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Research Article

Some Inequalities of the Grüss Type for the Numerical Radius of Bounded Linear Operators in Hilbert Spaces

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Some inequalities of the Grüss type for the numerical radius of bounded linear operators in Hilbert spaces are established.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [1, page 1]:

$$W(T)=\{\langle Tx,x\rangle,\ x\in H,\ \|x\|=1\}. \tag{1.1}$$

The *numerical radius* $w(T)$ of an operator T on H is given by [1, page 8]:

$$w(T)=\sup\{|\lambda|,\lambda\in W(T)\}=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T:H\to H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [1, page 9].

Theorem 1.1 (equivalent norm). *For any $T\in B(H)$, one has*

$$w(T)\le\|T\|\le 2w(T). \tag{1.3}$$

For other results on numerical radius (see [2, Chapter 11]).

We recall some classical results involving the numerical radius of two linear operators A, B.

The following general result for the product of two operators holds [1, page 37].

Theorem 1.2. *If A, B are two bounded linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$, then*

$$w(AB) \le 4w(A)w(B). \tag{1.4}$$

In the case that $AB = BA$, then

$$w(AB) \le 2w(A)w(B). \tag{1.5}$$

The following results are also well known [1, page 38].

Theorem 1.3. *If A is a unitary operator that commutes with another operator B, then*

$$w(AB) \le w(B). \tag{1.6}$$

If A is an isometry and $AB = BA$, then (1.6) *also holds true.*

We say that A and B *double commute*, if $AB = BA$ and $AB^* = B^*A$.

The following result holds [1, page 38].

Theorem 1.4 (double commute). *If the operators A and B double commute, then*

$$w(AB) \le w(B)\|A\|. \tag{1.7}$$

As a consequence of the above, one has [1, page 39] the following.

Corollary 1.5. *Let A be a normal operator commuting with B. Then*

$$w(AB) \le w(A)w(B). \tag{1.8}$$

For other results and historical comments on the above (see [1, pages 39–41]). For more results on the numerical radius, see [2].

In the recent survey paper [3], we provided other inequalities for the numerical radius of the product of two operators. We list here some of the results.

Theorem 1.6. *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$, then*

$$\begin{gathered} \left\| \frac{A^*A + B^*B}{2} \right\| \le w(B^*A) + \frac{1}{2}\|A - B\|^2, \\ \left\| \frac{A+B}{2} \right\|^2 \le \frac{1}{2}\left[\left\| \frac{A^*A + B^*B}{2} \right\| + w(B^*A) \right], \end{gathered} \tag{1.9}$$

respectively.

If more information regarding one of the operators is available, then the following results may be stated as well.

Theorem 1.7. *Let $A, B : H \to H$ be two bounded linear operators on H, and B is invertible such that, for a given $r > 0$,*

$$\|A - B\| \leq r. \tag{1.10}$$

Then

$$\begin{gathered}\|A\| \leq \|B^{-1}\| \left[w(B^*A) + \frac{1}{2} r^2 \right], \\ (0 \leq) \|A\| \|B\| - w(B^*A) \leq \frac{1}{2} r^2 + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{2\|B^{-1}\|^2},\end{gathered} \tag{1.11}$$

respectively.

Motivated by the natural questions that arise, in order to compare the quantity $w(AB)$ with other expressions comprising the norm or the numerical radius of the involved operators A and B (or certain expressions constructed with these operators), we establish in this paper some natural inequalities of the form

$$w(BA) \leq w(A)w(B) + K_1, \ \text{(additive Grüss'type inequality)}, \tag{1.12}$$

or

$$\frac{w(BA)}{w(A)w(B)} \leq K_2, \ \text{(multiplicative Grüss'type inequality)}, \tag{1.13}$$

where K_1 and K_2 are specified and desirably simple constants (depending on the given operators A and B).

Applications in providing upper bounds for the non-negative quantities

$$\|A\|^2 - w^2(A), \qquad w^2(A) - w(A^2), \tag{1.14}$$

and the *superunitary* quantities

$$\frac{\|A\|^2}{w^2(A)}, \qquad \frac{w^2(A)}{w(A^2)} \tag{1.15}$$

are also given.

2. Numerical radius inequalities of Grüss type

For the complex numbers α, β and the bounded linear operator T, we define the following transform:

$$C_{\alpha,\beta}(T) := (T^* - \overline{\alpha} I)(\beta I - T), \tag{2.1}$$

where by T^* we denote the adjoint of T.

We list some properties of the transform $C_{\alpha,\beta}(\cdot)$ that are useful in the following.

(i) For any $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$, we have

$$\begin{gathered} C_{\alpha,\beta}(I) = (1-\overline{\alpha})(\beta-1)I, \qquad C_{\alpha,\alpha}(T) = -(\alpha I - T)^*(\alpha I - T), \\ C_{\alpha,\beta}(\gamma T) = |\gamma|^2 C_{\alpha/\gamma,\beta/\gamma}(T), \quad \text{for each } \gamma \in \mathbb{C}\setminus\{0\}, \\ [C_{\alpha,\beta}(T)]^* = C_{\beta,\alpha}(T), \\ C_{\overline{\beta},\overline{\alpha}}(T^*) - C_{\alpha,\beta}(T) = T^*T - TT^*. \end{gathered} \tag{2.2}$$

(ii) The operator $T \in B(H)$ is normal, if and only if $C_{\overline{\beta},\overline{\alpha}}(T^*) = C_{\alpha,\beta}(T)$ for each $\alpha, \beta \in \mathbb{C}$.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is called *accretive*, if $\operatorname{Re}\langle Ty, y\rangle \geq 0$, for any $y \in H$.

Utilizing the following identity

$$\begin{aligned} \operatorname{Re}\langle C_{\alpha,\beta}(T)x, x\rangle &= \operatorname{Re}\langle C_{\beta,\alpha}(T)x, x\rangle \\ &= \frac{1}{4}|\beta-\alpha|^2 - \left\|\left(T - \frac{\alpha+\beta}{2}I\right)x\right\|^2, \end{aligned} \tag{2.3}$$

that holds for any scalars α, β, and any vector $x \in H$ with $\|x\| = 1$, we can give a simple characterization result that is useful in the following.

Lemma 2.1. *For $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$, the following statements are equivalent.*

(i) *The transform $C_{\alpha,\beta}(T)$ (or, equivalently $C_{\beta,\alpha}(T)$) is accretive.*

(ii) *The transform $C_{\overline{\alpha},\overline{\beta}}(T^*)$ (or, equivalently $C_{\overline{\beta},\overline{\alpha}}(T^*)$) is accretive.*

(iii) *One has the norm inequality*

$$\left\|T - \frac{\alpha+\beta}{2}\cdot I\right\| \leq \frac{1}{2}|\beta-\alpha|, \tag{2.4}$$

or, equivalently,

$$\left\|T^* - \frac{\overline{\alpha}+\overline{\beta}}{2}\cdot I\right\| \leq \frac{1}{2}|\beta-\alpha|. \tag{2.5}$$

Remark 2.2. In order to give examples of operators $T \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $C_{\alpha,\beta}(T)$ is accretive, it suffices to select a bounded linear operator S and the complex numbers z, w with the property that $\|S - zI\| \leq |w|$, and by choosing $T = S$, $\alpha = (1/2)(z+w)$, and $\beta = (1/2)(z-w)$, we observe that T satisfies (2.4), that is, $C_{\alpha,\beta}(T)$ is accretive.

The following results compare the quantities $w(AB)$ and $w(A)w(B)$ provided that some information about the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are available, where $\alpha, \beta, \gamma, \delta \in \mathbb{K}$.

Theorem 2.3. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be such that the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then*

$$w(BA) \leq w(A)w(B) + \frac{1}{4}|\beta-\alpha|\,|\gamma-\delta|. \tag{2.6}$$

Proof. Since $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then, on making use of Lemma 2.1, we have that

$$\left\|Ax - \frac{\alpha+\beta}{2}x\right\| \le \frac{1}{2}|\beta-\alpha|, \qquad \left\|B^*x - \frac{\overline{\gamma}+\overline{\delta}}{2}x\right\| \le \frac{1}{2}\left|\overline{\gamma}-\overline{\delta}\right|, \tag{2.7}$$

for any $x \in H$, $\|x\| = 1$.

Now, we make use of the following Grüss type inequality for vectors in inner product spaces obtained by the author in [4] (see also [5] or [6, page 43]).

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ such that

$$\operatorname{Re}\langle \beta e - u, u - \alpha e\rangle \ge 0, \qquad \operatorname{Re}\langle \delta e - v, v - \gamma e\rangle \ge 0, \tag{2.8}$$

or equivalently,

$$\left\|u - \frac{\alpha+\beta}{2}e\right\| \le \frac{1}{2}|\beta-\alpha|, \qquad \left\|v - \frac{\gamma+\delta}{2}e\right\| \le \frac{1}{2}|\delta-\gamma|, \tag{2.9}$$

then

$$|\langle u, v\rangle - \langle u, e\rangle\langle e, v\rangle| \le \frac{1}{4}|\beta-\alpha|\,|\delta-\gamma|. \tag{2.10}$$

Applying (2.10) for $u = Ax$, $v = B^*x$, and $e = x$ we deduce

$$|\langle BAx, x\rangle - \langle Ax, x\rangle\langle Bx, x\rangle| \le \frac{1}{4}|\beta-\alpha|\,|\delta-\gamma|, \tag{2.11}$$

for any $x \in H$, $\|x\| = 1$, which is an inequality of interest in itself.

Observing that

$$|\langle BAx, x\rangle| - |\langle Ax, x\rangle\langle Bx, x\rangle| \le |\langle BAx, x\rangle - \langle Ax, x\rangle\langle Bx, x\rangle|, \tag{2.12}$$

then by (2.10), we deduce the inequality

$$|\langle BAx, x\rangle| \le |\langle Ax, x\rangle\langle Bx, x\rangle| + \frac{1}{4}|\beta-\alpha|\,|\delta-\gamma|, \tag{2.13}$$

for any $x \in H$, $\|x\| = 1$. On taking the supremum over $\|x\| = 1$ in (2.13), we deduce the desired result (2.6). □

The following particular case provides an upper bound for the nonnegative quantity $\|A\|^2 - w(A)^2$ when some information about the operator A is available.

Corollary 2.4. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$(0 \le)\|A\|^2 - w^2(A) \le \frac{1}{4}|\beta-\alpha|^2. \tag{2.14}$$

Proof. Follows on applying Theorem 2.3 above for the choice $B = A^*$, taking into account that $C_{\alpha,\beta}(A)$ is accretive implies that $C_{\overline{\alpha},\overline{\beta}}(A^*)$ is the same and $w(A^*A) = \|A\|^2$. □

Remark 2.5. Let $A \in B(H)$ and $M > m > 0$be such that the transform $C_{m,M}(A) = (A^* - \mathrm{mI})(MI - A)$ is accretive. Then

$$(0 \le)\|A\|^2 - w^2(A) \le \frac{1}{4}(M - m)^2. \tag{2.15}$$

A sufficient simple condition for $C_{m,M}(A)$ to be accretive is that A is a self-adjoint operator on H and such that $MI \ge A \ge mI$ in the partial operator order of $B(H)$.

The following result may be stated as well.

Theorem 2.6. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be such that $\operatorname{Re}(\beta\overline{\alpha}) > 0$, $\operatorname{Re}(\delta\overline{\gamma}) > 0$ and the transforms $C_{\alpha,\beta}(A), C_{\gamma,\delta}(B)$ are accretive, then*

$$\begin{gathered}\frac{w(BA)}{w(A)w(B)} \le 1 + \frac{1}{4}\cdot\frac{|\beta-\alpha|\,|\delta-\gamma|}{[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})]^{1/2}},\\ w(BA) \le w(A)w(B) + \left[\left(|\alpha+\beta|-2[\operatorname{Re}(\beta\overline{\alpha})]^{1/2}\right)\times\left(|\delta+\gamma|-2[\operatorname{Re}(\delta\overline{\gamma})]^{1/2}\right)\right]^{1/2}[w(A)w(B)]^{1/2},\end{gathered} \tag{2.16}$$

respectively.

Proof. With the assumptions (2.8) (or, equivalently, (2.9) in the proof of Theorem 2.3) and if $\operatorname{Re}(\beta\overline{\alpha}) > 0$, $\operatorname{Re}(\delta\overline{\gamma}) > 0$ then

$$|\langle u,v\rangle - \langle u,e\rangle\langle e,v\rangle| \le \begin{cases}\dfrac{1}{4}\dfrac{|\beta-\alpha|\,|\delta-\gamma|}{[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})]^{1/2}}|\langle u,e\rangle\langle e,v\rangle|,\\ \left[\left(|\alpha+\beta|-2[\operatorname{Re}(\beta\overline{\alpha})]^{1/2}\right)\left(|\delta+\gamma|-2[\operatorname{Re}(\delta\overline{\gamma})]^{1/2}\right)\right]^{1/2}\\ \times[|\langle u,e\rangle\langle e,v\rangle|]^{1/2}.\end{cases} \tag{2.17}$$

The first inequality has been established in [7] (see [6, page 62]) while the second one can be obtained in a canonical manner from the reverse of the Schwarz inequality given in [8]. The details are omitted.

Applying (2.10) for $u = Ax$, $v = B^*x$, and $e = x$ we deduce

$$|\langle BAx,x\rangle - \langle Ax,x\rangle\langle Bx,x\rangle| \le \begin{cases}\dfrac{1}{4}\dfrac{|\beta-\alpha|\,|\delta-\gamma|}{[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})]^{1/2}}|\langle A,x\rangle\langle Bx,x\rangle|,\\ \left[\left(|\alpha+\beta|-2[\operatorname{Re}(\beta\overline{\alpha})]^{1/2}\right)\left(|\delta+\gamma|-2[\operatorname{Re}(\delta\overline{\gamma})]^{1/2}\right)\right]^{1/2}\\ \times[|\langle A,x\rangle\langle Bx,x\rangle|]^{1/2},\end{cases} \tag{2.18}$$

for any $x \in H$, $\|x\| = 1$, which are of interest in themselves.

A similar argument to that in the proof of Theorem 2.3 yields the desired inequalities (2.16). The details are omitted. □

Corollary 2.7. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that* $\operatorname{Re}(\beta\overline{\alpha}) > 0$ *and the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$(1 \leq)\frac{\|A\|^2}{w^2(A)} \leq 1 + \frac{1}{4}\cdot\frac{|\beta-\alpha|^2}{\operatorname{Re}(\beta\overline{\alpha})},$$
$$(0 \leq)\|A\|^2 - w^2(A) \leq (|\alpha+\beta| - 2[\operatorname{Re}(\beta\overline{\alpha})]^{1/2})w(A), \tag{2.19}$$

respectively.

The proof is obvious from Theorem 2.6 on choosing $B = A^*$ and the details are omitted.

Remark 2.8. Let $A \in B(H)$ and $M > m > 0$be such that the transform $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive. Then, on making use of Corollary 2.7, we may state the following simpler results:

$$(1 \leq)\frac{\|A\|}{w(A)} \leq \frac{1}{2}\cdot\frac{M+m}{\sqrt{Mm}},$$
$$(0 \leq)\|A\|^2 - w^2(A) \leq (\sqrt{M}-\sqrt{m})^2 w(A), \tag{2.20}$$

respectively. These two inequalities were obtained earlier by the author using a different approach (see [9]).

Problem 1. Find general examples of bounded linear operators realizing the equality case in each of inequalities (2.6), (2.16), respectively.

3. Some particular cases of interest

The following result is well known in the literature (see, e.g., [10]):

$$w(A^n) \leq w^n(A), \tag{3.1}$$

for each positive integer n and any operator $A \in B(H)$.

The following reverse inequalities for $n = 2$ can be stated.

Proposition 3.1. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$(0 \leq)w^2(A) - w(A^2) \leq \frac{1}{4}|\beta-\alpha|^2. \tag{3.2}$$

Proof. On applying inequality (2.11) from Theorem 2.3 for the choice $B = A$, we get the following inequality of interest in itself:

$$|\langle Ax, x\rangle^2 - \langle A^2x, x\rangle| \leq \frac{1}{4}|\beta-\alpha|^2, \tag{3.3}$$

for any $x \in H$, $\|x\| = 1$. Since obviously,

$$|\langle Ax, x\rangle|^2 - |\langle A^2x, x\rangle| \leq |\langle Ax, x\rangle^2 - \langle A^2x, x\rangle|, \tag{3.4}$$

then by (3.3), we get

$$|\langle Ax, x\rangle|^2 \leq |\langle A^2x, x\rangle| + \frac{1}{4}|\beta-\alpha|^2, \tag{3.5}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $\|x\| = 1$ in (3.5), we deduce the desired result (3.2). □

Remark 3.2. Let $A \in B(H)$ and $M > m > 0$ be such that the transform $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive. Then

$$(0 \leq) w^2(A) - w(A^2) \leq \frac{1}{4}(M-m)^2. \tag{3.6}$$

If $MI \geq A \geq mI$ in the partial operator order of $B(H)$, then (3.6) is valid.

Finally, we also have the following proposition.

Proposition 3.3. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that $\operatorname{Re}(\beta\overline{\alpha}) > 0$ and the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$\begin{gathered}(1 \leq) \frac{w^2(A)}{w(A^2)} \leq 1 + \frac{1}{4} \cdot \frac{|\beta - \alpha|^2}{\operatorname{Re}(\beta\overline{\alpha})}, \\ (0 \leq) w^2(A) - w(A^2) \leq \left(|\alpha + \beta| - 2[\operatorname{Re}(\beta\overline{\alpha})]^{1/2}\right) w(A),\end{gathered} \tag{3.7}$$

respectively.

Proof. On applying inequality (2.18) from Theorem 2.6 for the choice $B = A$, we get the following inequality of interest in itself:

$$|\langle Ax, x\rangle^2 - \langle A^2 x, x\rangle| \leq \begin{cases} \dfrac{1}{4} \cdot \dfrac{|\beta - \alpha|^2}{\operatorname{Re}(\beta\overline{\alpha})} |\langle A, x\rangle|^2, \\ \left(|\alpha + \beta| - 2[\operatorname{Re}(\beta\overline{\alpha})]^{1/2}\right) |\langle A, x\rangle|, \end{cases} \tag{3.8}$$

for any $x \in H$, $\|x\| = 1$.

Now, on making use of a similar argument to the one in the proof of Proposition 3.1, we deduce the desired results (3.7). The details are omitted. □

Remark 3.4. Let $A \in B(H)$ and $M > m > 0$ be such that the transform $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive. Then, on making use of Proposition 3.3, we may state the following simpler results:

$$\begin{gathered}(1 \leq) \frac{w^2(A)}{w(A^2)} \leq \frac{1}{4} \cdot \frac{(M+m)^2}{Mm}, \\ (0 \leq) w^2(A) - w(A^2) \leq \left(\sqrt{M} - \sqrt{m}\right)^2 w(A),\end{gathered} \tag{3.9}$$

respectively.

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