

## Research Article

# Subordination for Higher-Order Derivatives of Multivalent Functions

Rosihan M. Ali,<sup>1</sup> Abeer O. Badghaish,<sup>1,2</sup> and V. Ravichandran<sup>3</sup>

<sup>1</sup> School of Mathematical Sciences, Universiti Sains Malaysia (USM), 11800 Penang, Malaysia

<sup>2</sup> Mathematics Department, King Abdul Aziz University, P.O. Box 581, Jeddah 21421, Saudi Arabia

<sup>3</sup> Department of Mathematics, University of Delhi, Delhi 110 007, India

Correspondence should be addressed to Rosihan M. Ali, rosihan@cs.usm.my

Received 18 July 2008; Accepted 24 November 2008

Recommended by Vijay Gupta

Differential subordination methods are used to obtain several interesting subordination results and best dominants for higher-order derivatives of  $p$ -valent functions. These results are next applied to yield various known results as special cases.

Copyright © 2008 Rosihan M. Ali et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

## 1. Motivation and preliminaries

For a fixed  $p \in \mathbb{N} := \{1, 2, \dots\}$ , let  $\mathcal{A}_p$  denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad (1.1)$$

which are  $p$ -valent in the open unit disc  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$  and let  $\mathcal{A} := \mathcal{A}_1$ . Upon differentiating both sides of (1.1)  $q$ -times with respect to  $z$ , the following differential operator is obtained:

$$f^{(q)}(z) = \lambda(p; q) z^{p-q} + \sum_{k=1}^{\infty} \lambda(k+p; q) a_{k+p} z^{k+p-q}, \quad (1.2)$$

where

$$\lambda(p; q) := \frac{p!}{(p-q)!} \quad (p \geq q; p \in \mathbb{N}; q \in \mathbb{N} \cup \{0\}). \quad (1.3)$$

Several researchers have investigated higher-order derivatives of multivalent functions, see, for example, [1–10]. Recently, by the use of the well-known Jack's lemma [11, 12], Irmak and Cho [5] obtained interesting results for certain classes of functions defined by higher-order derivatives.

Let  $f$  and  $g$  be analytic in  $\mathbb{U}$ . Then  $f$  is *subordinate* to  $g$ , written as  $f(z) < g(z)$  ( $z \in \mathbb{U}$ ) if there is an analytic function  $w(z)$  with  $w(0) = 0$  and  $|w(z)| < 1$ , such that  $f(z) = g(w(z))$ . In particular, if  $g$  is univalent in  $\mathbb{U}$ , then  $f$  subordinate to  $g$  is equivalent to  $f(0) = g(0)$  and  $f(\mathbb{U}) \subseteq g(\mathbb{U})$ . A  $p$ -valent function  $f \in \mathcal{A}_p$  is *starlike* if it satisfies the condition  $(1/p)\Re(zf'(z)/f(z)) > 0$  ( $z \in \mathbb{U}$ ). More generally, let  $\phi(z)$  be an analytic function with positive real part in  $\mathbb{U}$ ,  $\phi(0) = 1$ ,  $\phi'(0) > 0$ , and  $\phi(z)$  maps the unit disc  $\mathbb{U}$  onto a region starlike with respect to 1 and symmetric with respect to the real axis. The classes  $S_p^*(\phi)$  and  $C_p(\phi)$  consist, respectively, of  $p$ -valent functions  $f$  *starlike* with respect to  $\phi$  and  $p$ -valent functions  $f$  *convex* with respect to  $\phi$  in  $\mathbb{U}$  given by

$$f \in S_p^*(\phi) \iff \frac{1}{p} \frac{zf'(z)}{f(z)} < \phi(z), \quad f \in C_p(\phi) \iff \frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \phi(z). \quad (1.4)$$

These classes were introduced and investigated in [13], and the functions  $h_{\phi,p}$  and  $k_{\phi,p}$ , defined, respectively, by

$$\begin{aligned} \frac{1}{p} \frac{zh'_{\phi,p}}{h_{\phi,p}} &= \phi(z) \quad (z \in \mathbb{U}, h_{\phi,p} \in \mathcal{A}_p), \\ \frac{1}{p} \left( 1 + \frac{zk''_{\phi,p}}{k'_{\phi,p}} \right) &= \phi(z) \quad (z \in \mathbb{U}, k_{\phi,p} \in \mathcal{A}_p), \end{aligned} \quad (1.5)$$

are important examples of functions in  $S_p^*(\phi)$  and  $C_p^*(\phi)$ . Ma and Minda [14] have introduced and investigated the classes  $S^*(\phi) := S_1^*(\phi)$  and  $C(\phi) := C_1(\phi)$ . For  $-1 \leq B < A \leq 1$ , the class  $S^*[A, B] = S^*((1 + Az)/(1 + Bz))$  is the class of Janowski starlike functions (cf. [15, 16]).

In this paper, corresponding to an appropriate subordinate function  $Q(z)$  defined on the unit disk  $\mathbb{U}$ , sufficient conditions are obtained for a  $p$ -valent function  $f$  to satisfy the subordination

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < Q(z), \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 < Q(z). \quad (1.6)$$

In the particular case when  $q = 1$  and  $p = 1$ , and  $Q(z)$  is a function with positive real part, the first subordination gives a sufficient condition for univalence of analytic functions, while the second subordination implication gives conditions for convexity of functions. If  $q = 0$  and  $p = 1$ , the second subordination gives conditions for starlikeness of functions. Thus results obtained in this paper give important information on the geometric properties of functions satisfying differential subordination conditions involving higher-order derivatives.

The following lemmas are needed to prove our main results.

**Lemma 1.1** (see [12, page 135, Corollary 3.4h.1]). *Let  $Q$  be univalent in  $\mathbb{U}$ , and  $\varphi$  be analytic in a domain  $D$  containing  $Q(\mathbb{U})$ . If  $zQ'(z) \cdot \varphi[Q(z)]$  is starlike, and  $P$  is analytic in  $\mathbb{U}$  with  $P(0) = Q(0)$  and  $P(\mathbb{U}) \subset D$ , then*

$$zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)] \implies P < Q, \quad (1.7)$$

and  $Q$  is the best dominant.

**Lemma 1.2** (see [12, page 135, Corollary 3.4h.2]). *Let  $Q$  be convex univalent in  $\mathbb{U}$ , and let  $\theta$  be analytic in a domain  $D$  containing  $Q(\mathbb{U})$ . Assume that*

$$\Re \left[ \theta'[Q(z)] + 1 + \frac{zQ''(z)}{Q'(z)} \right] > 0. \quad (1.8)$$

If  $P$  is analytic in  $\mathbb{U}$  with  $P(0) = Q(0)$  and  $P(\mathbb{U}) \subset D$ , then

$$zP'(z) + \theta[P(z)] < zQ'(z) + \theta[Q(z)] \implies P < Q, \quad (1.9)$$

and  $Q$  is the best dominant.

## 2. Main results

The first four theorems below give sufficient conditions for a differential subordination of the form

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < Q(z) \quad (2.1)$$

to hold.

**Theorem 2.1.** *Let  $Q(z)$  be univalent and nonzero in  $\mathbb{U}$ ,  $Q(0) = 1$ , and let  $zQ'(z)/Q(z)$  be starlike in  $\mathbb{U}$ . If a function  $f \in \mathcal{A}_p$  satisfies the subordination*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} < \frac{zQ'(z)}{Q(z)} + p - q, \quad (2.2)$$

then

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < Q(z), \quad (2.3)$$

and  $Q$  is the best dominant.

*Proof.* Define the analytic function  $P(z)$  by

$$P(z) := \frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}}. \quad (2.4)$$

Then a computation shows that

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = \frac{zP'(z)}{P(z)} + p - q. \quad (2.5)$$

The subordination (2.2) yields

$$\frac{zP'(z)}{P(z)} + p - q < \frac{zQ'(z)}{Q(z)} + p - q, \quad (2.6)$$

or equivalently

$$\frac{zP'(z)}{P(z)} < \frac{zQ'(z)}{Q(z)}. \quad (2.7)$$

Define the function  $\varphi$  by  $\varphi(w) := 1/w$ . Then (2.7) can be written as  $zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)]$ . Since  $Q(z) \neq 0$ ,  $\varphi(w)$  is analytic in a domain containing  $Q(\mathbb{U})$ . Also  $zQ'(z) \cdot \varphi(Q(z)) = zQ'(z)/Q(z)$  is starlike. The result now follows from Lemma 1.1.  $\square$

*Remark 2.2.* For  $f \in \mathcal{A}_p$ , Irmak and Cho [5, page 2, Theorem 2.1] showed that

$$\Re \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} < p - q \implies |f^{(q)}(z)| < \lambda(p; q)|z|^{p-q-1}. \quad (2.8)$$

However, it should be noted that the hypothesis of this implication cannot be satisfied by any function in  $\mathcal{A}_p$  as the quantity

$$\left. \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right|_{z=0} = p - q. \quad (2.9)$$

Theorem 2.1 is the correct formulation of their result in a more general setting.

**Corollary 2.3.** *Let  $-1 \leq B < A \leq 1$ . If  $f \in \mathcal{A}_p$  satisfies*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} < \frac{z(A-B)}{(1+Az)(1+Bz)} + p - q, \quad (2.10)$$

then

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < \frac{1 + Az}{1 + Bz}. \quad (2.11)$$

*Proof.* For  $-1 \leq B < A \leq 1$ , define the function  $Q$  by

$$Q(z) = \frac{1 + Az}{1 + Bz}. \quad (2.12)$$

Then a computation shows that

$$\begin{aligned} F(z) &:= \frac{zQ'(z)}{Q(z)} = \frac{(A - B)z}{(1 + Az)(1 + Bz)}, \\ h(z) &:= \frac{zF'(z)}{F(z)} = \frac{1 - ABz^2}{(1 + Az)(1 + Bz)}. \end{aligned} \quad (2.13)$$

With  $z = re^{i\theta}$ , note that

$$\begin{aligned} \Re(h(re^{i\theta})) &= \Re \frac{1 - ABr^2 e^{2i\theta}}{(1 + Are^{i\theta})(1 + Bre^{i\theta})} \\ &= \frac{(1 - ABr^2)(1 + ABr^2 + (A + B)r \cos \theta)}{|(1 + Are^{i\theta})(1 + Bre^{i\theta})|^2}. \end{aligned} \quad (2.14)$$

Since  $1 + ABr^2 + (A + B)r \cos \theta \geq (1 - Ar)(1 - Br) > 0$  for  $(A + B) \geq 0$ , and similarly,  $1 + ABr^2 + (A + B)r \cos \theta \geq (1 + Ar)(1 + Br) > 0$  for  $(A + B) \leq 0$ , it follows that  $\Re h(z) > 0$ , and hence  $zQ'(z)/Q(z)$  is starlike. The desired result now follows from Theorem 2.1.  $\square$

*Example 2.4.* (1) For  $0 < \beta < 1$ , choose  $A = \beta$  and  $B = 0$  in Corollary 2.3. Since  $w < \beta z / (1 + \beta z)$  is equivalent to  $|w| \leq \beta |1 - w|$ , it follows that if  $f \in \mathcal{A}_p$  satisfies

$$\left| \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + \frac{\beta^2}{1 - \beta^2} \right| < \frac{\beta}{1 - \beta^2}, \quad (2.15)$$

then

$$\left| \frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} - 1 \right| < \beta. \quad (2.16)$$

(2) With  $A = 1$  and  $B = 0$ , it follows from Corollary 2.3 that whenever  $f \in \mathcal{A}_p$  satisfies

$$\Re \left\{ \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right\} < \frac{1}{2}, \quad (2.17)$$

then

$$\left| \frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} - 1 \right| < 1. \quad (2.18)$$

Taking  $q = 0$  and  $Q(z) = h_{\phi, p}/z^p$ , Theorem 2.1 yields the following corollary.

**Corollary 2.5** (see [13]). *If  $f \in S_p^*(\phi)$ , then*

$$\frac{f(z)}{z^p} < \frac{h_{\phi, p}}{z^p}. \quad (2.19)$$

Similarly, choosing  $q = 1$  and  $Q(z) = k'_{\phi, p}/pz^{p-1}$ , Theorem 2.1 yields the following corollary.

**Corollary 2.6** (see [13]). *If  $f \in C_p^*(\phi)$ , then*

$$\frac{f'(z)}{z^{p-1}} < \frac{k'_{\phi, p}}{z^{p-1}}. \quad (2.20)$$

**Theorem 2.7.** *Let  $Q(z)$  be convex univalent in  $\mathbb{U}$  and  $Q(0) = 1$ . If  $f \in \mathcal{A}_p$  satisfies*

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} \cdot \left( \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right) < zQ'(z), \quad (2.21)$$

then

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < Q(z), \quad (2.22)$$

and  $Q$  is the best dominant.

*Proof.* Define the analytic function  $P(z)$  by  $P(z) := f^{(q)}(z)/\lambda(p; q)z^{p-q}$ . Then it follows from (2.5) that

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} \cdot \left( \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right) = zP'(z). \quad (2.23)$$

By assumption, it follows that

$$zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)], \quad (2.24)$$

where  $\varphi(w) = 1$ . Since  $Q(z)$  is convex, and  $zQ'(z) \cdot \varphi[Q(z)] = zQ'(z)$  is starlike, Lemma 1.1 gives the desired result.  $\square$

*Example 2.8.* When

$$Q(z) := 1 + \frac{z}{\lambda(p; q)}, \quad (2.25)$$

Theorem 2.7 is reduced to the following result in [5, page 4, Theorem 2.4]. For  $f \in \mathcal{A}_p$ ,

$$\left| f^{(q)}(z) \cdot \left( \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right) \right| \leq |z|^{p-q} \implies |f^{(q)}(z) - \lambda(p; q)z^{p-q}| \leq |z|^{p-q}. \quad (2.26)$$

In the special case  $q = 1$ , this result gives a sufficient condition for the multivalent function  $f(z)$  to be close-to-convex.

**Theorem 2.9.** *Let  $Q(z)$  be convex univalent in  $\mathbb{U}$  and  $Q(0) = 1$ . If  $f \in \mathcal{A}_p$  satisfies*

$$\frac{zf^{(q+1)}(z)}{\lambda(p; q)z^{p-q}} < zQ'(z) + (p - q)Q(z), \quad (2.27)$$

then

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < Q(z), \quad (2.28)$$

and  $Q$  is the best dominant.

*Proof.* Define the function  $P(z)$  by  $P(z) = f^{(q)}(z) / \lambda(p; q)z^{p-q}$ . It follows from (2.5) that

$$zP'(z) + (p - q)P(z) < zQ'(z) + (p - q)Q(z), \quad (2.29)$$

that is,

$$zP'(z) + \theta[P(z)] < zQ'(z) + \theta[Q(z)], \quad (2.30)$$

where  $\theta(w) = (p - q)w$ . The conditions in Lemma 1.2 are clearly satisfied. Thus  $f^{(q)}(z) / \lambda(p; q)z^{p-q} < Q(z)$ , and  $Q$  is the best dominant.  $\square$

Taking  $q = 0$ , Theorem 2.9 yields the following corollary.

**Corollary 2.10** (see [17, Corollary 2.11]). *Let  $Q(z)$  be convex univalent in  $\mathbb{U}$ , and  $Q(0) = 1$ . If  $f \in \mathcal{A}_p$  satisfies*

$$\frac{f'(z)}{z^{p-1}} < zQ'(z) + pQ(z), \quad (2.31)$$

then

$$\frac{f(z)}{z^p} \prec Q(z). \quad (2.32)$$

With  $p = 1$ , Corollary 2.10 yields the following corollary.

**Corollary 2.11** (see [17, Corollary 2.9]). *Let  $Q(z)$  be convex univalent in  $\mathbb{U}$ , and  $Q(0) = 1$ . If  $f \in \mathcal{A}$  satisfies*

$$f'(z) \prec zQ'(z) + Q(z), \quad (2.33)$$

then

$$\frac{f(z)}{z} \prec Q(z). \quad (2.34)$$

**Theorem 2.12.** *Let  $Q(z)$  be univalent and nonzero in  $\mathbb{U}$ ,  $Q(0) = 1$ , and  $zQ'(z)/Q^2(z)$  be starlike. If  $f \in \mathcal{A}_p$  satisfies*

$$\frac{\lambda(p; q)z^{p-q}}{f^{(q)}(z)} \cdot \left( \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right) \prec \frac{zQ'(z)}{Q^2(z)}, \quad (2.35)$$

then

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} \prec Q(z), \quad (2.36)$$

and  $Q$  is the best dominant.

*Proof.* Define the function  $P(z)$  by  $P(z) = f^{(q)}(z)/\lambda(p; q)z^{p-q}$ . It follows from (2.5) that

$$\frac{\lambda(p; q)z^{p-q}}{f^{(q)}(z)} \cdot \left( \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p - q) \right) = \frac{1}{P(z)} \cdot \frac{zP'(z)}{P(z)} = \frac{zP'(z)}{P^2(z)}. \quad (2.37)$$

By assumption,

$$\frac{zP'(z)}{P^2(z)} \prec \frac{zQ'(z)}{Q^2(z)}. \quad (2.38)$$

With  $\varphi(w) := 1/w^2$ , (2.38) can be written as  $zP'(z) \cdot \varphi[P(z)] \prec zQ'(z) \cdot \varphi[Q(z)]$ . The function  $\varphi(w)$  is analytic in  $\mathbb{C} - \{0\}$ . Since  $zQ'(z)\varphi[Q(z)]$  is starlike, it follows from Lemma 1.1 that  $P(z) \prec Q(z)$ , and  $Q(z)$  is the best dominant.  $\square$

The next four theorems give sufficient conditions for the following differential subordination

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 < Q(z) \quad (2.39)$$

to hold.

**Theorem 2.13.** *Let  $Q(z)$  be univalent and nonzero in  $\mathbb{U}$ ,  $Q(0) = 1$ ,  $Q(z) \neq q - p + 1$ , and  $zQ'(z)/[Q(z)(Q(z) + p - q - 1)]$  be starlike in  $\mathbb{U}$ . If  $f \in \mathcal{A}_p$  satisfies*

$$\frac{1 + (zf^{(q+2)}(z)/f^{(q+1)}(z)) - p + q + 1}{(zf^{(q+1)}(z)/f^{(q)}(z)) - p + q + 1} < 1 + \frac{zQ'(z)}{Q(z)(Q(z) + p - q - 1)}, \quad (2.40)$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 < Q(z), \quad (2.41)$$

and  $Q$  is the best dominant.

*Proof.* Let the function  $P(z)$  be defined by

$$P(z) = \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1. \quad (2.42)$$

Upon differentiating logarithmically both sides of (2.42), it follows that

$$\frac{zP'(z)}{P(z) + p - q - 1} = 1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)}. \quad (2.43)$$

Thus

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q + 1 = \frac{zP'(z)}{P(z) + p - q - 1} + P(z). \quad (2.44)$$

The equations (2.42) and (2.44) yield

$$\frac{1 + (zf^{(q+2)}(z)/f^{(q+1)}(z)) - p + q + 1}{(zf^{(q+1)}(z)/f^{(q)}(z)) - p + q + 1} = \frac{zP'(z)}{P(z)(P(z) + p - q - 1)} + 1. \quad (2.45)$$

If  $f \in \mathcal{A}_p$  satisfies the subordination (2.40), (2.45) gives

$$\frac{zP'(z)}{P(z)(P(z) + p - q - 1)} < \frac{zQ'(z)}{Q(z)(Q(z) + p - q - 1)}, \quad (2.46)$$

that is,

$$zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)] \quad (2.47)$$

with  $\varphi(w) := 1/w(w + p - q - 1)$ . The desired result is now established by an application of Lemma 1.1.  $\square$

Theorem 2.13 contains a result in [18, page 122, Corollary 4] as a special case. In particular, we note that Theorem 2.13 with  $p = 1$ ,  $q = 0$ , and  $Q(z) = (1 + Az)/(1 + Bz)$  for  $-1 \leq B < A \leq 1$  yields the following corollary.

**Corollary 2.14** (see [18, page 123, Corollary 6]). *Let  $-1 \leq B < A \leq 1$ . If  $f \in \mathcal{A}$  satisfies*

$$\frac{1 + (zf''(z)/f'(z))}{zf'(z)/f(z)} < 1 + \frac{(A - B)z}{(1 + Az)^2}, \quad (2.48)$$

then  $f \in S^*[A, B]$ .

For  $A = 0$ ,  $B = b$  and  $A = 1$ ,  $B = -1$ , Corollary 2.14 gives the results of Obradović and Tuneski [19].

**Theorem 2.15.** *Let  $Q(z)$  be univalent and nonzero in  $\mathbb{U}$ ,  $Q(0) = 1$ ,  $Q(z) \neq q - p + 1$ , and let  $zQ'(z)/[Q(z) + p - q - 1]$  be starlike in  $\mathbb{U}$ . If  $f \in \mathcal{A}_p$  satisfies*

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} < \frac{zQ'(z)}{Q(z) + p - q - 1}, \quad (2.49)$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 < Q(z), \quad (2.50)$$

and  $Q$  is the best dominant.

*Proof.* Let the function  $P(z)$  be defined by (2.42). It follows from (2.43) and the hypothesis that

$$\frac{zP'(z)}{P(z) + p - q - 1} < \frac{zQ'(z)}{Q(z) + p - q - 1}. \quad (2.51)$$

Define the function  $\varphi$  by  $\varphi(w) := 1/(w + p - q - 1)$ . Then (2.51) can be written as

$$zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)]. \quad (2.52)$$

Since  $\varphi(w)$  is analytic in a domain containing  $Q(\mathbb{U})$ , and  $zQ'(z) \cdot \varphi[Q(z)]$  is starlike, the result follows from Lemma 1.1.  $\square$

**Theorem 2.16.** Let  $Q(z)$  be a convex function in  $\mathbb{U}$ , and  $Q(0) = 1$ . If  $f \in \mathcal{A}_p$  satisfies

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \left[ 2 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right] < zQ'(z) + Q(z) + p - q - 1, \quad (2.53)$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 < Q(z), \quad (2.54)$$

and  $Q$  is the best dominant.

*Proof.* Let the function  $P(z)$  be defined by (2.42). Using (2.43), it follows that

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \left( 1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right) = zP'(z), \quad (2.55)$$

and, therefore,

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \left( 2 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right) = zP'(z) + P(z) + p - q - 1. \quad (2.56)$$

By assumption,

$$zP'(z) + P(z) + p - q - 1 < zQ'(z) + Q(z) + p - q - 1, \quad (2.57)$$

or

$$zP'(z) + \theta[P(z)] < zQ'(z) + \theta[Q(z)], \quad (2.58)$$

where the function  $\theta(w) = w + p - q + 1$ . The proof is completed by applying Lemma 1.2.  $\square$

**Theorem 2.17.** Let  $Q(z)$  be a convex function in  $\mathbb{U}$ , with  $Q(0) = 1$ . If  $f \in \mathcal{A}_p$  satisfies

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \left[ 1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right] < zQ'(z), \quad (2.59)$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 < Q(z), \quad (2.60)$$

and  $Q$  is the best dominant.

*Proof.* Let the function  $P(z)$  be defined by (2.42). It follows from (2.43) that  $zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)]$ , where  $\varphi(w) = 1$ . The result follows easily from Lemma 1.1.  $\square$

### Acknowledgment

This work was supported in part by the FRGS and Science Fund research grants, and was completed while the third author was visiting USM.

### References

- [1] O. Altıntaş, "Neighborhoods of certain  $p$ -valently analytic functions with negative coefficients," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 47–53, 2007.
- [2] O. Altıntaş, H. Irmak, and H. M. Srivastava, "Neighborhoods for certain subclasses of multivalently analytic functions defined by using a differential operator," *Computers & Mathematics with Applications*, vol. 55, no. 3, pp. 331–338, 2008.
- [3] M. P. Chen, H. Irmak, and H. M. Srivastava, "Some multivalent functions with negative coefficients defined by using a differential operator," *Panamerican Mathematical Journal*, vol. 6, no. 2, pp. 55–64, 1996.
- [4] H. Irmak, "A class of  $p$ -valently analytic functions with positive coefficients," *Tamkang Journal of Mathematics*, vol. 27, no. 4, pp. 315–322, 1996.
- [5] H. Irmak and N. E. Cho, "A differential operator and its applications to certain multivalently analytic functions," *Hacetatepe Journal of Mathematics and Statistics*, vol. 36, no. 1, pp. 1–6, 2007.
- [6] H. Irmak, S. H. Lee, and N. E. Cho, "Some multivalently starlike functions with negative coefficients and their subclasses defined by using a differential operator," *Kyungpook Mathematical Journal*, vol. 37, no. 1, pp. 43–51, 1997.
- [7] M. Nunokawa, "On the multivalent functions," *Indian Journal of Pure and Applied Mathematics*, vol. 20, no. 6, pp. 577–582, 1989.
- [8] Y. Polatoğlu, "Some results of analytic functions in the unit disc," *Publications de l'Institut Mathématique*, vol. 78, no. 92, pp. 79–85, 2005.
- [9] H. Silverman, "Higher order derivatives," *Chinese Journal of Mathematics*, vol. 23, no. 2, pp. 189–191, 1995.
- [10] T. Yaguchi, "The radii of starlikeness and convexity for certain multivalent functions," in *Current Topics in Analytic Function Theory*, pp. 375–386, World Scientific, River Edge, NJ, USA, 1992.
- [11] I. S. Jack, "Functions starlike and convex of order  $\alpha$ ," *Journal of the London Mathematical Society. Second Series*, vol. 3, pp. 469–474, 1971.
- [12] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Application*, vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.
- [13] R. M. Ali, V. Ravichandran, and S. K. Lee, "Subclasses of multivalent starlike and convex functions," to appear in *Bulletin of the Belgian Mathematical Society - Simon Stevin*.
- [14] W. C. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," in *Proceedings of the Conference on Complex Analysis*, Conference Proceedings and Lecture Notes in Analysis, I, pp. 157–169, International Press, Tianjin, China, 1994.
- [15] W. Janowski, "Some extremal problems for certain families of analytic functions. I," *Annales Polonici Mathematici*, vol. 28, pp. 297–326, 1973.
- [16] Y. Polatoğlu and M. Bolcal, "The radius of convexity for the class of Janowski convex functions of complex order," *Matematichki Vesnik*, vol. 54, no. 1-2, pp. 9–12, 2002.
- [17] Ö. Ö. Kılıç, "Sufficient conditions for subordination of multivalent functions," *Journal of Inequalities and Applications*, vol. 2008, Article ID 374756, 8 pages, 2008.
- [18] V. Ravichandran and M. Darus, "On a criteria for starlikeness," *International Mathematical Journal*, vol. 4, no. 2, pp. 119–125, 2003.
- [19] M. Obradowiç and N. Tuneski, "On the starlike criteria defined by Silverman," *Zeszyty Naukowe Politechniki Rzeszowskiej. Matematyka*, vol. 181, no. 24, pp. 59–64, 2000.