T_p$, we have

$$\int_X \chi(x)\xi^x e^{xt}\,dx = \frac{t\sum_{i=0}^{d-1}\chi(i)\xi^i e^{it}}{\xi^d e^{dt}-1} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}\frac{t^n}{n!}, \tag{2.1}$$

where $B_{n,\chi,\xi}$ are the nth generalized twisted Bernoulli numbers attached to χ. We also see that the generalized twisted Bernoulli polynomials attached to χ are given by

$$\int_X \chi(y)\xi^y e^{(x+y)t}\,dy = \frac{t\sum_{i=0}^{d-1}\chi(i)\xi^i e^{it}}{\xi^d e^{dt}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}(x)\frac{t^n}{n!}. \tag{2.2}$$

By (2.1) and (2.2), we see that

$$\int_X \chi(x)\xi^x x^n\,dx = B_{n,\chi,\xi}, \qquad \int_X \chi(y)\xi^y (x+y)^n\,dy = B_{n,\chi,\xi}(x). \tag{2.3}$$

From (2.3), we derive that

$$B_{n,\chi,\xi}(x) = \sum_{l=0}^{n}\binom{n}{l}B_{l,\chi,\xi}x^{n-l}. \tag{2.4}$$

By (1.5) and (2.3), we see that

$$\int_X \chi(x)\xi^x e^{xt}dx = \frac{1}{d}\sum_{a=0}^{d-1}\chi(a)e^{at}\xi^a\int_{\mathbb{Z}_p}\xi^{dx}e^{dxt}dx = \frac{1}{d}\sum_{a=0}^{d-1}\chi(a)\xi^a\frac{dt}{\xi^d e^{dt}-1}e^{(a/d)dt}. \tag{2.5}$$

From (2.2) and (2.5), we obtain that

$$\int_X \chi(x)\xi^x e^{xt}dx = \sum_{n=0}^{\infty}\left\{d^{n-1}\sum_{a=0}^{d-1}\chi(a)\xi^a B_{n,\xi^d}\left(\frac{a}{d}\right)\right\}\frac{t^n}{n!}. \tag{2.6}$$

Thus we have the following theorem from (2.1) and (2.6).

Theorem 2.1. *For $\xi \in T_p$, one has*

$$B_{n,\chi,\xi} = d^{n-1}\sum_{a=0}^{d-1}\chi(a)\xi^a B_{n,\xi^d}\left(\frac{a}{d}\right). \tag{2.7}$$

By (1.3) and (1.5), we have that for $n \in \mathbb{N}$,

$$\int_X f(x+n)dx = \int_X f(x)dx + \sum_{i=0}^{n-1}f'(i), \tag{2.8}$$

where $f'(i) = df(x)/dx|_{x=i}$. Taking $f(x) = \chi(x)\xi^x e^{xt}$ in (2.8), it follows that

$$\begin{aligned}&\frac{1}{t}\left(\int_X \chi(x)\xi^{nd+x}e^{(nd+x)t}dx - \int_X \chi(x)\ \xi^x e^{xt}dx\right)\\ &\quad= \frac{nd\int_X \chi(x)\ \xi^x e^{xt}dx}{\int_X \xi^{ndx}e^{ndxt}dx} = \frac{\xi^{nd}e^{ndt}-1}{\xi^d e^{dt}-1}\left(\sum_{i=0}^{d-1}\chi(i)\xi^i e^{it}\right).\end{aligned} \tag{2.9}$$

Thus, we have

$$\frac{1}{t}\left(\int_X \chi(x)\xi^{nd+x}e^{(nd+x)t}dx - \int_X \chi(x)\xi^x e^{xt}dx\right) = \sum_{k=0}^{\infty}\left(\sum_{l=0}^{nd-1}\chi(l)\ \xi^l l^k\right)\frac{t^k}{k!}. \tag{2.10}$$

For $k \in \mathbb{Z}_+$, let us define the p-adic functional $K(\chi,\xi,k:n)$ as follows:

$$K(\chi,\xi,k:n) = \sum_{l=0}^{n}\chi(l)\xi^l l^k. \tag{2.11}$$

By (2.10) and (2.11), we see that for $k,n,d \in \mathbb{N}$,

$$\int_X \chi(x)\xi^{nd+x}(nd+x)^k dx - \int_X \chi(x)\xi^x x^k dx = kK(\chi,\xi,k-1:nd-1). \tag{2.12}$$

From (2.3) and (2.12), we have the following result.

Theorem 2.2. *For $\xi \in T_p$ and $k, n, d \in \mathbb{N}$, one has*

$$\xi^{nd} B_{k,\chi,\xi}(nd) - B_{k,\chi,\xi} = kK\left(\chi, \xi, k-1 : nd-1\right). \tag{2.13}$$

Let $w_1, w_2, d \in \mathbb{N}$. Then we have that

$$\begin{aligned} &\frac{d\int_X\int_X \chi(x_1)\chi(x_2)\ \xi^{w_1x_1+w_2x_2} e^{(w_1x_1+w_2x_2)t} dx_1 dx_2}{\int_X \xi^{dw_1w_2x} e^{dw_1w_2xt} dx} \\ &\quad = \frac{t\left(\xi^{dw_1w_2} e^{dw_1w_2t} - 1\right)}{\left(\xi^{w_1d} e^{dw_1t} - 1\right)\left(\xi^{w_2d} e^{dw_2t} - 1\right)} \left(\sum_{a=0}^{d-1} \chi(a) \xi^{w_1a} e^{w_1at}\right)\left(\sum_{b=0}^{d-1} \chi(b) \xi^{w_2b} e^{w_2bt}\right). \end{aligned} \tag{2.14}$$

By (2.9), (2.10), and (2.11), we see that

$$\frac{w_1 d\int_X \chi(x) \xi^x e^{xt} dx}{\int_X \xi^{dw_1x} e^{dw_1xt} dx} = \sum_{k=0}^{\infty} K\left(\chi, \xi, k : dw_1 - 1\right) \frac{t^k}{k!}. \tag{2.15}$$

Now let us define the p-adic functional $Y_{\chi,\xi}(w_1, w_2)$ as follows:

$$Y_{\chi,\xi}(w_1, w_2) = \frac{d\int_X\int_X \chi(x_1)\chi(x_2) \xi^{w_1x_1+w_2x_2} e^{(w_1x_1+w_2x_2+w_1w_2x)t} dx_1 dx_2}{\int_X \xi^{dw_1w_2x_3} e^{dw_1w_2x_3t} dx_3}. \tag{2.16}$$

Then it follows from (2.14) that

$$Y_{\chi,\xi}(w_1, w_2) = \frac{t\left(\xi^{dw_1w_2} e^{dw_1w_2t} - 1\right) e^{w_1w_2xt}}{\left(\xi^{w_1d} e^{dw_1t} - 1\right)\left(\xi^{w_2d} e^{dw_2t} - 1\right)} \left(\sum_{a=0}^{d-1} \chi(a) \xi^{w_1a} e^{w_1at}\right)\left(\sum_{b=0}^{d-1} \chi(b) \xi^{w_2b} e^{w_2bt}\right). \tag{2.17}$$

By (2.15) and (2.16), we obtain that

$$\begin{aligned} Y_{\chi,\xi}(w_1, w_2) &= \left(\frac{1}{w_1}\int_X \chi(x_1) \xi^{w_1x_1} e^{w_1(x_1+w_2x)t} dx_1\right)\left(\frac{dw_1\int_X \chi(x_2) \xi^{w_2x_2} e^{w_2x_2t} dx_2}{\int_X \xi^{dw_1w_2x} e^{dw_1w_2xt} dx}\right) \\ &= \sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\binom{l}{i} B_{i,\chi,\xi^{w_1}}(w_2x) K\left(\chi, \xi^{w_2}, l-i : dw_1 - 1\right) w_1^{i-1} w_2^{l-i}\right)\frac{t^l}{l!}. \end{aligned} \tag{2.18}$$

On the other hand, the symmetric property of $Y_{\chi,\xi}(w_1,w_2)$ shows that

$$\begin{aligned}
Y_{\chi,\xi}(w_1,w_2) &= \left(\frac{1}{w_2}\int_X \chi(x_2)\xi^{w_2x_2}e^{w_2(x_2+w_1x)t}dx_2\right)\left(\frac{dw_2\int_X \chi(x_1)\xi^{w_1x_1}e^{w_1x_1t}dx_1}{\int_X \xi^{dw_1w_2x}e^{dw_1w_2xt}dx}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\binom{l}{i}B_{i,\chi,\xi^{w_2}}(w_1x)K\left(\chi,\xi^{w_1},l-i:dw_2-1\right)w_2^{i-1}w_1^{l-i}\right)\frac{t^l}{l!}.
\end{aligned} \tag{2.19}$$

Comparing the coefficients on the both sides of (2.18) and (2.19), we have the following theorem.

Theorem 2.3. *Let $\xi \in T_p$ and $d, w_1, w_2 \in \mathbb{N}$. Then one has*

$$\begin{aligned}
&\sum_{i=0}^{l}\binom{l}{i}B_{i,\chi,\xi^{w_1}}(w_2x)K\left(\chi,\xi^{w_2},l-i:dw_1-1\right)w_1^{i-1}w_2^{l-i} \\
&\quad= \sum_{i=0}^{l}\binom{l}{i}B_{i,\chi,\xi^{w_2}}(w_1x)K\left(\chi,\xi^{w_1},l-i:dw_2-1\right)w_2^{i-1}w_1^{l-i}.
\end{aligned} \tag{2.20}$$

We also derive some identities for the generalized twisted Bernoulli numbers. Taking $x = 0$ in Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let $\xi \in T_p$ and $d, w_1, w_2 \in \mathbb{N}$. Then one has*

$$\sum_{i=0}^{l}\binom{l}{i}B_{i,\chi,\xi^{w_1}}K\left(\chi,\xi^{w_2},l-i:dw_1-1\right)w_1^{i-1}w_2^{l-i}=\sum_{i=0}^{l}\binom{l}{i}B_{i,\chi,\xi^{w_2}}K\left(\chi,\xi^{w_1},l-i:dw_2-1\right)w_2^{i-1}w_1^{l-i}. \tag{2.21}$$

Now we will derive another identities for the generalized twisted Bernoulli polynomials using the symmetric property of $Y_{\chi,\xi}(w_1,w_2)$. From (1.2), (2.15) and (2.17), we see that

$$\begin{aligned}
Y_{\chi,\xi}(w_1,w_2) &= \left(\frac{e^{w_1w_2xt}}{w_1}\int_X \chi(x_1)\xi^{w_1x_1}e^{w_1x_1t}dx_1\right)\left(\frac{dw_1\int_X \chi(x_2)\xi^{w_2x_2}e^{w_2x_2t}dx_2}{\int_X \xi^{dw_1w_2x}e^{dw_1w_2xt}dx}\right) \\
&= \frac{1}{w_1}\sum_{i=0}^{dw_1-1}\chi(i)\xi^{w_2i}\int_X \chi(x_1)\xi^{w_1x_1}e^{w_1(x_1+w_2x+(w_2/w_1)i)t}dx_1 \\
&= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{dw_1-1}\chi(i)\xi^{w_2i}B_{k,\chi,\xi^{w_1}}\left(w_2x+\frac{w_2}{w_1}i\right)w_1^{k-1}\right)\frac{t^k}{k!}.
\end{aligned} \tag{2.22}$$

From the symmetric property of $Y_{\chi,\xi}(w_1, w_2)$, we also see that

$$Y_{\chi,\xi}(w_1, w_2) = \left(\frac{e^{w_1 w_2 xt}}{w_2}\int_X \chi(x_2)\xi^{w_2x_2}e^{w_2x_2t}dx_2\right)\left(\frac{dw_2\int_X \chi(x_1)\xi^{w_1x_1}e^{w_1x_1t}dx_1}{\int_X \xi^{dw_1w_2x}e^{dw_1w_2xt}dx}\right)$$

$$= \frac{1}{w_2}\sum_{i=0}^{dw_2-1}\chi(i)\xi^{w_1 i}\int_X \chi(x_2)\xi^{w_2x_2}e^{w_2(x_2+w_1x+(w_1/w_2)i)t}dx_2 \quad (2.23)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{dw_2-1}\chi(i)\xi^{w_1 i}B_{k,\chi,\xi^{w_2}}\left(w_1x + \frac{w_1}{w_2}i\right)w_2^{k-1}\right)\frac{t^k}{k!}.$$

Comparing the coefficients on the both sides of (2.22) and (2.23), we obtain the following theorem.

Theorem 2.5. *Let $\xi \in T_p$ and $d, w_1, w_2 \in \mathbb{N}$. Then one has*

$$\sum_{i=0}^{dw_1-1}\chi(i)\xi^{w_2 i}B_{k,\chi,\xi^{w_1}}\left(w_2x + \frac{w_2}{w_1}i\right)w_1^{k-1} = \sum_{i=0}^{dw_2-1}\chi(i)\xi^{w_1 i}B_{k,\chi,\xi^{w_2}}\left(w_1x + \frac{w_1}{w_2}i\right)w_2^{k-1}. \quad (2.24)$$

If we take $x = 0$ in Theorem 2.5, we also derive the interesting identity for the generalized twisted Bernoulli numbers as follows: for $d, w_1, w_2 \in \mathbb{N}$,

$$\sum_{i=0}^{dw_1-1}\chi(i)\xi^{w_2 i}B_{k,\chi,\xi^{w_1}}\left(\frac{w_2}{w_1}i\right)w_1^{k-1} = \sum_{i=0}^{dw_2-1}\chi(i)\xi^{w_1 i}B_{k,\chi,\xi^{w_2}}\left(\frac{w_1}{w_2}i\right)w_2^{k-1}. \quad (2.25)$$

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