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Research Article

Multidimensional Hilbert-Type Inequalities with a Homogeneous Kernel

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We consider the Hilbert-type inequalities with nonconjugate parameters. The obtaining of the best possible constants in the case of nonconjugate parameters remains still open. Our generalization will include a general homogeneous kernel. Also, we obtain the best possible constants in the case of conjugate parameters when the parameters satisfy appropriate conditions. We also compare our results with some known results.

1. Introduction

Let $1/p + 1/q = 1$ $(p > 1)$, $f, g \geq 0$,

$$0 < \int_0^\infty f^p(x)dx < \infty, \qquad 0 < \int_0^\infty g^q(x)dx < \infty. \tag{1.1}$$

The well-known Hardy-Hilbert's integral inequality (see [1]) is given by

$$\iint_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(x)dx\right)^{1/q}, \tag{1.2}$$

and an equivalent form is given by

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \int_0^\infty f^p(x)dx, \tag{1.3}$$

where the constant factors $\pi/\sin(\pi/p)$ and $[\pi/\sin(\pi/p)]^p$ are the best possible.

During the previous decades, the Hilbert-type inequalities were discussed by many authors, who either reproved them using various techniques or applied and generalized them in many different ways. For example, we refer to a paper of Yang (see [2]). If $n \in \mathbb{N} \setminus \{1\}$, $p_i > 1$, $\sum_{i=1}^{n}(1/p_i) = 1$, $s > 0$, $f_i \geq 0$, satisfy

$$0 < \int_0^\infty x^{p_i-s-1} f_i^{p_i}(x)dx < \infty \quad (i = 1,2,\dots,n), \tag{1.4}$$

then

$$\int_{(0,\infty)^n} \frac{\prod_{i=1}^{n} f_i(x_i)}{\left(\sum_{j=1}^{n} x_j\right)^s} dx_1 \cdots dx_n < \frac{1}{\Gamma(s)} \prod_{i=1}^{n} \Gamma\left(\frac{s}{p_i}\right) \left(\int_0^\infty x^{p_i-s-1} f_i^{p_i}(x)dx\right)^{1/p_i}, \tag{1.5}$$

where the constant factor $(1/\Gamma(s))\prod_{i=1}^{n}\Gamma(s/p_i)$ is the best possible.

Our generalization will include a general homogeneous kernel $K(x_1,\dots,x_k) : (\mathbb{R}_+{}^n)^k \to \mathbb{R}$, where $k \geq 2$, with k being nonconjugate parameters. The techniques that will be used in the proofs are mainly based on classical real analysis, especially on the well-known Hölder's inequality and on Fubini's theorem. The obtaining of the best possible constants in the case of nonconjugate parameters seems to be a very difficult problem and it remains still open.

Let us recall the definition of nonconjugate exponents (see [3]). Let p and q be real parameters, such that

$$p > 1, \quad q > 1, \quad \frac{1}{p} + \frac{1}{q} \geq 1, \tag{1.6}$$

and let p' and q', respectively, be their conjugate exponents, that is, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Further, define

$$\lambda = \frac{1}{p'} + \frac{1}{q'} \tag{1.7}$$

and note that $0 < \lambda \leq 1$ for all p and q values as in (1.6). In particular, $\lambda = 1$ holds if and only if $q = p'$, that is, only when p and q are mutually conjugate. Otherwise, $0 < \lambda < 1$, and in such cases p and q will be referred to as nonconjugate exponents.

Considering p, q, and λ as in (1.6) and (1.7), Hardy et al. [1], proved that there exists a constant $C_{p,q}$, dependent only on the parameters p and q, such that the following Hilbert-type inequality holds for all nonnegative functions $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$:

$$\iint_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx\, dy \leq C_{p,q} \|f\|_{L^p(\mathbb{R}_+)} \|g\|_{L^q(\mathbb{R}_+)}. \tag{1.8}$$

Conventions

Throughout this paper we suppose that all the functions are nonnegative and measurable, so that all integrals converge. We also introduce the following notations:

$$\mathbb{R}_+{}^n = \{x = (x_1, x_2, \ldots, x_n);\ x_1, x_2, \ldots, x_n > 0\},$$
$$|x|_\alpha = (x_1{}^\alpha + x_2{}^\alpha + \cdots + x_n{}^\alpha)^{1/\alpha}, \quad \alpha > 0, \tag{1.9}$$

and let $|\mathbb{S}^{n-1}|_\alpha = 2^n\Gamma^n(1/\alpha)/\alpha^{n-1}\Gamma(n/\alpha)$ be an area of unit sphere in $\mathbb{R}^n$ in view of α–norm.

2. Main Results

Before presenting our idea and results, we repeat the notion of general nonconjugate exponents from [3]. Let $p_i,\ i = 1, 2, \ldots, k$, be the real parameters which satisfy

$$\sum_{i=1}^{k} \frac{1}{p_i} \geq 1, \quad p_i > 1,\ i = 1, 2, \ldots, k. \tag{2.1}$$

Further, the parameters p_i', $i = 1, 2, \ldots, k$ are defined by the equations

$$\frac{1}{p_i} + \frac{1}{p_i'} = 1, \quad i = 1, 2, \ldots, k. \tag{2.2}$$

Since $p_i > 1,\ i = 1, 2, \ldots, k$, it is obvious that $p_i' > 1,\ i = 1, 2, \ldots, k$. We define

$$\lambda := \frac{1}{k-1} \sum_{i=1}^{k} \frac{1}{p_i'}. \tag{2.3}$$

It is easy to deduce that $0 < \lambda \leq 1$. Also, we introduce the parameters $q_i,\ i = 1, 2, \ldots, k$, defined by the relations

$$\frac{1}{q_i} = \lambda - \frac{1}{p_i'}, \quad i = 1, 2, \ldots, k. \tag{2.4}$$

In order to obtain our results we need to require

$$q_i > 0, \quad i = 1, 2, \ldots, k. \tag{2.5}$$

It is easy to see that the above conditions do not automatically apply (2.5). Further, it follows

$$\lambda = \sum_{i=1}^{k} \frac{1}{q_i}, \quad \frac{1}{q_i} + 1 - \lambda = \frac{1}{p_i}, \quad i = 1, 2, \ldots, k. \tag{2.6}$$

Of course, if $\lambda = 1$, then $\sum_{i=1}^{k}(1/p_i) = 1$; so the conditions (2.1)–(2.4) reduce to the case of conjugate parameters.

Results in this section will be based on the following general form of Hardy-Hilbert's inequality proven in [4]. All the measures are assumed to be σ-finite on some Ω measure space.

Theorem 2.1. *Let $k, n \in \mathbb{N}$, $k \geq 2$, and λ, p_i, p_i', q_i, $i = 1, 2, \ldots, k$, be real numbers satisfying* (2.1)–(2.5). *Let $K : \Omega^k \to \mathbb{R}$ and $\phi_{ij} : \Omega \to \mathbb{R}$, $i, j = 1, \ldots, k$, be nonnegative measurable functions such that $\prod_{i,j=1}^{k}\phi_{ij}(x_j) = 1$. Then, for any nonnegative measurable functions f_i, $i = 1, 2, \ldots, k$, the following inequalities hold and are equivalent:*

$$\int_{\Omega^k} K^{\lambda}(x_1, \ldots, x_k)\prod_{i=1}^{k} f_i(x_i)d\mu_1(x_1)\cdots d\mu_k(x_k) \leq \prod_{i=1}^{k}\left(\int_{\Omega}(\phi_{ii}F_i f_i)^{p_i}(x_i)d\mu_i(x_i)\right)^{1/p_i}, \tag{2.7}$$

$$\left(\int_{\Omega}\left(\frac{1}{(\phi_{kk}F_k)(x_k)}\int_{\Omega^{k-1}} K^{\lambda}(x_1, \ldots, x_k)\prod_{i=1}^{k-1} f_i(x_i)d\mu_1(x_1)\cdots d\mu_{k-1}(x_{k-1})\right)^{p_k'} d\mu_k(x_k)\right)^{1/p_k'} \leq \prod_{i=1}^{k-1}\left(\int_{\Omega}(\phi_{ii}F_i f_i)^{p_i}(x_i)d\mu_i(x_i)\right)^{1/p_i}, \tag{2.8}$$

where

$$F_i(x_i) = \left(\int_{\Omega^{k-1}} K(x_1, \ldots, x_k) \cdot \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j)d\mu_1(x_1)\cdots d\mu_{i-1}(x_{i-1})d\mu_{i+1}(x_{i+1})\cdots d\mu_k(x_k)\right)^{1/q_i}, \quad i = 1, \ldots, k. \tag{2.9}$$

In the same paper the authors discussed the case of equality in inequalities (2.7) and (2.8). They proved that the equality holds in (2.7) (and analogously in (2.8)) if and only if

$$f_i(x_i) = C_i\phi_{ii}(x_i)^{q_i/(1-\lambda q_i)}F_i(x_i)^{(1-\lambda)q_i}, \quad C_i \geq 0, \ i = 1, \ldots, k. \tag{2.10}$$

In the following theorem we give the most important case where $\Omega = \mathbb{R}_+{}^n$, the measures μ_i, $i = 1, \ldots, k$, are Lebesgue measures, $K_\alpha : (0, \infty)^k \to \mathbb{R}$ is a nonnegative homogeneous function of degree $-s$, $s > 0$, and the functions ϕ_{ij} represent the form $\phi_{ij}(x_j) = |x_j|_\alpha^{A_{ij}}$ where $A_{ij} \in \mathbb{R}$, $i, j = 1, \ldots, n$. In order to obtain the generalizations of some known results we define

$$k_\alpha(\beta_1, \ldots, \beta_{k-1}) := \int_{(0,\infty)^{k-1}} K_\alpha(1, t_1, \ldots, t_{k-1})t_1^{\beta_1}\cdots t_{k-1}^{\beta_{k-1}}dt_1\cdots dt_{k-1}, \tag{2.11}$$

where we suppose that $k_\alpha(\beta_1, \ldots, \beta_{k-1}) < \infty$ for $\beta_1, \ldots, \beta_{k-1} > -1$ and $\beta_1 + \cdots + \beta_{k-1} + k < s + 1$.

Due to technical reasons, we introduce real parameters A_{ij}, $i, j = 1, 2, \ldots, k$ satisfying

$$\sum_{i=1}^{k} A_{ij} = 0, \quad j = 1, 2, \ldots, k. \tag{2.12}$$

We also define

$$\alpha_i = \sum_{j=1}^{k} A_{ij}, \quad i = 1, 2, \ldots, k. \tag{2.13}$$

Theorem 2.2. *Let $k, n \in \mathbb{N}$, $k \geq 2$, and λ, p_i, p_i', q_i, $i = 1, 2, \ldots, k$, be real numbers satisfying* (2.1)–(2.5)*. Let $K_\alpha : (0, \infty)^k \to \mathbb{R}$ be nonnegative measurable homogeneous function of degree $-s$, $s > 0$, and let A_{ij}, $i, j = 1, \ldots, k$, and α_i, $i = 1, \ldots, k$ be real parameters satisfying* (2.12) *and* (2.13)*. If $f_i : {\mathbb{R}_+}^n \to \mathbb{R}$, $f_i \neq 0$, $i = 1, \ldots, k$ are nonnegative measurable functions, then the following inequalities hold and are equivalent:*

$$\int_{({\mathbb{R}_+}^n)^k} K_\alpha^\lambda(|x_1|_\alpha, \ldots, |x_k|_\alpha) \prod_{i=1}^{k} f_i(x_i)\, dx_1 \cdots dx_k < L \prod_{i=1}^{k} \left(\int_{{\mathbb{R}_+}^n} |x_i|_\alpha^{p_i/q_i[(k-1)n-s]+p_i\alpha_i} f_i^{p_i}(x_i)\, dx_i \right)^{1/p_i},$$

$$\begin{aligned} &\int_{{\mathbb{R}_+}^n} |x_k|_\alpha^{-(p_k'/q_k)[(k-1)n-s]-p_k'\alpha_k} \left(\int_{({\mathbb{R}_+}^n)^{k-1}} K_\alpha^\lambda(|x_1|_\alpha, \ldots, |x_k|_\alpha) \cdot \prod_{i=1}^{k-1} f_i(x_i)\, dx_1 \cdots dx_{k-1} \right)^{p_k'} dx_k \\ &\quad < L^{p_k'} \prod_{i=1}^{k-1} \left(\int_{{\mathbb{R}_+}^n} |x_i|_\alpha^{(p_i/q_i)[(k-1)n-s]+p_i\alpha_i} f_i^{p_i}(x_i)\, dx_i \right)^{p_k'/p_i}, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} L = {} & \frac{\left|\mathbb{S}^{n-1}\right|_\alpha^{(k-1)\lambda}}{2^{(k-1)n\lambda}} k_\alpha\left(n - 1 + q_1 A_{12}, \ldots, n - 1 + q_1 A_{1k}\right)^{1/q_1} \\ & \cdot k_\alpha\left(s - (k-1)n - 1 - q_2(\alpha_2 - A_{22}), n - 1 + q_2 A_{23}, \ldots, n - 1 + q_2 A_{2k}\right)^{1/q_2} \\ & \cdots k_\alpha\left(n - 1 + q_k A_{k2}, \ldots, n - 1 + q_k A_{k,k-1}, s - (k-1)n - 1 - q_k(\alpha_k - A_{kk})\right)^{1/q_k}, \end{aligned} \tag{2.15}$$

$q_i A_{ij} > -n$, $i \neq j$ *and* $q_i(A_{ii} - \alpha_i) > (k-1)n - s$.

Proof. Set $K(x_1,\ldots,x_k) = K_\alpha(|x_1|_\alpha,\ldots,|x_k|_\alpha)$ and $\phi_{ij}(x_j) = |x_j|^{A_{ij}}$ in Theorem 2.1, where $\sum_{i=1}^k A_{ij} = 0$ for every $j = 1,\ldots,k$. It is enough to calculate the functions $F_i(x_i)$, $i = 1,\ldots,k$. By using the n-dimensional spherical coordinates we find

$$\begin{aligned} F_1^{q_1}(x_1) &= \int_{(\mathbb{R}_+{}^n)^{k-1}} K_\alpha(|x_1|_\alpha,\ldots,|x_k|_\alpha)\prod_{j=2}^k |x_j|^{q_1A_{1j}}dx_2\cdots dx_k \\ &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}}\int_{(0,\infty)^{k-1}} K_\alpha(|x_1|_\alpha,t_2,\ldots,t_k)\prod_{j=2}^k t_j^{n-1+q_1A_{1j}}dt_2\cdots dt_k. \end{aligned} \tag{2.16}$$

Using homogeneity of the function K_α and the substitutions $u_i = t_i/|x_1|_\alpha$, $i = 2,\ldots,k$, we have

$$\begin{aligned} F_1^{q_1}(x_1) &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}}\int_{(0,\infty)^{k-1}} |x_1|_\alpha^{-s}K_\alpha(1,u_2,\ldots,u_k)\cdot\prod_{j=2}^k (|x_1|_\alpha u_j)^{n-1+q_1A_{1j}}|x_1|_\alpha^{k-1}du_2\cdots du_k \\ &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}}|x_1|_\alpha^{(k-1)n-s+q_1(\alpha_1-A_{11})}k_\alpha(n-1+q_1A_{12},\ldots,n-1+q_1A_{1k}). \end{aligned} \tag{2.17}$$

Similarly, by applying the n-dimensional spherical coordinates and homogeneity of the function K_α we have

$$\begin{aligned} F_2^{q_2}(x_2) &= \int_{(\mathbb{R}_+{}^n)^{k-1}} K_\alpha(|x_1|_\alpha,\ldots,|x_k|_\alpha)\prod_{j=1,j\neq 2}^k |x_j|^{q_2A_{2j}}dx_1dx_3\cdots dx_k \\ &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}}\int_{(0,\infty)^{k-1}} t_1^{-s}K_\alpha\left(1,\frac{|x_2|_\alpha}{t_1},\frac{t_3}{t_1},\ldots,\frac{t_k}{t_1}\right)\cdot\prod_{j=1,j\neq 2}^k t_j^{n-1+q_2A_{2j}}dt_1dt_3\cdots dt_k. \end{aligned} \tag{2.18}$$

Using the change of variables

$$t_1 = |x_2|_\alpha u_2^{-1}, \qquad t_i = |x_2|_\alpha u_2^{-1}u_i, \quad i = 3,\ldots,k, \quad \text{so } \frac{\partial(t_1,t_3,\ldots,t_k)}{\partial(u_2,u_3,\ldots,u_k)} = |x_2|_\alpha^{k-1}u_2^{-k}, \tag{2.19}$$

where $\partial(t_1, t_3, \ldots, t_k)/\partial(u_2, u_3, \ldots, u_k)$ denotes the Jacobian of the transformation, we have

$$\begin{aligned} F_2^{q_2}(x_2) &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}} |x_2|_\alpha^{(k-1)n-s+q_2(\alpha_2-A_{22})} \\ &\quad \cdot \int_{(0,\infty)^{k-1}} K_\alpha(1, u_2, \ldots, u_k) u_2^{s-(k-1)n-q_2(\alpha_2-A_{22})} \prod_{j=3}^{k} u_j^{n-1+q_2A_{2j}} du_2 \cdots du_k \\ &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}} |x_2|_\alpha^{(k-1)n-s-q_2(\alpha_2-A_{22})} \\ &\quad \cdot k_\alpha\big(s-(k-1)n-1-q_2(\alpha_2-A_{22}), n-1+q_2A_{23}, \ldots, n-1+q_2A_{2k}\big). \end{aligned} \tag{2.20}$$

In a similar manner we obtain

$$\begin{aligned} F_i^{q_i}(x_i) &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}} |x_i|_\alpha^{(k-1)n-s+q_i(\alpha_i-A_{ii})} \\ &\quad \cdot k_\alpha\big(n-1+q_iA_{i2}, \ldots, n-1+q_iA_{i,i-1}, s-(k-1)n-1-q_i(\alpha_i-A_{ii}), \\ &\qquad n-1+q_iA_{i,i+1}, \ldots, n-1+q_iA_{ik}\big) \end{aligned} \tag{2.21}$$

for $i = 3, \ldots, k$. This gives inequalities (2.14) with inequality sign $\leq$. Condition (2.10) immediately gives that nontrivial case of equality in (2.14) leads to the divergent integrals. This completes the proof. □

Remark 2.3. Note that the kernel $K_\alpha(|x_1|_\alpha, \ldots, |x_k|_\alpha) = \left(\sum_{i=1}^k |x_i|_\alpha^\beta\right)^{-s}$ is a homogeneous function of degree $-\beta s$. In this case we have

$$\begin{aligned} k_\alpha(\beta_1, \ldots, \beta_{k-1}) &= \int_{(0,\infty)^{k-1}} \frac{\prod_{i=1}^{k-1} t_i^{\beta_i}}{\left(1+\sum_{i=1}^{k-1} t_i^{\beta}\right)^s} dt_1 \cdots dt_{k-1} \\ &= \frac{1}{\beta^{k-1}\Gamma(s)} \Gamma\left(s - \sum_{i=1}^{k-1} \frac{\beta_i+1}{\beta}\right) \prod_{i=1}^{k-1} \Gamma\left(\frac{\beta_i+1}{\beta}\right), \end{aligned} \tag{2.22}$$

where we used the well-known formula for gamma function (see, e.g., [5, Lemma 5.1]). Now, by using Theorem 2.2 and (2.22) we obtain the result of Krnić et al. (see [6]).

3. The Best Possible Constants in the Conjugate Case

In this section we consider the inequalities in Theorem 2.2. In such a way we shall obtain the best possible constants for some general cases.

It follows easily that Theorem 2.2 in the conjugate case ($\lambda = 1$, $p_i = q_i$) becomes as follows.

Theorem 3.1. *Let $k,\ n \in \mathbb{N},\ k \geq 2$ and let $p_1, \ldots, p_k$ be conjugate parameters such that $p_i > 1,\ i = 1, \ldots, k$. Let $K_\alpha : (0,\infty)^k \to \mathbb{R}$ be nonnegative measurable homogeneous function of degree $-s$, $s > 0$, and let $A_{ij},\ i,j = 1,\ldots,k$, and $\alpha_i,\ i = 1,\ldots,k$ be real parameters satisfying* (2.12) *and* (2.13). *If $f_i : {\mathbb{R}_+}^n \to \mathbb{R}$, $f_i \neq 0,\ i = 1,\ldots,k$ are nonnegative measurable functions, then the following inequalities hold and are equivalent:*

$$\int_{({\mathbb{R}_+}^n)^k} K_\alpha(|x_1|_\alpha,\ldots,|x_k|_\alpha)\prod_{i=1}^{k} f_i(x_i)dx_1\cdots dx_k < M\prod_{i=1}^{k}\left(\int_{{\mathbb{R}_+}^n}|x_i|_\alpha^{(k-1)n-s+p_i\alpha_i} f_i^{p_i}(x_i)dx_i\right)^{1/p_i},$$

$$\int_{{\mathbb{R}_+}^n}|x_k|_\alpha^{(1-p_k')[(k-1)n-s]-p_k'\alpha_k}\left(\int_{({\mathbb{R}_+}^n)^{k-1}} K_\alpha(|x_1|_\alpha,\ldots,|x_k|_\alpha)\cdot\prod_{i=1}^{k-1} f_i(x_i)dx_1\cdots dx_{k-1}\right)^{p_k'}dx_k$$
$$< M^{p_k'}\prod_{i=1}^{k-1}\left(\int_{{\mathbb{R}_+}^n}|x_i|_\alpha^{(k-1)n-s+p_i\alpha_i} f_i^{p_i}(x_i)dx_i\right)^{p_k'/p_i}, \tag{3.1}$$

where

$$\begin{aligned} M = {} & \frac{\left|\mathbb{S}^{n-1}\right|_\alpha^{(k-1)}}{2^{(k-1)n}} k_\alpha\left(n-1+p_1A_{12},\ldots,n-1+p_1A_{1k}\right)^{1/p_1} \\ & \cdot k_\alpha\left(s-(k-1)n-1-p_2(\alpha_2-A_{22}),n-1+p_2A_{23},\ldots,n-1+p_2A_{2k}\right)^{1/p_2} \\ & \cdots k_\alpha\left(n-1+p_kA_{k2},\ldots,n-1+p_kA_{k,k-1},s-(k-1)n-1-p_k(\alpha_k-A_{kk})\right)^{1/p_k}, \end{aligned} \tag{3.2}$$

$p_iA_{ij} > -n,\ i \neq j$ *and* $p_i(A_{ii}-\alpha_i) > (k-1)n-s$.

To obtain a case of the best inequality it is natural to impose the following conditions on the parameters A_{ij}:

$$n + p_jA_{ji} = s-(k-1)n-p_i(\alpha_i-A_{ii}), \quad j \neq i,\ i,j \in \{1,2,\ldots,k\}. \tag{3.3}$$

In that case the constant M from Theorem 3.1 is simplified to the following form:

$$M^* = \frac{\left|\mathbb{S}^{n-1}\right|_\alpha^{(k-1)}}{2^{(k-1)n}} k_\alpha\left(n-1+\widetilde{A}_2,\ldots,n-1+\widetilde{A}_k\right), \tag{3.4}$$

where

$$\widetilde{A}_i = p_1A_{1i} \quad \text{for } i \neq 1, \qquad \widetilde{A}_1 = p_kA_{k1}. \tag{3.5}$$

Further, by using (3.4) and (3.5), the inequalities (3.1) with the parameters A_{ij}, satisfying the relation (3.3), become

$$\int_{(\mathbb{R}_+{}^n)^k} K_\alpha(|x_1|_\alpha,\dots,|x_k|_\alpha)\prod_{i=1}^{k} f_i(x_i)dx_1\cdots dx_k < M^*\prod_{i=1}^{k}\left(\int_{\mathbb{R}_+{}^n}|x_i|_\alpha^{-n-p_i\widetilde{A}_i} f_i^{p_i}(x_i)dx_i\right)^{1/p_i}, \tag{3.6}$$

$$\left[\int_{\mathbb{R}_+{}^n}|x_k|_\alpha^{(1-p_k')(-n-p_k\widetilde{A}_k)}\left(\int_{(\mathbb{R}_+{}^n)^{k-1}} K_\alpha(|x_1|_\alpha,\dots,|x_k|_\alpha)\cdot\prod_{i=1}^{k-1} f_i(x_i)dx_1\cdots dx_{k-1}\right)^{p_k'}dx_k\right]^{1/p_k'}$$
$$< M^*\prod_{i=1}^{k-1}\left(\int_{\mathbb{R}_+{}^n}|x_i|_\alpha^{-n-p_i\widetilde{A}_i} f_i^{p_i}(x_i)dx_i\right)^{1/p_i}. \tag{3.7}$$

Theorem 3.2. *Suppose that the real parameters A_{ij}, $i,j=1,\dots,k$ satisfy conditions in Theorem 3.1 and conditions given in* (3.3). *If the kernel $K_\alpha(t_1,\dots,t_k)$ is as in Theorem 3.1 and for every $i=2,\dots,k$*

$$K_\alpha(1,t_2,\dots,t_i,\dots,t_k)\le CK_\alpha(1,t_2,\dots,0,\dots,t_k),\quad 0\le t_i\le 1,\ t_j\ge 0,\ j\ne i \tag{3.8}$$

for some $C>0$, then the constant M^* is the best possible in inequalities (3.6) *and* (3.7).

Proof. Let us suppose that the constant factor M^* given by (3.4) is not the best possible in the inequality (3.6). Then, there exists a positive constant $M_1<M^*$, such that (3.6) is still valid when we replace M^* by M_1.

We define the real functions $\widetilde{f}_{i,\varepsilon}:\mathbb{R}^n\mapsto\mathbb{R}$ by the formulas

$$\widetilde{f}_{i,\varepsilon}(x_i)=\begin{cases}0, & |x_i|_\alpha<1,\\ |x_i|_\alpha{}^{\widetilde{A}_i-\varepsilon/p_i}, & |x_i|_\alpha\ge 1,\end{cases}\qquad i=1,\dots,k, \tag{3.9}$$

where $0<\varepsilon<\min_{1\le i\le k}\{p_i+p_i\widetilde{A}_i\}$. Now, we shall put these functions in inequality (3.6). By using the n-dimensional spherical coordinates, the right-hand side of the inequality (3.6) becomes

$$M_1\prod_{i=1}^{k}\left[\int_{|x_i|_\alpha\ge 1}|x_i|_\alpha^{-n-\varepsilon}dx_i\right]^{1/p_i}=\frac{M_1|\mathbb{S}^{n-1}|_\alpha}{2^n}\int_1^\infty t^{-1-\varepsilon}dt=\frac{M_1|\mathbb{S}^{n-1}|_\alpha}{2^n\varepsilon}. \tag{3.10}$$

Further, let J denotes the left-hand side of the inequality (3.6), for the above choice of the functions $\widetilde{f}_{i,\varepsilon}$. By applying the n-dimensional spherical coordinates and the substitutions $u_i = t_i/t_1,\ i \neq 2$, we find

$$\begin{aligned} J &= \int_{|x_1|_\alpha \geq 1} \cdots \int_{|x_k|_\alpha \geq 1} K_\alpha(|x_1|_\alpha, \ldots, |x_k|_\alpha) \prod_{i=1}^{k} |x_i|_\alpha^{\widetilde{A}_i - \varepsilon/p_i} dx_1 \cdots dx_k \\ &= \frac{|\mathbb{S}^{n-1}|_\alpha^k}{2^{kn}} \int_1^\infty \cdots \int_1^\infty K_\alpha(t_1, \ldots, t_k) \prod_{i=1}^{k} t_i^{n-1+\widetilde{A}_i - \varepsilon/p_i} dt_1 \cdots dt_k \\ &= \frac{|\mathbb{S}^{n-1}|_\alpha^k}{2^{kn}} \int_1^\infty t_1^{-1-\varepsilon/\beta} \left(\int_{1/t_1}^\infty \cdots \int_{1/t_1}^\infty K_\alpha(1, u_2, \ldots, u_k) \prod_{i=2}^{k} u_i^{n-1+\widetilde{A}_i - \varepsilon/p_i} du_2 \cdots du_k \right) dt_1. \end{aligned} \tag{3.11}$$

Now, it is easy to see that the following inequality holds:

$$\begin{aligned} J \geq{}& \frac{|\mathbb{S}^{n-1}|_\alpha^k}{2^{kn}} \int_1^\infty t_1^{-1-\varepsilon} \left(\int_0^\infty \cdots \int_0^\infty K_\alpha(1, u_2, \ldots, u_k) \prod_{i=2}^{k} u_i^{n-1+\widetilde{A}_i - \varepsilon/p_i} du_2 \cdots du_k \right) dt_1 \\ &- \frac{|\mathbb{S}^{n-1}|_\alpha^k}{2^{kn}} \int_1^\infty t_1^{-1-\varepsilon} \sum_{j=2}^{k} I_j(t_1) dt_1, \end{aligned} \tag{3.12}$$

where for $j = 2, \ldots, k$, $I_j(t_1)$ is defined by

$$I_j(t_1) = \int_{D_j} K_\alpha(1, u_2, \ldots, u_k) \prod_{i=2}^{k} u_i^{n-1+\widetilde{A}_i - \varepsilon/p_i}\, du_2 \cdots du_k, \tag{3.13}$$

satisfying $D_j = \{(u_2, \ldots, u_k);\ 0 < u_j < 1/t_1,\ 0 < u_l < \infty,\ l \neq j\}$. Without losing generality, we only estimate the integral $I_2(t_1)$. For $k = 2$ we have

$$\begin{aligned} I_2(t_1) &= \int_0^{1/t_1} K_\alpha(1, u_2) u_2^{n-1+\widetilde{A}_2 - \varepsilon/p_2} du_2 \leq C \int_0^{1/t_1} u_2^{n-1+\widetilde{A}_2 - \varepsilon/p_2} du_2 \\ &= C \left(n + \widetilde{A}_2 - \frac{\varepsilon}{p_2} \right)^{-1} t_1^{\varepsilon/p_2 - n - \widetilde{A}_2}, \end{aligned} \tag{3.14}$$

and for $k > 2$ we find

$$I_2(t_1) \le C\left[\int_{(0,\infty)^{k-2}} K_\alpha(1,0,u_3,\ldots,u_k)\prod_{i=3}^{k} u_i^{n-1+\widetilde{A}_i-\varepsilon/p_i}\, du_3\cdots du_k\right]\int_0^{1/t_1} u_2^{n-1+\widetilde{A}_2-\varepsilon/p_2} du_2$$
$$= C\left(n-\frac{\varepsilon}{p_2}+\widetilde{A}_2\right)^{-1} t_1^{\varepsilon/p_2-\widetilde{A}_2-n} k_\alpha\left(n-1+\widetilde{A}_3-\frac{\varepsilon}{p_3},\ldots,n-1+\widetilde{A}_k-\frac{\varepsilon}{p_k}\right), \tag{3.15}$$

where $k_\alpha(n-1+\widetilde{A}_3-\varepsilon/p_3,\ldots,n-1+\widetilde{A}_k-\varepsilon/p_k)$ is well defined since obviously $\widetilde{A}_3+\cdots+\widetilde{A}_k < s-(k-2)n$. Hence, we have $I_j(t_1) \le t_1^{\varepsilon/p_j-n-\widetilde{A}_j} O_j(1)$, for $\varepsilon \to 0^+$, $j \in \{2,\ldots,k\}$, and consequently

$$\int_1^\infty t_1^{-1-\varepsilon}\sum_{j=2}^{k} I_j(t_1)dt_1 \le O(1). \tag{3.16}$$

We conclude, by using (3.10), (3.12), and (3.16), that $M^* \le M_1$ which is an obvious contradiction. It follows that the constant M^* in (3.6) is the best possible.

Finally, the equivalence of the inequalities (3.6) and (3.7) means that the constant M^* is also the best possible in the inequality (3.7). That completes the proof. □

Remark 3.3. If we put $k = 2$, $K_\alpha(x,y) = \ln(|x|_\alpha/|y|_\alpha)/(|x|_\alpha^s - |y|_\alpha^s)$, $\widetilde{A}_1 = s/q - n$ and $\widetilde{A}_2 = s/p - n$ in the inequalities (3.6) and (3.7) applying Theorem 3.2, we obtain the result of Baoju Sun (see [7]). Further, by putting $n = 1$ in Theorems 3.1 and 3.2 we obtain appropriate results from [8]. More precisely, the inequality (3.6) becomes

$$\int_{(0,\infty)^k} K_\alpha(x_1,\ldots,x_k)\prod_{i=1}^{k} f_i(x_i)dx_1\cdots dx_k < M^*\prod_{i=1}^{k}\left(\int_0^\infty x_i^{-1-p_i\widetilde{A}_i} f_i^{p_i}(x_i)dx_i\right)^{1/p_i}. \tag{3.17}$$

If the kernel $K_\alpha(x_1,\ldots,x_k)$ and the parameters A_{ij} satisfy the conditions from Theorem 3.2, then the constant $M^* = k_\alpha(\widetilde{A}_2,\ldots,\widetilde{A}_k)$ is the best possible. For example, setting $K_\alpha(x_1,\ldots,x_k) = (x_1+\cdots+x_k)^{-s}$, $s > 0$, $\widetilde{A}_i = (s-p_i)/p_i$, $i = 2,\ldots,k$, in the inequality (3.17), we obtain Yang's result (1.5) from introduction.

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