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Research Article

On Pečarić-Rajić-Dragomir-Type Inequalities in Normed Linear Spaces

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We establish some generalizations of the recent Pečarić-Rajić-Dragomir-type inequalities by providing upper and lower bounds for the norm of a linear combination of elements in a normed linear space. Our results provide new estimates on inequalities of this type.

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1. Introduction

In the recent paper [1], Pečarić and Rajić proved the following inequality for n nonzero vectors x_k, $k \in \{1, \dots, n\}$ in the real or complex normed linear space $(X, \|\cdot\|)$:

$$\max_{k\in\{1,\dots,n\}} \left\{ \frac{1}{\|x_k\|} \left[\left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} \left| \|x_j\| - \|x_k\| \right| \right] \right\} \le \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \le \min_{k\in\{1,\dots,n\}} \left\{ \frac{1}{\|x_k\|} \left[\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} \left| \|x_j\| - \|x_k\| \right| \right] \right\} \tag{1.1}$$

and showed that this inequality implies the following refinement of the generalised triangle inequality obtained by Kato et al. in [2]:

$$\min_{k\in\{1,\ldots,n\}}\{\|x_k\|\}\left[n-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right]\leq\sum_{j=1}^{n}\|x_j\|-\left\|\sum_{j=1}^{n}x_j\right\|\leq\max_{k\in\{1,\ldots,n\}}\{\|x_k\|\}\left[n-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right]. \tag{1.2}$$

The inequality (1.2) can also be obtained as a particular case of Dragomir's result established in [3]:

$$\begin{aligned}\max_{1\leq j\leq n}\{\|x_j\|\}\left[\sum_{j=1}^{n}\|x_j\|^{p-1}-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|^p\right]&\geq\sum_{j=1}^{n}\|x_j\|^p-n^{1-p}\left\|\sum_{j=1}^{n}x_j\right\|^p\\&\geq\min_{1\leq j\leq n}\{\|x_j\|\}\left[\sum_{j=1}^{n}\|x_j\|^{p-1}-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|^p\right],\end{aligned} \tag{1.3}$$

where $p\geq 1$ and $n\geq 2$.

Notice that, in [3], a more general inequality for convex functions has been obtained as well.

Recently, the following inequality which is more general than (1.1) was given by Dragomir [4]:

$$\begin{aligned}&\max_{k\in\{1,\ldots,n\}}\left\{|\alpha_k|\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}|\alpha_j-\alpha_k|\|x_j\|\right\}\\&\quad\leq\left\|\sum_{j=1}^{n}\alpha_jx_j\right\|\leq\min_{k\in\{1,\ldots,n\}}\left\{|\alpha_k|\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}|\alpha_j-\alpha_k|\|x_j\|\right\}.\end{aligned} \tag{1.4}$$

The main aim of this paper is to establish further generalizations of these Pečarić-Rajić-Dragomir-type inequalities (1.1), (1.2), (1.3), and (1.4) by providing upper and lower bounds for the norm of a linear combination of elements in the normed linear space. Our results provide new estimates on such type of inequalities.

2. Main Results

Theorem 2.1. *Let* $(X, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$*. If* $\alpha_{i_1,\dots,i_n} \in \mathbb{K}$ *and* $x_{i_1,\dots,i_n} \in X$ *for* $i_1,\dots,i_n \in \{1,\dots,n\}$ *with* $n \geq 2$*, then*

$$\begin{aligned}
&\max_{\substack{k_j\in\{1,\dots,n\}\\ j=1,\dots,n}}\left\{|\alpha_{k_1,\dots,k_n}|\left\|\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}x_{i_1,\dots,i_n}\right\|-\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}|\alpha_{i_1,\dots,i_n}-\alpha_{k_1,\dots,k_n}|\|x_{i_1,\dots,i_n}\|\right\}\\
&\quad\leq\left\|\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}\alpha_{i_1,\dots,i_n}x_{i_1,\dots,i_n}\right\| \\
&\quad\leq\min_{\substack{k_j\in\{1,\dots,n\}\\ j=1,\dots,n}}\left\{|\alpha_{k_1,\dots,k_n}|\left\|\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}x_{i_1,\dots,i_n}\right\|+\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}|\alpha_{i_1,\dots,i_n}-\alpha_{k_1,\dots,k_n}|\|x_{i_1,\dots,i_n}\|\right\}.
\end{aligned} \tag{2.1}$$

Proof. Observe that, for any fixed $k_j \in \{1,\dots,n\}$, $j = 1,\dots,n$, we have

$$\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}\alpha_{i_1,\dots,i_n}x_{i_1,\dots,i_n}=\alpha_{k_1,\dots,k_n}\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}x_{i_1,\dots,i_n}+\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}(\alpha_{i_1,\dots,i_n}-\alpha_{k_1,\dots,k_n})x_{i_1,\dots,i_n}. \tag{2.2}$$

Taking the norm in (2.2) and utilizing the triangle inequality, we have

$$\begin{aligned}
\left\|\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}\alpha_{i_1,\dots,i_n}x_{i_1,\dots,i_n}\right\|&\leq\left\|\alpha_{k_1,\dots,k_n}\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}x_{i_1,\dots,i_n}\right\|+\left\|\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}(\alpha_{i_1,\dots,i_n}-\alpha_{k_1,\dots,k_n})x_{i_1,\dots,i_n}\right\|\\
&\leq|\alpha_{k_1,\dots,k_n}|\left\|\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}x_{i_1,\dots,i_n}\right\|+\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}|\alpha_{i_1,\dots,i_n}-\alpha_{k_1,\dots,k_n}|\|x_{i_1,\dots,i_n}\|,
\end{aligned} \tag{2.3}$$

which, on taking the minimum over $k_j \in \{1,\dots,n\}$, $j = 1,\dots,n$, produces the second inequality in (2.1).

Next, by (2.2) we have obviously

$$\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}\alpha_{i_1,\dots,i_n}x_{i_1,\dots,i_n}=\alpha_{k_1,\dots,k_n}\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}x_{i_1,\dots,i_n}-\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n}(\alpha_{k_1,\dots,k_n}-\alpha_{i_1,\dots,i_n})x_{i_1,\dots,i_n}. \tag{2.4}$$

On utilizing the continuity property of the norm we also have

$$\begin{aligned}
&\left\| \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} \alpha_{i_1,\ldots,i_n} x_{i_1,\ldots,i_n} \right\| \\
&\geq \left| \left\| \alpha_{k_1,\ldots,k_n} \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} x_{i_1,\ldots,i_n} \right\| - \left\| \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} \left(\alpha_{i_1,\ldots,i_n} - \alpha_{k_1,\ldots,k_n} \right) x_{i_1,\ldots,i_n} \right\| \right| \\
&\geq \left\| \alpha_{k_1,\ldots,k_n} \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} x_{i_1,\ldots,i_n} \right\| - \left\| \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} \left(\alpha_{i_1,\ldots,i_n} - \alpha_{k_1,\ldots,k_n} \right) x_{i_1,\ldots,i_n} \right\| \\
&\geq \left| \alpha_{k_1,\ldots,k_n} \right| \left\| \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} x_{i_1,\ldots,i_n} \right\| - \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} \left| \alpha_{i_1,\ldots,i_n} - \alpha_{k_1,\ldots,k_n} \right| \left\| x_{i_1,\ldots,i_n} \right\|,
\end{aligned} \tag{2.5}$$

which, on taking the maximum over $k_j \in \{1,\ldots,n\}$, $j = 1,\ldots,n$, produces the first part of (2.1) and the theorem is completely proved. □

Remark 2.2. (i) In case the multi-indices $i_1,\ldots,i_n$ and $k_1,\ldots,k_n$ reduce to single indices j and k, respectively, after suitable modifications, (2.1) reduces to inequality (1.4) obtained by Dragomir in [4].

(ii) Furthermore, if $x_j \in X \setminus \{0\}$ for $j \in \{1,\ldots,n\}$ and $\alpha_k = 1/\|x_k\|$, $k \in \{1,\ldots,n\}$ with $n \geq 2$, the inequality reduces further to inequality (1.1) obtained by Pečarić and Rajić in [1].

(iii) Further to (ii), if $n = 2$, writing $x_1 = x$ and $x_2 = -y$, we have

$$\frac{\|x-y\| - \left| \|x\| - \|y\| \right|}{\min\{\|x\|, \|y\|\}} \leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\|x-y\| + \left| \|x\| - \|y\| \right|}{\max\{\|x\|, \|y\|\}}, \tag{2.6}$$

which holds for any nonzero vectors $x, y \in X$.

The first inequality in (2.6) was obtained by Mercer in [5].

The second inequality in (2.6) has been obtained by Maligranda in [6]. It provides a refinement of the *Massera-Schäffer inequality* [7]:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\max\{\|x\|, \|y\|\}}, \tag{2.7}$$

which, in turn, is a refinement of the *Dunkl-Williams inequality* [8]:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x-y\|}{\|x\| + \|y\|}. \tag{2.8}$$

Theorem 2.3. *Let* $(X, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$. *If* $\alpha_{j_1,\ldots,j_n} \in \mathbb{K}$ *and* $x_{j_1,\ldots,j_n} \in X \setminus \{0\}$ *for* $j_1, \ldots, j_n \in \{1, \ldots, n\}$ *with* $n \geq 2$, *then*

$$\begin{aligned}
&\max_{\substack{k_i\in\{1,\ldots,n\}\\ i=1,\ldots,n}} \left\{ \frac{1}{\|x_{k_1,\ldots,k_n}\|} \left[\left\| \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} x_{j_1,\ldots,j_n} \right\| - \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \left| \|x_{j_1,\ldots,j_n}\| - \|x_{k_1,\ldots,k_n}\| \right| \right] \right\} \\
&\leq \left\| \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \frac{x_{j_1,\ldots,j_n}}{\|x_{j_1,\ldots,j_n}\|} \right\| \qquad (2.9) \\
&\leq \min_{\substack{k_i\in\{1,\ldots,n\}\\ i=1,\ldots,n}} \left\{ \frac{1}{\|x_{k_1,\ldots,k_n}\|} \left[\left\| \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} x_{j_1,\ldots,j_n} \right\| + \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \left| \|x_{j_1,\ldots,j_n}\| - \|x_{k_1,\ldots,k_n}\| \right| \right] \right\}.
\end{aligned}$$

This follows immediately from Theorem 2.1 by requiring $x_{j_1,\ldots,j_n} \neq 0$ for $j_i = 1, \ldots, n$, and letting $\alpha_{k_1,\ldots,k_n} = 1/\|x_{k_1\cdots k_n}\|$ for $k_i = 1, \ldots, n$; $n \geq 2$.

A somewhat surprising consequence of Theorem 2.3 is the following version.

Theorem 2.4. *Let* $(X, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$. *If* $x_{j_1,\ldots,j_n} \in X \setminus \{0\}$ *for* $j_1, \ldots, j_n \in \{1, \ldots, n\}$ *with* $n \geq 2$, *then*

$$\begin{aligned}
&\left\| \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} x_{j_1,\ldots,j_n} \right\| + \left(n^n - \left\| \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \frac{x_{j_1,\ldots,j_n}}{\|x_{j_1,\ldots,j_n}\|} \right\| \right) \min_{\substack{j_i=1,\ldots,n\\ i=1,\ldots,n}} \|x_{j_1,\ldots,j_n}\| \\
&\leq \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \|x_{j_1,\ldots,j_n}\| \qquad (2.10) \\
&\leq \left\| \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} x_{j_1,\ldots,j_n} \right\| + \left(n^n - \left\| \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \frac{x_{j_1,\ldots,j_n}}{\|x_{j_1,\ldots,j_n}\|} \right\| \right) \max_{\substack{j_i=1,\ldots,n\\ i=1,\ldots,n}} \|x_{j_1,\ldots,j_n}\|.
\end{aligned}$$

Proof. Letting $\|x_{i_1,\ldots,i_n}\| = \max_{j_i=1,\ldots,n,\ i=1,\ldots,n} \|x_{j_1,\ldots,j_n}\|$ and by using the second inequality in (2.9), we have

$$\begin{aligned}
\left\| \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \frac{x_{j_1,\ldots,j_n}}{\|x_{j_1,\ldots,j_n}\|} \right\| &\leq \frac{1}{\|x_{i_1,\ldots,i_n}\|} \left(\left\| \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} x_{j_1,\ldots,j_n} \right\| + \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \left| \|x_{j_1,\ldots,j_n}\| - \|x_{i_1,\ldots,i_n}\| \right| \right) \\
&= \frac{1}{\|x_{i_1,\ldots,i_n}\|} \left(\left\| \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} x_{j_1,\ldots,j_n} \right\| + n^n \|x_{i_1,\ldots,i_n}\| - \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \|x_{j_1,\ldots,j_n}\| \right).
\end{aligned} \qquad (2.11)$$

Hence

$$\|x_{i_1,\ldots,i_n}\| \left\| \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \frac{x_{j_1,\ldots,j_n}}{\|x_{j_1,\ldots,j_n}\|} \right\| \leq \left\| \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} x_{j_1,\ldots,j_n} \right\| + n^n \|x_{i_1,\ldots,i_n}\| - \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \|x_{j_1,\ldots,j_n}\|. \qquad (2.12)$$

Then it follows that

$$\begin{aligned}\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\|x_{j_1,\ldots,j_n}\| &\le \left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}x_{j_1,\ldots,j_n}\right\| + \left(n^n - \left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\frac{x_{j_1,\ldots,j_n}}{\|x_{j_1,\ldots,j_n}\|}\right\|\right)\|x_{i_1,\ldots,i_n}\| \\ &= \left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}x_{j_1,\ldots,j_n}\right\| + \left(n^n - \left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\frac{x_{j_1,\ldots,j_n}}{\|x_{j_1,\ldots,j_n}\|}\right\|\right)\max_{\substack{j_i=1,\ldots,n\\ i=1,\ldots,n}}\|x_{j_1,\ldots,j_n}\|.\end{aligned} \tag{2.13}$$

On the other hand, letting $\|x_{k_1,\ldots,k_n}\| = \min_{j_i=1,\ldots,n,\ i=1,\ldots,n}\|x_{j_1,\ldots,j_n}\|$ and by using the first inequality in (2.9), we have

$$\begin{aligned}\left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\frac{x_{j_1,\ldots,j_n}}{\|x_{j_1,\ldots,j_n}\|}\right\| &\ge \frac{1}{\|x_{k_1,\ldots,k_n}\|}\left(\left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}x_{j_1,\ldots,j_n}\right\| - \sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\left|\|x_{j_1,\ldots,j_n}\| - \|x_{k_1,\ldots,k_n}\|\right|\right) \\ &= \frac{1}{\|x_{k_1,\ldots,k_n}\|}\left(\left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}x_{j_1,\ldots,j_n}\right\| + n^n\|x_{k_1,\ldots,k_n}\| - \sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\|x_{j_1,\ldots,j_n}\|\right).\end{aligned} \tag{2.14}$$

Hence

$$\|x_{k_1,\ldots,k_n}\|\left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\frac{x_{j_1,\ldots,j_n}}{\|x_{j_1,\ldots,j_n}\|}\right\| \ge \left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}x_{j_1,\ldots,j_n}\right\| + n^n\|x_{k_1,\ldots,k_n}\| - \sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\|x_{j_1,\ldots,j_n}\|, \tag{2.15}$$

from which we get

$$\begin{aligned}\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\|x_{j_1,\ldots,j_n}\| &\ge \left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}x_{j_1,\ldots,j_n}\right\| + \left(n^n - \left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\frac{x_{j_1,\ldots,j_n}}{\|x_{j_1,\ldots,j_n}\|}\right\|\right)\|x_{k_1,\ldots,k_n}\| \\ &= \left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}x_{j_1,\ldots,j_n}\right\| + \left(n^n - \left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\frac{x_{j_1,\ldots,j_n}}{\|x_{j_1,\ldots,j_n}\|}\right\|\right)\min_{\substack{j_i=1,\ldots,n\\ i=1,\ldots,n}}\|x_{j_1,\ldots,j_n}\|.\end{aligned} \tag{2.16}$$

This completes the proof. □

Remark 2.5. In case the multi-indices $j_1,\ldots,j_n$ and $k_1,\ldots,k_n$ reduce to single indices j and k, respectively, after suitable modifications, (2.10) reduces to inequality (1.2) obtained in [2] by Kato et al.

Theorem 2.6. *Let* $(X, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$*. If* $x_{j_1,\ldots,j_n} \in X \setminus \{0\}$ *for* $j_1, \ldots, j_n \in \{1, \ldots, n\}$ *with* $n \geq 2$ *and* $p \geq 1$*, then*

$$\begin{aligned}
&\min_{\substack{1\leq j_i\leq n\\ i=1,\ldots,n}}\left\{\left\|x_{j_1,\ldots,j_n}\right\|\right\}\left[\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\left\|x_{j_1,\ldots,j_n}\right\|^{p-1}-\left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\frac{x_{j_1,\ldots,j_n}}{\left\|x_{j_1,\ldots,j_n}\right\|}\right\|^{p}\right]\\
&\quad\leq\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\left\|x_{j_1,\ldots,j_n}\right\|^{p}-n^{n(1-p)}\left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}x_{j_1,\ldots,j_n}\right\|^{p}\\
&\quad\leq\max_{\substack{1\leq j_i\leq n\\ i=1,\ldots,n}}\left\{\left\|x_{j_1,\ldots,j_n}\right\|\right\}\left[\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\left\|x_{j_1,\ldots,j_n}\right\|^{p-1}-\left\|\sum_{j_1=1}^{n}\cdots\sum_{j_n=1}^{n}\frac{x_{j_1,\ldots,j_n}}{\left\|x_{j_1,\ldots,j_n}\right\|}\right\|^{p}\right].
\end{aligned}\tag{2.17}$$

This follows much in the line as the proofs of Theorem 2.1 and Theorem 2.4, and so it is omitted here.

Remark 2.7. In case the multi-index $j_1, \ldots, j_n$ reduces to a single index j, after suitable modifications, (2.17) reduces to inequality (1.3) obtained by Dragomir in [3].

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