

Research Article

Local Regularity Results for Minima of Anisotropic Functionals and Solutions of Anisotropic Equations

Gao Hongya,^{1,2} Qiao Jinjing,¹ Wang Yong,³ and Chu Yuming⁴

¹ College of Mathematics and Computer Science, Hebei University, Baoding 071002, China

² Study Center of Mathematics of Hebei Province, Shijiazhuang 050016, China

³ Department of Mathematics, Chengde Teachers College for Nationalities, Chengde 067000, China

⁴ Faculty of Science, Huzhou Teachers College, Huzhou 313000, China

Correspondence should be addressed to Gao Hongya, hongya-gao@sohu.com

Received 13 July 2007; Accepted 21 November 2007

Recommended by Alberto Cabada

This paper gives some local regularity results for minima of anisotropic functionals $I(u; \Omega) = \int_{\Omega} f(x, u, Du) dx$, $u \in W_{\text{loc}}^{1, q_i}(\Omega)$ and for solutions of anisotropic equations $-\text{div} \mathcal{A}(x, u, Du) = -\sum_{i=1}^N (\partial f / \partial x_i)$, $u \in W_{\text{loc}}^{1, q_i}(\Omega)$ which can be regarded as generalizations of the classical results.

Copyright © 2008 Gao Hongya et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Let Ω be an open bounded subset of \mathbf{R}^N , $N \geq 2$. Let $q_i > 1$, $i = 1, \dots, N$. Denote

$$q = \max_{1 \leq i \leq N} q_i, \quad p = \min_{1 \leq i \leq N} q_i, \quad \bar{q} : \frac{1}{\bar{q}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{q_i}. \quad (1.1)$$

Throughout this paper, we will make use of the anisotropic Sobolev space

$$W_{\text{loc}}^{1, q_i}(\Omega) = \left\{ v \in L_{\text{loc}}^q(\Omega) : \frac{\partial v}{\partial x_i} \in L_{\text{loc}}^{q_i}(\Omega), \forall i = 1, \dots, N \right\}. \quad (1.2)$$

Let $x_0 \in \Omega$ and $t > 0$, we denote by B_t the ball of radius t centered at x_0 . For functions u and $k > 0$, let $A_k = \{x \in \Omega : |u(x)| > k\}$, $A_{k,t} = A_k \cap B_t$. Moreover, if $p > 1$, then p' is always the real number $p/(p-1)$, and if $s < N$, s^* is always the real number satisfying $1/s^* = 1/s - 1/N$.

This paper mainly considers the functions u minimizing the anisotropic functionals

$$I(u; \Omega) = \int_{\Omega} f(x, u, Du) dx, \quad u \in W_{\text{loc}}^{1, q_i}(\Omega) \quad (1.3)$$

and weak solutions of the anisotropic equations

$$-\operatorname{div} \mathcal{A}(x, u, Du) = -\sum_{i=1}^N \frac{\partial f_i}{\partial x_i}, \quad u \in W_{\text{loc}}^{1, q_i}(\Omega). \quad (1.4)$$

We refer to the classical books by Ladyženskaya and Ural'ceva [1], Morrey [2], Gilbarg and Trudinger [3], and Giaquinta [4] for some details of isotropic cases.

For isotropic cases, global L^s -summability was proved in the 1960s by Stampacchia [5] for solutions of linear elliptic equations. This result was extended by Boccardo and Giachetti to the nonlinear case in [6]. For anisotropic cases, Giachetti and Porzio recently proved in [7] the local L^s -summability for minima of anisotropic functionals and weak solutions of anisotropic nonlinear elliptic equations. Precisely, the authors considered the minima of functionals whose prototype is (1.3), f is a Carathéodory function satisfying the growth conditions

$$a \sum_{i=1}^N |\xi_i|^{q_i} \leq f(x, s, \xi) \leq b \sum_{i=1}^N |\xi_i|^{q_i} + \varphi_1(x), \quad (1.5)$$

where the function $\varphi_1 \in L_{\text{loc}}^r(\Omega)$ with $1 < r < N/\bar{q}$. The authors also considered the local solutions $u \in W_{\text{loc}}^{1, q_i}(\Omega)$ of the anisotropic equations (1.4), where $\mathcal{A} : \Omega \times \mathbf{R} \times \mathbf{R}^N \rightarrow \mathbf{R}^N$ is a Carathéodory function satisfying the following structural conditions:

$$\begin{aligned} \mathcal{A}(x, u, \xi) \cdot \xi &\geq m_0 \sum_{i=1}^N |\xi_i|^{q_i}, \\ |\mathcal{A}_j(x, u, \xi)| &\leq m_1 \left(h(x) + \sum_{i=1}^N |\xi_i|^{q_i} \right)^{1-1/q_j}, \quad j = 1, \dots, N, \end{aligned} \quad (1.6)$$

where $m_l, l = 0, 1$ are positive constants, the function h is in $L_{\text{loc}}^1(\Omega)$ and the functions f_i belong, respectively, to the spaces $L_{\text{loc}}^{(q_i)'}(\Omega)$. Under the above conditions, the authors obtained some local regularity results.

The aim of the present paper is to prove the local regularity property for minima of the anisotropic functionals of type (1.3) with the more general growth conditions than (1.5), that is, we assume the integrand f satisfies the following growth conditions:

$$\sum_{i=1}^N |\xi_i|^{q_i} - b|u|^\alpha - \varphi_0(x) \leq f(x, u, \xi) \leq a \sum_{i=1}^N |\xi_i|^{q_i} + b|u|^\alpha + \varphi_1(x), \quad (1.7)$$

where

$$\begin{aligned} \varphi_0 &\in L_{\text{loc}}^{r_1}(\Omega), \quad \varphi_1 \in L_{\text{loc}}^{r_2}(\Omega), \quad r_1, r_2 > 1, \quad a \geq 1, b \geq 0, \\ p &\leq \alpha < p^*, \quad q < \bar{q}^*, \quad \bar{q} < N, \quad 1 < \min\{r_1, r_2\} < \frac{N}{\bar{q}}. \end{aligned} \quad (1.8)$$

We also consider weak solutions of the type (1.4) with more general growth conditions than (1.6), that is, we assume the operator \mathcal{A} satisfies the following coercivity and growth conditions:

$$\mathcal{A}(x, u, \xi) \cdot \xi \geq b_0 \sum_{i=1}^N |\xi_i|^{q_i} - b_1 |u|^{\alpha_1} - \varphi_2(x), \quad (1.9)$$

$$|\mathcal{A}(x, u, \xi)| \leq b_2 \sum_{i=1}^N |\xi_i|^{q_i-1} + b_3 |u|^{\alpha_2} + k(x), \quad (1.10)$$

where $b_0 \geq 1$, $b_i > 0$, $i = 1, 2, 3$, $q < \bar{q}^*$, $\bar{q} < N$, $p \leq \alpha_1 < p^*$, $p - 1 \leq \alpha_2 \leq N(p - 1)/(N - p)$, $\varphi_2 \in L_{\text{loc}}^{r_0}(\Omega)$ with $r_0 > 1$, $k \in L_{\text{loc}}^{r_{N+1}}(\Omega)$, $f_i \in L_{\text{loc}}^{r_i}(\Omega)$, $i = 1, \dots, N$.

Remark 1.1. Notice that we have confined ourselves to the case $\bar{q} < N$ because when such inequality is violated, every function in $W_{\text{loc}}^{1, q_i}(\Omega)$ is trivially in $L_{\text{loc}}^s(\Omega)$ (for every fixed $s < \infty$) by [7, Lemma 3.2].

Remark 1.2. Since we have assumed in (1.7), (1.9), and (1.10) that the integrand f and the operator \mathcal{A} satisfy some growth conditions depending on u , in the proof of the local regularity results, we have to estimate the integral of some power of $|u|$ by means of $|Du|$. To do this, we will make use of the Sobolev inequality that has been used in [8].

2. Preliminary lemmas

In order to prove the local L^s -integrability of the local unbounded minima of the anisotropic functionals and weak solutions of anisotropic equations, we need a useful lemma from [7].

Lemma 2.1. *Let $u \in W_{\text{loc}}^{1, q_i}(\Omega)$, $\phi_0 \in L_{\text{loc}}^r(\Omega)$, where q , \bar{q} , and r satisfy*

$$1 < r < \frac{N}{\bar{q}}, \quad q < \bar{q}^*, \quad \bar{q} < N. \quad (2.1)$$

Assume that the following integral estimates hold:

$$\int_{A_{k, \tau}} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \leq c_0 \left[\int_{A_{k, t}} \phi_0 dx + (t - \tau)^{-\gamma} \int_{A_{k, t}} \sum_{i=1}^N |u|^{q_i} dx \right] \quad (2.2)$$

for every $k \in \mathbb{N}$ and $R_0 \leq \tau < t \leq R_1$, where c_0 is a positive constant that depends only on N , q_i , r , R_0 , R_1 and $|\Omega|$ and γ is a real positive constant. Then $u \in L_{\text{loc}}^s(\Omega)$, where

$$s = \frac{\bar{q}^* q}{q - \bar{q}^* (1 - 1/r)}. \quad (2.3)$$

One will also need a lemma from [8].

Lemma 2.2. *Let $f(t)$ be a nonnegative bounded function defined for $0 \leq T_0 \leq t \leq T_1$. Suppose that for $T_0 \leq t < s \leq T_1$,*

$$f(t) \leq A(s - t)^{-\gamma} + B + \theta f(s), \quad (2.4)$$

where A , B , γ , θ are nonnegative constants, and $\theta < 1$. Then there exists a constant c , depending only on γ and θ such that for every ϱ , R , $T_0 \leq \varrho < R \leq T_1$, one has

$$f(\varrho) \leq c[A(R - \varrho)^{-\gamma} + B]. \quad (2.5)$$

3. Minima of anisotropic functionals

In this section, we prove a local regularity result for minima of anisotropic functionals.

Definition 3.1. By a local minimum of the anisotropic functional I in (1.3), we mean a function $u \in W_{\text{loc}}^{1,q_i}(\Omega)$, such that for every function $\psi \in W^{1,q_i}(\Omega)$ with $\text{supp } \psi \subset\subset \Omega$, it holds that

$$I(u; \text{supp } \psi) \leq I(u + \psi; \text{supp } \psi). \quad (3.1)$$

Theorem 3.2. *Assume that the functional I satisfies the conditions (1.7). If u is a local minimum of I , then it belongs to $L_{\text{loc}}^s(\Omega)$, where*

$$s = \frac{\bar{q}^* q}{q - \bar{q}^* (1 - 1/\min\{r_1, r_2\})}. \quad (3.2)$$

Proof. Owing to Lemma 2.1, it is sufficient to prove that u satisfies the integral estimates (2.2) with $\gamma = q$ and $\phi_0 = \varphi_0 + \varphi_1$. Let $B_{R_1} \subset\subset \Omega$ and $0 \leq R_0 \leq \tau < t \leq R_1$ be arbitrarily but fixed. It is no loss of generality to assume that $R_1 - R_0 < 1$. For $k > 0$, let

$$A_k^+ = \{x \in \Omega : u(x) > k\}, \quad A_k^- = \{x \in \Omega : u(x) < -k\}. \quad (3.3)$$

It is obvious that $A_k = A_k^+ \cup A_k^-$. Denote $A_{k,t}^+ = A_k^+ \cap B_t$ and $A_{k,t}^- = A_k^- \cap B_t$. Let $w = \max(u - k, 0)$. Choose $\psi = -\eta w$ in (3.1), where η is a cut-off function such that

$$\text{supp } \eta \subset B_t, \quad 0 \leq \eta \leq 1, \quad \eta = 1 \text{ in } B_\tau, \quad |D\eta| \leq 2(t - \tau)^{-1}. \quad (3.4)$$

We obtain from the minimality of u that

$$\begin{aligned} \int_{B_t} f(x, u, Du) dx &\leq \int_{B_t} f(x, u + \psi, Du + D\psi) dx \\ &= \int_{A_{k,t}^+} f(x, u - \eta w, Du - D(\eta w)) dx + \int_{B_t \cap \{u \leq k\}} f(x, u, Du) dx. \end{aligned} \quad (3.5)$$

This implies that

$$\int_{A_{k,t}^+} f(x, u, Du) dx \leq \int_{A_{k,t}^+} f(x, u - \eta w, Du - D(\eta w)) dx. \quad (3.6)$$

By (1.7), we obtain

$$\begin{aligned} &\int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ &\leq b \int_{A_{k,t}^+} u^\alpha dx + \int_{A_{k,t}^+} \varphi_0 dx + a \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} - \frac{\partial(\eta w)}{\partial x_i} \right|^{q_i} dx + b \int_{A_{k,t}^+} (u - \eta w)^\alpha dx + \int_{A_{k,t}^+} \varphi_1 dx. \end{aligned} \quad (3.7)$$

We first estimate the 3rd term on the right-hand side of (3.7). Using the elementary inequality

$$(a + b)^q \leq 2^{q-1}(a^q + b^q), \quad a, b \geq 0, \quad q \geq 1, \quad (3.8)$$

we obtain

$$\begin{aligned} a \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} - \frac{\partial(\eta w)}{\partial x_i} \right|^{q_i} dx &= a \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} - \frac{\partial(\eta w)}{\partial x_i} \right|^{q_i} dx \\ &\leq 2^{q-1} a \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N \left[(1 - \eta)^{q_i} \left| \frac{\partial u}{\partial x_i} \right|^{q_i} + \left| \frac{\partial \eta}{\partial x_i} \right|^{q_i} w^{q_i} \right] dx \\ &\leq 2^{q-1} a \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{2^{2q-1} a}{(t - \tau)^{q_i}} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N w^{q_i} dx \\ &\leq 2^{q-1} a \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{2^{2q-1} a}{(t - \tau)^q} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N u^{q_i} dx \end{aligned} \quad (3.9)$$

since $w^{q_i} \leq u^{q_i}$ in $A_{k,t}^+$ and $t - \tau < 1$. The summation of the 1st and the 4th terms on the right-hand side of (3.7) can be estimated as

$$b \int_{A_{k,t}^+} u^\alpha dx + b \int_{A_{k,t}^+} (u - \eta w)^\alpha dx \leq 2b \int_{A_{k,t}^+} u^\alpha dx. \quad (3.10)$$

Substituting (3.9) and (3.10) into (3.7) yields

$$\begin{aligned} &\int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ &\leq \int_{A_{k,t}^+} (\varphi_0 + \varphi_1) dx + 2b \int_{A_{k,t}^+} u^\alpha dx + 2^{q-1} a \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{2^{2q-1} a}{(t - \tau)^q} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N u^{q_i} dx. \end{aligned} \quad (3.11)$$

We know from [8] that if $\tilde{u} \in W^{1,p}(B_t)$ and $|\text{supp } \tilde{u}| \leq (1/2)|B_t|$, we then have the Sobolev inequality

$$\left(\int_{B_t} \tilde{u}^{p^*} dx \right)^{p/p^*} \leq c_1(N, p) \int_{B_t} |D\tilde{u}|^p dx. \quad (3.12)$$

Let

$$\tilde{u} = \begin{cases} u, & x \in A_{k,t}^+ \\ 0, & x \in \Omega \setminus A_{k,t}^+. \end{cases} \quad (3.13)$$

By assumption, $p \leq \alpha < p^*$, which implies

$$\begin{aligned} \int_{A_{k,t}^+} u^\alpha dx &= \int_{B_t} \tilde{u}^\alpha dx \leq \|\tilde{u}\|_{p^*}^{\alpha-p} |B_t|^{1-\alpha/p^*} \left(\int_{B_t} \tilde{u}^{p^*} dx \right)^{p/p^*} \\ &\leq c_1 \|\tilde{u}\|_{p^*}^{\alpha-p} |B_t|^{1-\alpha/p^*} \int_{B_t} |D\tilde{u}|^p dx \\ &\leq c_1 \|\tilde{u}\|_{p^*}^{\alpha-p} |B_t|^{1-\alpha/p^*} \max\{1, 2^{p/2-1}\} \int_{B_t} \sum_{i=1}^N \left| \frac{\partial \tilde{u}}{\partial x_i} \right|^{q_i} dx \\ &= c_1 \|\tilde{u}\|_{p^*}^{\alpha-p} |B_t|^{1-\alpha/p^*} \max\{1, 2^{p/2-1}\} \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx, \end{aligned} \quad (3.14)$$

provided that $|\text{supp } \tilde{u}|_{B_t} \leq (1/2)|B_t|$. We can choose T so small that for $t \leq T$ we get

$$c_1 \|\tilde{u}\|_{p^*}^{\alpha-p} |B_t|^{1-\alpha/p^*} \max\{1, 2^{p/2-1}\} \leq \frac{1}{4b}. \quad (3.15)$$

It is obvious that

$$k^{p^*} |A_k^+| \leq \|\tilde{u}\|_{p^*, \Omega'}^{p^*}, \quad (3.16)$$

and therefore, there exists a constant k_0 , such that for $k \geq k_0$, we have

$$|A_k^+| \leq \frac{1}{2} |B_{T/2}|. \quad (3.17)$$

For such values of k we then have $|\text{supp } \tilde{u}| < (1/2)|B_{T/2}|$ and therefore, if $T/2 \leq t \leq T$,

$$\int_{A_{k,t}^+} u^\alpha dx \leq \frac{1}{4b} \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx. \quad (3.18)$$

Thus, from (3.11) and, we get

$$\begin{aligned} &\int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ &\leq 2 \int_{A_{k,t}^+} (\varphi_0 + \varphi_1) dx + 2^q a \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{2^{2q} a}{(t-\tau)^q} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N |u|^{q_i} dx. \end{aligned} \quad (3.19)$$

Suppose now $T/2 \leq \rho \leq \tau < t \leq R \leq T$, we get

$$\begin{aligned} &\int_{A_{k,\rho}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ &\leq 2 \int_{A_{k,R}^+} (\varphi_0 + \varphi_1) dx + 2^q a \int_{A_{k,t}^+ \setminus A_{k,\rho}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{2^{2q} a}{(t-\tau)^q} \int_{A_{k,R}^+} \sum_{i=1}^N |u|^{q_i} dx. \end{aligned} \quad (3.20)$$

Adding to both sides $2^q a$ times the left-hand side, we get eventually

$$\begin{aligned} & \int_{A_{k,\rho}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ & \leq \frac{2}{2^q a + 1} \int_{A_{k,R}^+} (\varphi_0 + \varphi_1) dx + \frac{2^q a}{2^q a + 1} \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{2^{2q} a}{2^q a + 1} \cdot \frac{1}{(t - \tau)^q} \int_{A_{k,R}^+} \sum_{i=1}^N |u|^{q_i} dx, \end{aligned} \quad (3.21)$$

we can now apply Lemma 2.2 to conclude that

$$\int_{A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \leq c \left\{ \frac{2}{2^q a + 1} \int_{A_{k,t}^+} (\varphi_0 + \varphi_1) dx + \frac{2^{2q} a}{2^q a + 1} \cdot \frac{1}{(t - \tau)^q} \int_{A_{k,t}^+} \sum_{i=1}^N |u|^{q_i} dx \right\}, \quad (3.22)$$

where c depends only on q and a .

Since $-u$ minimizes the functional

$$\tilde{F}(v; \Omega) = \int_{\Omega} \tilde{f}(x, v, Dv) dx, \quad (3.23)$$

where $\tilde{f}(x, v, p) = f(x, -v, -p)$ satisfies the same growth conditions (1.7), inequality (3.22) holds with u replaced by $-u$. We then conclude that

$$\int_{A_{k,\tau}^-} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \leq c \left\{ \frac{2}{2^q a + 1} \int_{A_{k,t}^-} (\varphi_0 + \varphi_1) dx + \frac{2^{2q} a}{2^q a + 1} \cdot \frac{1}{(t - \tau)^q} \int_{A_{k,t}^-} \sum_{i=1}^N |u|^{q_i} dx \right\}. \quad (3.24)$$

Adding (3.22) and (3.24) yields

$$\int_{A_{k,\tau}} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \leq c \left\{ \frac{2}{2^q a + 1} \int_{A_{k,t}} (\varphi_0 + \varphi_1) dx + \frac{2^{2q} a}{2^q a + 1} \cdot \frac{1}{(t - \tau)^q} \int_{A_{k,t}} \sum_{i=1}^N |u|^{q_i} dx \right\}. \quad (3.25)$$

This shows that u satisfies estimates (2.2) with $\gamma = q$ and $\phi_0 = \varphi_0 + \varphi_1$. Theorem 3.2 follows from Lemma 2.1. \square

4. Local solutions of anisotropic equations

In this section, we prove a local regularity result for weak solutions of anisotropic equations. Let $u \in W_{\text{loc}}^{1,q_i}(\Omega)$ be a local solution of the anisotropic equation (1.4), where $\mathcal{A} : \Omega \times \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a Carathéodory function satisfying the structural conditions (1.9) and (1.10).

Definition 4.1. By a weak solution of (1.4) we mean a function $u \in W_{\text{loc}}^{1,q_i}(\Omega)$, such that for every function $\varphi \in W^{1,q_i}(\Omega)$ with $\text{supp } \varphi \subset\subset \Omega$ it holds

$$\int_{\text{supp } \varphi} \mathcal{A}(x, u, Du) \cdot D\varphi dx = \int_{\text{supp } \varphi} f \cdot D\varphi dx, \quad (4.1)$$

where $f = (f_1, f_2, \dots, f_N)$.

Theorem 4.2. Under the previous assumptions (1.9) and (1.10), if one assumes that $\varphi_2 \in L_{\text{loc}}^{r_0}(\Omega)$, $f_i \in L_{\text{loc}}^{r_i}(\Omega)$, $i = 1, 2, \dots, N$, $k \in L_{\text{loc}}^{r_{N+1}}(\Omega)$, and r_i , $i = 0, \dots, N + 1$ satisfy

$$1 < r = \min_{1 \leq i \leq N} \left\{ \frac{r_i}{q_i'}, r_0, \frac{r_{N+1}}{p'} \right\} < \frac{N}{\bar{q}}, \quad (4.2)$$

then $u \in L_{\text{loc}}^s(\Omega)$, where

$$s = \frac{\bar{q}^* q}{q - \bar{q}^*(1 - 1/r)}. \quad (4.3)$$

Proof. By virtue of Lemma 2.1, it is sufficient to prove that u satisfies the integral estimates (2.2) with $\gamma = q$ and $\phi_0 = \varphi_2 + |k|^{p'} + \sum_{i=1}^N |f_i|^{q_i}$. Let $B_{R_1} \subset\subset \Omega$ and $0 \leq R_0 \leq \tau < t \leq R_1$ be arbitrarily but fixed. Assume again that $R_1 - R_0 < 1$. Let $w = \max\{u - k, 0\}$. Choose $\varphi = \eta w$ as a test function in (4.1), where the cut-off function η satisfies the conditions (3.4). We obtain from Definition 4.1 that

$$\int_{A_{k,t}^+} \mathcal{A}(x, u, Du) \cdot D(\eta w) dx = \int_{A_{k,t}^+} f \cdot D(\eta w) dx. \quad (4.4)$$

We now estimate the integrals in (4.4). Applying the assumption (1.9), we deduce from (4.4) that

$$\begin{aligned} b_0 \int_{A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx &\leq b_1 \int_{A_{k,t}^+} |u|^{\alpha_1} dx + \int_{A_{k,t}^+} \varphi_2 dx + \int_{A_{k,t}^+} f \cdot Du dx + \frac{2}{t - \tau} \int_{A_{k,t}^+} |f| w dx \\ &+ \frac{2}{t - \tau} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} |\mathcal{A}(x, u, Du)| w dx. \end{aligned} \quad (4.5)$$

The 3rd term on the right-hand side of the above inequality can be estimated as

$$\int_{A_{k,t}^+} f \cdot Du dx = \int_{A_{k,t}^+} \sum_{i=1}^N f_i \cdot \frac{\partial u}{\partial x_i} dx \leq \varepsilon \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \sum_{i=1}^N C(\varepsilon, q_i) \int_{A_{k,t}^+} |f_i|^{q_i} dx. \quad (4.6)$$

By Young's inequality, the 4th term on the right-hand side of inequality (4.5) can be estimated as

$$\frac{2}{t - \tau} \int_{A_{k,t}^+} |f| w dx \leq \frac{2^q}{(t - \tau)^q} \int_{A_{k,t}^+} \sum_{i=1}^N (u - k)^{q_i} dx + \sum_{i=1}^N \int_{A_{k,t}^+} |f_i|^{q_i} dx. \quad (4.7)$$

By (1.10), the last term on the right-hand side of (4.5) can be estimated as

$$\frac{2}{t - \tau} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} |\mathcal{A}(x, u, Du)| w dx \leq \frac{2}{t - \tau} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \left[b_2 \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i - 1} + b_3 |u|^{\alpha_2} + k \right] w dx = I_1 + I_2 + I_3. \quad (4.8)$$

By Young's inequality, we derive that

$$I_1 \leq b_2 \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{b_2 2^q}{(t-\tau)^q} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N (u-k)^{q_i} dx. \quad (4.9)$$

Hölder's inequality and Young's inequality yield

$$\begin{aligned} I_2 &\leq b_3 \varepsilon \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} |u|^{\alpha_2 p'} dx + \frac{C(\varepsilon, p) 2^p}{(t-\tau)^p} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} (u-k)^p dx \\ &\leq b_3 \varepsilon \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} |u|^{\alpha_2 p'} dx + \frac{C(\varepsilon, p) 2^p}{N(t-\tau)^q} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N (u-k)^{q_i} dx, \end{aligned} \quad (4.10)$$

where ε is a positive constant to be determined later. Further,

$$I_3 \leq \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} |k|^{p'} dx + \frac{2^q}{N(t-\tau)^q} \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N (u-k)^{q_i} dx. \quad (4.11)$$

Combining (4.6)–(4.11) with (4.5) yields

$$\begin{aligned} &b_0 \int_{A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ &\leq \int_{A_{k,t}^+} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^N (C(\varepsilon, q_i) + 1) |f_i|^{q_i} \right) dx + b_1 \int_{A_{k,t}^+} |u|^{\alpha_1} dx + b_3 \varepsilon \int_{A_{k,t}^+} |u|^{\alpha_2 p'} dx \\ &\quad + \varepsilon \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + b_2 \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ &\quad + (b_2 + C(\varepsilon, p) + 2) \frac{2^q}{(t-\tau)^q} \int_{A_{k,t}^+} \sum_{i=1}^N (u-k)^{q_i} dx. \end{aligned} \quad (4.12)$$

Since $p \leq \alpha_1 < p^*$, then as in the proof of Theorem 3.2, we know that there exist a sufficiently small T and a sufficiently large k_0 , such that for all $T/2 \leq t \leq T$ and $k \geq k_0$, we have

$$\int_{A_{k,t}^+} |u|^{\alpha_1} dx \leq \frac{1}{2b_1} \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx. \quad (4.13)$$

Similarly, since $p-1 \leq \alpha_2 \leq N(p-1)/(n-p)$, then $p \leq \alpha_2 p' \leq p^*$, therefore

$$\int_{A_{k,t}^+} |u|^{\alpha_2 p'} dx \leq C \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx. \quad (4.14)$$

Thus, from (4.12)–(4.14) we can derive that

$$\begin{aligned} b_0 \int_{A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx &\leq \int_{A_{k,t}^+} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^N (C(\varepsilon, q_i) + 1) |f_i|^{q_i} \right) dx \\ &+ \left(\frac{1}{2} + (Cb_3 + 1)\varepsilon \right) \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + b_2 \int_{A_{k,t}^+ \setminus A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ &+ (b_2 + C(\varepsilon, p) + 2) \frac{2^q}{(t - \tau)^q} \int_{A_{k,t}^+} \sum_{i=1}^N (u - k)^{q_i} dx. \end{aligned} \quad (4.15)$$

Adding to both sides

$$b_2 \int_{A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx, \quad (4.16)$$

we get eventually

$$\begin{aligned} \int_{A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx &\leq \frac{1}{b_0 + b_2} \int_{A_{k,t}^+} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^N (C(\varepsilon, q_i) + 1) |f_i|^{q_i} \right) dx \\ &+ \left(\frac{1}{2} + (Cb_3 + 1)\varepsilon \right) \frac{1}{b_0 + b_2} \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{b_2}{b_0 + b_2} \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \\ &+ (b_2 + C(\varepsilon, p) + 2) \frac{1}{b_0 + b_2} \frac{2^q}{(t - \tau)^q} \int_{A_{k,t}^+} \sum_{i=1}^N (u - k)^{q_i} dx. \end{aligned} \quad (4.17)$$

Choosing ε small enough, such that

$$\theta = \frac{1/2 + (Cb_3 + 1)\varepsilon + b_2}{b_0 + b_2} < 1, \quad (4.18)$$

(4.17) implies that

$$\int_{A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \leq C \int_{A_{k,t}^+} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^N |f_i|^{q_i} \right) dx + \theta \int_{A_{k,t}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{C}{(t - \tau)^q} \int_{A_{k,t}^+} \sum_{i=1}^N (u - k)^{q_i} dx. \quad (4.19)$$

Suppose now that $T/2 \leq \varrho \leq \tau < t \leq R \leq T$, we get

$$\begin{aligned} \int_{A_{k,\varrho}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx &\leq C \int_{A_{k,t}^+} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^N |f_i|^{q_i} \right) dx \\ &+ \theta \int_{A_{k,R}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx + \frac{C}{(R - \varrho)^q} \int_{A_{k,t}^+} \sum_{i=1}^N (u - k)^{q_i} dx. \end{aligned} \quad (4.20)$$

Applying Lemma 2.2, we conclude that

$$\int_{A_{k,\tau}^+} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \leq cC \int_{A_{k,t}^+} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^N |f_i|^{q_i'} \right) dx + \frac{cC}{(t-\tau)^q} \int_{A_{k,t}^+} \sum_{i=1}^N (u-k)^{q_i} dx. \quad (4.21)$$

Since $-u$ is a weak solution of

$$-\operatorname{div} \tilde{\mathcal{A}}(x, u, Du) = -\sum_{i=1}^N \frac{\partial f_i}{\partial x_i}, \quad (4.22)$$

where $\tilde{\mathcal{A}}(x, s, \xi) = \mathcal{A}(x, -s, -\xi)$ satisfies the same conditions (1.9) and (1.10), inequality (4.21) holds with u replaced by $-u$. We then conclude that

$$\int_{A_{k,\tau}^-} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \leq cC \int_{A_{k,t}^-} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^N |f_i|^{q_i'} \right) dx + \frac{cC}{(t-\tau)^q} \int_{A_{k,t}^-} \sum_{i=1}^N (u-k)^{q_i} dx. \quad (4.23)$$

Adding (4.21) with (4.23) yields

$$\int_{A_{k,\tau}} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \leq cC \int_{A_{k,t}} \left(\varphi_2 + |k|^{p'} + \sum_{i=1}^N |f_i|^{q_i'} \right) dx + \frac{cC}{(t-\tau)^q} \int_{A_{k,t}} \sum_{i=1}^N (u-k)^{q_i} dx. \quad (4.24)$$

Thus, u satisfies (2.2) with $\phi_0 = \varphi_2 + |k|^{p'} + \sum_{i=1}^N |f_i|^{q_i'}$ and $\alpha = q$. Theorem 4.2 follows from Lemma 2.1. \square

Acknowledgments

Gao Hongya is supported by Special Fund of Mathematics Research of Natural Science Foundation of Hebei Province (no. 07M003) and Doctoral Foundation of the Department of Education of Hebei Province (B2004103). Chu Yuming is supported by NSF of Zhejiang Province (No. Y607128) and NSFC (no. 10771195). This research was done when the first author was visiting Institute of Mathematics, Nankai University. He wished to thank the Institute for support and hospitality.

References

- [1] O. A. Ladyženskaya and N. N. Ural'ceva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, NY, USA, 1968.
- [2] C. B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer, Berlin, Germany, 1968.
- [3] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, Germany, 1977.
- [4] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of Mathematics Studies, no. 105, Princeton University Press, Princeton, NJ, USA, 1983.
- [5] G. Stampacchia, *Équations Elliptiques du Second Ordre à Coefficients Discontinus*, Séminaire de Mathématiques Supérieures, Les Presses de l'Université de Montréal, Montreal, Quebec, Canada, 1966.
- [6] L. Boccardo and D. Giachetti, "Alcune osservazioni sulla regolarità delle soluzioni di problemi nonlineari e applicazioni," *Ricerche di Matematica*, vol. 34, pp. 309–323, 1985.
- [7] D. Giachetti and M. M. Porzio, "Local regularity results for minima of functionals of the calculus of variation," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 39, no. 4, pp. 463–482, 2000.
- [8] M. Giaquinta and E. Giusti, "On the regularity of the minima of variational integrals," *Acta Mathematica*, vol. 148, no. 1, pp. 31–46, 1982.