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Research Article

Note on the q-Extension of Barnes' Type Multiple Euler Polynomials

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We construct the q-Euler numbers and polynomials of higher order, which are related to Barnes' type multiple Euler polynomials. We also derive many properties and formulae for our q-Euler polynomials of higher order by using the multiple integral equations on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then one normally assumes $|1-q|_p < 1$. We use the following notations:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}, \tag{1.1}$$

for all $x \in \mathbb{Z}_p$ (see [1–6]).

Let d a fixed positive odd integer with $(p,d)=1$. For $N\in\mathbb{N}$, we set

$$X = X_d = \frac{\lim_{\overrightarrow{N}}\mathbb{Z}}{dp^N\mathbb{Z}}, \qquad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<dp\\ (a,p)=1}} \left(a + dp\,\mathbb{Z}_p\right), \tag{1.2}$$
$$a + dp^N\mathbb{Z}_p = \left\{x\in X \mid x\equiv a \left(\operatorname{mod} dp^N\right)\right\},$$

where $a\in\mathbb{Z}$ lies in $0\le a<dp^N$. The fermionic p-adic q-measures on $\mathbb{Z}_p$ are defined as

$$\mu_{-q}\left(a+dp^N\mathbb{Z}_p\right) = \frac{(-q)^a}{\left[dp^N\right]_{-q}}, \tag{1.3}$$

(see [5]).

We say that f is a uniformly differentiable function at a point $a\in\mathbb{Z}_p$ and write $f\in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x,y)=(f(x)-f(y))/(x-y)$ have a limit $f'(a)$ as $(x,y)\to(a,a)$. For $f\in UD(\mathbb{Z}_p)$, let us begin with expression

$$\frac{1}{\left[p^N\right]_{-q}}\sum_{0\le j<p^N}(-q)^j f(j) = \sum_{0\le j<p^N} f(j)\mu_{-q}\left(j+p^N\mathbb{Z}_p\right), \tag{1.4}$$

which represents a q-analogue of Riemann sums for f in the fermionic sense (see [4, 5]). The integral of f on $\mathbb{Z}_p$ is defined by the limit of these sums (as $n\to\infty$) if this limit exists. The fermionic invariant p-adic q-integral of function $f\in UD(\mathbb{Z}_p)$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = \lim_{N\to\infty}\frac{[2]_q}{1+q^{p^N}}\sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{1.5}$$

Note that if $f_n\to f$ in $UD(\mathbb{Z}_p)$, then

$$\int_{\mathbb{Z}_p} f_n(x)\,d\mu_{-q}(x) \longrightarrow \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x), \qquad \int_X f(x)\,d\mu_{-q}(x) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x). \tag{1.6}$$

The Barnes' type Euler polynomials are considered as follows:

$$2^k\prod_{j=1}^{k}\left(\frac{1}{e^{w_j t}+1}\right)e^{xt} = \sum_{n=0}^{\infty}E_n^{(k)}(x\mid w_1,\dots,w_k)\frac{t^n}{n!}, \tag{1.7}$$

where $w_1,w_2,\dots,w_k\in\mathbb{Z}$ (cf. [7]).

From (1.5), we can derive the fermionic invariant integral on $\mathbb{Z}_p$ as follows:

$$\lim_{q \to 1} I_{-q}(f) = I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x). \tag{1.8}$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$, one has

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \tag{1.9}$$

By (1.9), we see that

$$e^{xt} \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} e^{(w_1 x_1 + \cdots + w_k x_k)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = 2^k \prod_{j=1}^{k} \left(\frac{1}{e^{w_j t} + 1} \right). \tag{1.10}$$

From (1.10), we note that

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} (x + w_1 x_1 + \cdots + w_k x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = E_n^{(k)}(x \mid w_1, \ldots, w_k). \tag{1.11}$$

In the view point of (1.11), we try to study the q-extension of Barnes' type Euler polynomials by using the q-extension of fermionic p-adic invariant integral on $\mathbb{Z}_p$.

The purpose of this paper is to construct the q-Euler numbers and polynomials of higher order, which are related to Barnes' type multiple Euler numbers and polynomials. Also, we give many properties and formulae for our q-Euler polynomials of higher order. Finally, we give the generating function for these q-Euler polynomials of higher order.

2. Barnes' Type Multiple q-Euler Polynomials

Let $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k \in \mathbb{Z}$. For $w \in \mathbb{Z}_p$ and $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, we define the Barnes' type multiple q-Euler polynomials as follows:

$$E_{n,q}^{(k)}(w \mid a_1, \ldots, a_k; b_1, \ldots, b_k) = \int_{\mathbb{Z}_p^k} q^{\sum_{j=1}^{k}(b_j - 1)x_j} \left[w + \sum_{j=1}^{k} a_j x_j \right]_q^n d\mu_{-q}(x), \tag{2.1}$$

where

$$\int_{\mathbb{Z}_p^k} f(x_1,\ldots,x_k)\,d\mu_{-q}(x) = \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\text{-times}} f(x_1,\ldots,x_k)\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k) \tag{2.2}$$

(see [1, 5]).

In the special case $w=0$, $E_n^{(k)}(a_1,\ldots,a_k;b_1,\ldots,b_k)$ are called the Barnes' type multiple q-Euler numbers. From (2.1), one has

$$\begin{aligned}
&\int_{\mathbb{Z}_p^k} q^{\sum_{j=1}^k (b_j-1)x_j}\left[w+\sum_{j=1}^k a_j x_j\right]_q^n d\mu_{-q}(x) \\
&\quad= \frac{1}{(1-q)^n}\sum_{r=0}^n \binom{n}{r}(-q^w)^r \lim_{N\to\infty}\left(\frac{1+q}{1+q^{p^N}}\right)\sum_{x_1,\ldots,x_k=0}^{p^N-1} q^{\sum_{j=1}^k (a_j r+b_j)x_j}(-1)^{x_1+\cdots+x_k} \\
&\quad= \frac{1}{(1-q)^n}\sum_{r=0}^n \binom{n}{r}(-q^w)^r [2]_q^k \prod_{j=1}^k\left(\frac{1}{1+q^{a_j r+b_j}}\right).
\end{aligned} \tag{2.3}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *Let $w\in\mathbb{Z}_p$ and $k\in\mathbb{N}$. For $a_1,a_2,\ldots,a_k,b_1,b_2,\ldots,b_k\in\mathbb{Z}$, one has*

$$E_{n,q}^{(k)}(w\mid a_1,\ldots,a_k;b_1,\ldots,b_k) = \frac{[2]_q^k}{(1-q)^n}\sum_{r=0}^n\binom{n}{r}(-q^w)^r\prod_{j=1}^k\left(\frac{1}{1+q^{a_j r+b_j}}\right). \tag{2.4}$$

By (1.7), we easily see that

$$\lim_{q\to 1} E_{n,q}^{(k)}(w\mid a_1,\ldots,a_k;b_1,\ldots,b_k) = E_n^{(k)}(w\mid a_1,\ldots,a_k). \tag{2.5}$$

From (1.7), we can derive

$$\begin{aligned}
&\int_{\mathbb{Z}_p^k} q^{\sum_{j=1}^k (b_j-1)x_j}\left[\sum_{j=1}^k a_j x_j\right]_q^n d\mu_{-q}(x) \\
&\quad= (q-1)\int_{\mathbb{Z}_p^k} q^{\sum_{j=1}^k (b_j-a_j-1)x_j}\left[\sum_{j=1}^k a_j x_j\right]_q^{n+1} d\mu_{-q}(x) + \int_{\mathbb{Z}_p^k} q^{\sum_{j=1}^k (b_j-a_j-1)x_j}\left[\sum_{j=1}^k a_j x_j\right]_q^n d\mu_{-q}(x).
\end{aligned} \tag{2.6}$$

By (2.6), one has

$$E_{n,q}^{(k)}(a_1,\dots,a_k;b_1,\dots,b_k) = (q-1)E_{n+1,q}^{(k)}(a_1,\dots,a_k;b_1-a_1,\dots,b_k-a_k) + E_{n,q}^{(k)}(a_1,\dots,a_k;b_1-a_1,\dots,b_k-a_k). \tag{2.7}$$

Hence we obtain the following theorem.

Theorem 2.2. *For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has*

$$E_{n,q}^{(k)}(a_1,\dots,a_k;b_1,\dots,b_k) = (q-1)E_{n+1,q}^{(k)}(a_1,\dots,a_k;b_1-a_1,\dots,b_k-a_k) + E_{n,q}^{(k)}(a_1,\dots,a_k;b_1-a_1,\dots,b_k-a_k). \tag{2.8}$$

It is not difficult to show that the following integral equation is satisfied:

$$\begin{aligned}\sum_{j=0}^{i}\binom{i}{j}(q-1)^j\int_{\mathbb{Z}_p^k}[a_1x_1+\cdots+a_kx_k]_q^{n-i+j}q^{\sum_{l=1}^{k}(b_l-1)x_l}d\mu_{-q}(x)\\ =\sum_{j=0}^{i}\binom{i-m}{j}(q-1)^j\int_{\mathbb{Z}_p^k}[a_1x_1+\cdots+a_kx_k]_q^{n-i+j}q^{\sum_{l=1}^{k}(b_l+ma_l-1)x_l}d\mu_{-q}(x),\end{aligned} \tag{2.9}$$

where $m \in \mathbb{N}$ with $i \ge m$. By (2.9), we obtain the following theorem.

Theorem 2.3. *Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. For $i \in \mathbb{N}$ with $i \ge m$, one has*

$$\begin{aligned}\sum_{j=0}^{i}\binom{i}{j}(q-1)^jE_{n-i+j,q}^{(k)}(a_1,\dots,a_k;b_1,\dots,b_k)\\ =\sum_{j=0}^{i}\binom{i-m}{j}(q-1)^jE_{n-i+j,q}^{(k)}(a_1,\dots,a_k;b_1+ma_1,\dots,b_k+ma_k).\end{aligned} \tag{2.10}$$

For the special case $k=1$ in Theorem 2.3, one has

$$\begin{aligned}\sum_{j=0}^{n}\binom{n}{j}(q-1)^jE_{j,q}^{(1)}(a_1;b_1) &= \sum_{j=0}^{n}\binom{n-m}{j}(q-1)^jE_{j,q}^{(1)}(a_1;b_1+ma_1)\\ &=\int_{\mathbb{Z}_p}q^{(na_1+b_1-1)x}d\mu_{-q}(x)=\frac{[2]_q}{1+q^{na_1+b_1}}.\end{aligned} \tag{2.11}$$

By (2.1), (2.3), and (2.9), we obtain the following *a* corollary.

Corollary 2.4. *For $n, k \in \mathbb{N}$ and $w \in \mathbb{Z}_p$, one has*

$$\begin{aligned} E_{n,q}^{(k)}(w \mid a_1, \ldots, a_k; b_1, \ldots, b_k) &= \frac{[2]_q^k}{(1-q)^n} \sum_{r=0}^{n} \binom{n}{r} (-q^w)^r \prod_{j=1}^{k} \left(\frac{1}{1+q^{a_j r + b_j}} \right) \\ &= \sum_{i=0}^{n} \binom{n}{i} [w]_q^{n-i} q^{wi} E_{i,q}^{(k)}(w \mid a_1, \ldots, a_k; b_1, \ldots, b_k). \end{aligned} \tag{2.12}$$

From (2.3), we note that

$$\begin{aligned} & q^w \int_{\mathbb{Z}_p^k} \left[w + \sum_{j=1}^{k} a_j x_j \right]_q^m q^{\sum_{j=1}^{k} (b_j - 1) x_j} d\mu_{-q}(x) \\ &\quad = (q-1) \int_{\mathbb{Z}_p^k} \left[w + \sum_{j=1}^{k} a_j x_j \right]_q^{m+1} q^{\sum_{j=1}^{k} (b_j - a_j - 1) x_j} d\mu_{-q}(x) \\ &\qquad + \int_{\mathbb{Z}_p^k} \left[w + \sum_{j=1}^{k} a_j x_j \right]_q^m q^{\sum_{j=1}^{k} (b_j - a_j - 1) x_j} d\mu_{-q}(x), \\ & \int_{X^k} \left[w + \sum_{j=1}^{k} a_j x_j \right]_q^m q^{\sum_{j=1}^{k} (b_j - 1) x_j} d\mu_{-q}(x) \\ &\quad = [d]_q^m [2]_q^k \sum_{i_1, \ldots, i_k = 0}^{d-1} q^{\sum_{j=1}^{k} b_j i_j} \\ &\qquad \times (-1)^{i_1 + \cdots + i_k} \int_{\mathbb{Z}_p^k} \left[\frac{w + \sum_{j=1}^{k} a_j i_j}{d} + \sum_{j=1}^{k} a_j x_j \right]_{q^d}^m q^{d \sum_{j=1}^{k} (b_j - 1) x_j} d\mu_{-q^d}(x), \end{aligned} \tag{2.13}$$

where d is an odd positive integer. By (2.13), we obtain the following theorem.

Theorem 2.5. *For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, one has*

$$\begin{aligned} & E_{m,q}^{(k)}(w \mid a_1, \ldots, a_k; b_1, \ldots, b_k) \\ &\quad = [d]_q^m [2]_q^k \sum_{i_1, \ldots, i_k = 0}^{d-1} q^{\sum_{j=1}^{k} b_j i_j} (-1)^{i_1 + \cdots + i_k} E_{m,q^d}^{(k)} \left(\frac{w + \sum_{j=1}^{k} a_j i_j}{d} \mid a_1, \ldots, a_k; b_1, \ldots, b_k \right), \\ & q^w E_{m,q}^{(k)}(w \mid a_1, \ldots, a_k; b_1, \ldots, b_k) \\ &\quad = (q-1) E_{m+1,q}^{(k)}(w \mid a_1, \ldots, a_k; b_1 - a_1, \ldots, b_k - a_k) \\ &\qquad + E_{m,q}^{(k)}(w \mid a_1, \ldots, a_k; b_1 - a_1, \ldots, b_k - a_k). \end{aligned} \tag{2.14}$$

Remark 2.6. Let

$$F_q(w,t) = \sum_{n=0}^{\infty} E_{n,q}^{(k)}(w \mid a_1,\dots,a_k; b_1,\dots,b_k)\frac{t^n}{n!}. \tag{2.15}$$

From (2.4), we can easily derive the following equation:

$$\begin{aligned} F_q(w,(q-1)t) &= \sum_{m=0}^{\infty}(q-1)^m E_{m,q}^{(k)}(w \mid a_1,\dots,a_k; b_1,\dots,b_k)\frac{t^m}{m!} \\ &= [2]_q^k\ e^{-t}\sum_{i=0}^{\infty}\left(\prod_{j=1}^{k}\frac{1}{1+q^{a_j i+b_j}}\right)q^{wi}\frac{t^i}{i!}. \end{aligned} \tag{2.16}$$

By differentiating both sides of (2.16) with respect to t and comparing coefficients on both sides, one has

$$\begin{aligned} &q^w E_{m,q}^{(k)}(w \mid a_1,\dots,a_k; b_1,\dots,b_k) - E_{m,q}^{(k)}(w \mid a_1,\dots,a_k; b_1-a_1,\dots,b_k-a_k) \\ &\quad= (q-1)E_{m+1,q}^{(k)}(w \mid a_1,\dots,a_k; b_1-a_1,\dots,b_k-a_k). \end{aligned} \tag{2.17}$$

The inversion formula of Equation (2.4) at $w=0$ is given by

$$\sum_{i=0}^{m}\binom{m}{i}(q-1)^i\int_{\mathbb{Z}_p^k}[a_1x_1+\cdots+a_kx_k]_q^i q^{\sum_{j=1}^{k}(b_j-1)x_j}d\mu_{-q}(x) = \int_{\mathbb{Z}_p^k}q^{\sum_{j=1}^{k}(ma_j+b_j-1)x_j}d\mu_{-q}(x). \tag{2.18}$$

Thus, one has

$$\sum_{i=0}^{m}\binom{m}{i}(q-1)^i E_{i,q}^{(k)}(a_1,\dots,a_k; b_1,\dots,b_k) = [2]_q^k\prod_{j=1}^{k}\left(\frac{1}{1+q^{ma_j+b_j}}\right). \tag{2.19}$$

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