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Research Article

Some Remarks Concerning Quasiconformal Extensions in Several Complex Variables

Paula Curt[1] and Gabriela Kohr[2]

[1] *Department of Statistics, Forecasts and Mathematics, Faculty of Economics and Business Administration, Babeş-Bolyai University, 58-60 Teodor Mihali Street, 400591 Cluj-Napoca, Romania*
[2] *Department of Mathematics, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 1 M. Kogălniceanu Street, 400084 Cluj-Napoca, Romania*

Correspondence should be addressed to Gabriela Kohr, gkohr@math.ubbcluj.ro

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Let B be the unit ball in $\mathbb{C}^n$ with respect to the Euclidean norm. In this paper, we obtain a sufficient condition for a normalized quasiregular mapping $f \in H(B)$ to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself. In the last section we consider the asymptotical case of this result and we obtain certain applications.

1. Introduction

Becker [1] proved that if $0 \le q < 1$; and f is a holomorphic function on the unit disc U which satisfies the condition

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{q}{1-|z|^2}, \quad z \in U, \tag{1.1}$$

then f is univalent on U and extends to a quasiconformal homeomorphism of $\mathbb{R}^2$ onto itself.

This result was generalized by Pfaltzgraff [2] (cf. [3]) to several complex variables. He proved that if $0 \le q < 1$ and $f \in H(B)$ is a quasiregular mapping, which satisfies the condition

$$\left(1-\|z\|^2\right)\left\|\left[Df(z)\right]^{-1}D^2f(z)(z,\cdot)\right\| \le q, \quad z \in B, \tag{1.2}$$

then f is biholomorphic on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Recently, the problem of quasiconformal extensions for quasiregular holomorphic mappings on the unit ball in $\mathbb{C}^n$ has been studied by Hamada and Kohr ([4–6]; see also [7]), Curt ([8–10]), Curt and Kohr [11].

In this paper we will generalize certain results due to Pfaltzgraff [2], Curt ([8, 9]), Hamada and Kohr [5].

2. Notations and preliminary results

Let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1, \dots, z_n)$ with the usual inner product $\langle z, w\rangle = \sum_{i=1}^{n} z_i \overline{w}_i$ and Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. Let B denote the open unit ball in $\mathbb{C}^n$ and let U be the unit disc in $\mathbb{C}$. Also let $\overline{B}$ be the closed unit ball in $\mathbb{C}^n$ and let $\overline{\mathbb{R}^m} = \mathbb{R}^m \cup \{\infty\}$ be the one point compactification of $\mathbb{R}^m$.

Let $\mathcal{H}(\Omega)$ be the set of holomorphic mappings from a domain Ω in $\mathbb{C}^n$ into $\mathbb{C}^n$. If $f \in \mathcal{H}(B)$, let $J_f(z) = \det Df(z)$ be the complex jacobian determinant of f at z. Also let $\mathcal{L}(\mathbb{C}^n)$ be the space of continuous linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard operator norm

$$\|A\| = \sup\{\|Az\| : \|z\| = 1\}, \tag{2.1}$$

and let I be the identity in $\mathcal{L}(\mathbb{C}^n)$. A mapping $f \in \mathcal{H}(B)$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$.

If $f \in H(B)$, let $Df(z)$ be the Fréchet derivative of f at $z \in B$ given by

$$Df(z) = \left(\frac{\partial f_j}{\partial z_k}(z)\right)_{1\le j,k\le n}. \tag{2.2}$$

Also let $D^2 f(z)$ be the second Fréchet derivative of f at $z \in B$. Clearly $D^2 f(z)(z, \cdot)$ is the linear operator from $\mathbb{C}^n$ into $\mathbb{C}^n$ that is obtained by restricting to $\{z\} \times \mathbb{C}^n$ the symmetric bilinear operator $D^2 f(z)$. Then

$$D^2 f(z)(z, \cdot) = \left(\sum_{m=1}^{n} \frac{\partial^2 f_j}{\partial z_k \partial z_m}(z) z_m\right)_{1\le j,k\le n}. \tag{2.3}$$

We say that a mapping $f \in \mathcal{H}(B)$ is K-quasiregular, $K \ge 1$, if

$$\|Df(z)\|^n \le K|J_f(z)|, \quad z \in B. \tag{2.4}$$

A mapping $f \in \mathcal{H}(B)$ is called quasiregular if f is K-quasiregular for some $K \ge 1$. It is well known that quasiregular holomorphic mappings are locally biholomorphic.

Definition 2.1. Let G and G' be domains in $\mathbb{R}^m$. A homeomorphism $f : G \to G'$ is said to be K-quasiconformal if it is differentiable a.e., absolutely continuous on lines (ACL) and

$$\|Df(x)\|^m \le K|\det Df(x)| \quad \text{a.e. } x \in G, \tag{2.5}$$

where $Df(x)$ denotes the real Jacobian matrix of f; and K is a constant.

Note that a K-quasiregular biholomorphic mapping is K^2-quasiconformal.

If $f, g \in \mathcal{H}(B)$, we say that f is subordinate to g (and write $f \prec g$) if there exists a Schwarz mapping v (i.e., $v \in \mathcal{H}(B)$ and $\|v(z)\| \le \|z\|,\ z \in B$) such that $f(z) = g(v(z)),\ z \in B$.

Definition 2.2. A mapping $L : B \times [0,\infty) \to \mathbb{C}^n$ is called a subordination chain if the following conditions hold:

(i) $L(0,t) = 0$ and $L(\cdot,t) \in \mathcal{H}(B)$ for $t \geq 0$;

(ii) $L(\cdot,s) \prec L(\cdot,t)$ for $0 \leq s \leq t < \infty$.

If $L(z,t)$ is a subordination chain such that $L(\cdot,t)$ is biholomorphic on B for $t \in [0,\infty)$, then we say that $L(z,t)$ is a univalent subordination chain (or a Loewner chain). In this case there exists a biholomorphic Schwarz mapping $v = v(z,s,t)$ (which is called the transition mapping associated with $L(z,t)$) such that

$$L(z,s) = L\big(v(z,s,t),t\big), \quad z \in B,\ 0 \leq s \leq t. \tag{2.6}$$

If $L(z,t)$ is a univalent subordination chain such that $DL(0,t) = e^t I$, we say that $L(z,t)$ is a normalized subordination chain (or a normalized Loewner chain).

An important role in our discussion is played by the n-dimensional version of the Carathéodory set (i.e., the class of holomorphic functions on the unit disc with positive real part):

$$\begin{aligned}\mathcal{N} &= \left\{h \in \mathcal{H}(B) : h(0) = 0,\ \Re\langle h(z), z\rangle > 0,\ z \in B\setminus\{0\}\right\},\\ \mathcal{M} &= \left\{h \in \mathcal{N},\ \ Dh(0) = I\right\}.\end{aligned} \tag{2.7}$$

The authors ([12, Theorem 1.10] and [13, Theorem 2.3]) proved that normalized univalent subordination chains satisfy the generalized Loewner differential equation.

Theorem 2.3. *Let $L : B \times [0,\infty) \to \mathbb{C}^n$ be a normalized univalent subordination chain. Then there exists a mapping $h = h(z,t) : B \times [0,\infty) \to \mathbb{C}^n$ such that $h(\cdot,t) \in \mathcal{M}$ for $t \geq 0$, $h(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in B$, and*

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t)h(z,t), \quad \text{a.e. } t \geq 0,\ \forall z \in B. \tag{2.8}$$

Using an elementary change of variable, it is not difficult to reformulate the above result in the case of nonnormalized subordination chains $L(z,t) = a(t)z + \cdots$, where $a : [0,\infty) \to \mathbb{C}$, $a \in C^1([0,\infty))$, $a(0) = 1$, and $a(t) \to \infty$ as $t \to \infty$ (see [10, 14]).

Theorem 2.4. *Let $L(z,t) : B \times [0,\infty) \to \mathbb{C}^n$ be a Loewner chain such that $L(z,t) = a(t)z + \cdots$, where $a \in C^1([0,\infty))$, $a(0) = 1$, and $\lim_{t\to\infty}|a(t)| = \infty$. Then there exists a mapping $h = h(z,t) : B \times [0,\infty) \to \mathbb{C}^n$ such that $h(\cdot,t) \in \mathcal{N}$ for $t \geq 0$, $h(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in B$, and*

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t)h(z,t), \quad \text{a.e. } t \geq 0,\ \forall z \in B. \tag{2.9}$$

Definition 2.5 ([15]). Let $F = F(u,v) : B \times \mathbb{C}^n \to \mathbb{C}^n$ be a mapping of class C^1 with $F(0,0) = 0$. We say that F satisfies the conditions (P) if the following assumptions hold.

(i) $F(e^{-t}z, e^t z) \in \mathcal{H}(B)$, for $t \geq 0$.

(ii) $D_v F(u,v)$ is invertible, for all $(u,v) \in B \times \mathbb{C}^n$.

(iii) For each $t \geq 0$, there exists a complex number $a(t) \neq 0$, with $a(0) = 1$, such that

$$e^{-t}D_u F(0,0) + e^t D_v F(0,0) = a(t)I. \tag{2.10}$$

Here $D_uF(u,v)$ $(D_vF(u,v))$ is the $n\times n$ matrix for which the (i,j) entry is given by

$$\frac{\partial F_i}{\partial u_j}(u,v)\ \left(\frac{\partial F_i}{\partial v_j}(u,v)\right). \tag{2.11}$$

(iv) $\{[e^{-t}D_uF(0,0)+e^{t}D_vF(0,0)]^{-1}F(e^{-t}z,e^{t}z)\}_{t\geq 0}$ is a normal family on B.

Recently Hamada and Kohr [5, Theorem 3.2] (see also [11, Theorem 2.4]) proved the following result.

Theorem 2.6. *Let $L=L(z,t):B\times[0,\infty)\to\mathbb{C}^n$ be a normalized univalent subordination chain. Assume the following conditions hold:*

(i) *there exists $K>0$ such that $L(\cdot,t)$ is K-quasiregular for each $t\in[0,\infty)$;*

(ii) *there exist some constants $M>0$ and $\alpha\in[0,1)$ such that*

$$\|DL(z,t)\|\leq\frac{e^{t}M}{(1-\|z\|)^{\alpha}},\quad z\in B,\ t\in[0,\infty); \tag{2.12}$$

(iii) *there exists a sequence $\{t_m\}_{m\in\mathbb{N}}$, $t_m>0$, $\lim_{m\to\infty}t_m=\infty$, and a mapping $F\in\mathcal{H}(B)$ such that*

$$\lim_{m\to\infty}e^{-t_m}L(z,t_m)=F(z) \tag{2.13}$$

locally uniformly on B.

Moreover, assume that the mapping $h(z,t)$ defined by Theorem 2.3 satisfies the following conditions:

(iv) *there exists a constant $C>0$ such that*

$$C\|z\|^2\leq\Re\langle h(z,t),z\rangle,\quad z\in B,\ t\in[0,\infty); \tag{2.14}$$

(v) *there exists a constant $C_1>0$ such that*

$$\|h(z,t)\|\leq C_1,\quad z\in B,\ t\in[0,\infty). \tag{2.15}$$

Then $f=L(\cdot,0)$ extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

In this paper we continue the work begun in [5, 6, 8, 9, 11, 16]; and we obtain a sufficient condition for a normalized quasiregular holomorphic mapping on B, which can be imbedded as the first element of a nonnormalized univalent subordination chain, to be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself. We also obtain certain applications of this result, including the n-dimensional versions of the quasiconformal extension results due to Becker and Ahlfors-Becker.

3. Main results

We begin this section with the following result.

Theorem 3.1. *Let* $L(z,t) : B \times [0,\infty) \to \mathbb{C}^n$, $L(z,t) = a(t)z + \cdots$ *be a Loewner chain such that* $a(\cdot) \in C^1([0,\infty))$, $a(0) = 1$ *and* $\lim_{t\to\infty}|a(t)| = \infty$. *Assume that the following conditions hold:*

(i) *there exists* $K > 0$ *such that* $L(\cdot,t)$ *is* K*-quasiregular for each* $t \geq 0$;

(ii) *there exist some constants* $M > 0$ *and* $\alpha \in [0,1)$ *such that*

$$\|DL(z,t)\| \leq \frac{M|a(t)|}{(1-\|z\|)^{\alpha}}, \quad z \in B,\ t \in [0,\infty); \tag{3.1}$$

(iii) *there exists a sequence* $\{t_m\}_{m\in\mathbb{N}}$, $t_m > 0$, $\lim_{m\to\infty} t_m = \infty$, *and a mapping* $F \in \mathcal{H}(B)$ *such that*

$$\lim_{m\to\infty} \frac{L(z,t_m)}{a(t_m)} = F(z) \tag{3.2}$$

locally uniformly on B.

Further, assume that the mapping $h(z,t)$ *defined by Theorem 2.4 satisfies the following conditions:*

(iv) *there exists a constant* $C > 0$ *such that*

$$C\|z\|^2 \leq \Re\langle h(z,t), z\rangle, \quad z \in B,\ t \in [0,\infty); \tag{3.3}$$

(v) *there exists a constant* $C_1 > 0$ *such that*

$$\|h(z,t)\| \leq C_1, \quad z \in B,\ t \in [0,\infty). \tag{3.4}$$

Then $f = L(\cdot,0)$ *extends to a quasiconformal homeomorphism of* $\mathbb{R}^{2n}$ *onto itself.*

Proof. Since $L(z,t)$ is a Loewner chain, it follows that $L(z,s) \prec L(z,t)$ for $z \in B$ and $0 \leq s \leq t < \infty$. Hence $|a(\cdot)|$ is increasing by Schwarz's lemma. Moreover, taking into account the condition (3.3) and the fact that $Dh(0,t) = (a'(t)/a(t))I$ for $t \geq 0$, it is not difficult to deduce that

$$\Re\left(\frac{a'(t)}{a(t)}\right) \geq C, \quad t \in [0,\infty). \tag{3.5}$$

Indeed, fix $w \in \partial B$ and $t \geq 0$. Let $q_t : U \to \mathbb{C}$ be given by

$$q_t(\zeta) = \begin{cases} \dfrac{1}{\zeta}\langle h(\zeta w,t), w\rangle, & 0 < |\zeta| < 1, \\ \dfrac{a'(t)}{a(t)}, & \zeta = 0. \end{cases} \tag{3.6}$$

Then q_t is a holomorphic function on U, and in view of the relation (3.3) we deduce that $\Re q_t(\zeta) \geq C$ for $0 < |\zeta| < 1$. Hence, we must have $\Re q_t(0) \geq C$, that is, $\Re(a'(t)/a(t)) \geq C$, as claimed.

As in the proof of [17, Theorem 2] (see also [14]), we use the change of parameter $\theta(t) = \arg a(t)$, $t^* = \ln|a(t)|$, in order to pass from the nonnormalized subordination chain $L(z,t)$ to the normalized subordination chain $L^*(z,t^*)$ given by

$$L^*(z,t^*) = L(e^{-i\theta(t)}z,t), \quad z \in B,\ t \in [0,\infty). \tag{3.7}$$

Also let $h^* = h^*(z,t^*) : B \times [0,\infty) \to \mathbb{C}^n$ be given by

$$h^*(z,t^*) = \frac{1}{\Re(a'(t)/a(t))}\left[h(ze^{-i\theta(t)},t)e^{i\theta(t)} - i\frac{d\theta(t)}{dt}z\right], \quad z \in B,\ t^* \in [0,\infty). \tag{3.8}$$

In the proof of [17, Theorem 2] (see also [14]), it was shown that $L^*(z,t^*)$ is a normalized subordination chain, which satisfies the Loewner differential equation

$$\frac{\partial L^*}{\partial t^*}(z,t^*) = DL^*(z,t^*)h^*(z,t^*), \quad \text{a.e. } t^* \geq 0,\ \forall z \in B. \tag{3.9}$$

We next prove that the mapping $L^* = L^*(z,t^*)$ satisfies assumptions of Theorem 2.6. Indeed, since $L(\cdot,t)$ is K-quasiregular for $t \in [0,\infty)$, we easily deduce that

$$\begin{aligned}\left\|DL^*(z,t^*)\right\|^n &= \left\|DL(e^{-i\theta(t)}z,t)\right\|^n \leq K\left|\det DL(e^{-i\theta(t)}z,t)\right| \\ &= K\left|\det DL^*(z,t^*)\right|, \quad z \in B,\ t^* \in [0,\infty),\end{aligned} \tag{3.10}$$

and hence $L^*(z,t^*)$ is also K-quasiregular on B for $t^* \in [0,\infty)$.

Taking into account condition (ii) in the hypothesis, we deduce that

$$\left\|DL^*(z,t^*)\right\| = \left\|DL(e^{-i\theta(t)}z,t)\right\| \leq \frac{Me^{t^*}}{(1-\|z\|)^{\alpha}}, \quad t^* \geq 0,\ z \in B. \tag{3.11}$$

Hence L^* satisfies assumptions (i) and (ii) of Theorem 2.6.

On the other hand, in view of condition (iv), we deduce that

$$\begin{aligned}\Re\langle h^*(z,t^*),z\rangle &= \frac{1}{\Re(a'(t)/a(t))}\Re\left\langle h(ze^{-i\theta(t)},t)e^{i\theta(t)} - i\frac{d\theta(t)}{dt}z, z\right\rangle \\ &= \frac{1}{\Re(a'(t)/a(t))}\Re\langle h(ze^{-i\theta(t)},t), e^{-i\theta(t)}z\rangle \\ &\geq \frac{C\|z\|^2}{\Re(a'(t)/a(t))} \geq \frac{C}{\sup_{t\in[0,\infty)}\Re(a'(t)/a(t))}\|z\|^2.\end{aligned} \tag{3.12}$$

We next prove that

$$\sup_{t\in[0,\infty)} \Re\frac{a'(t)}{a(t)} < \infty. \tag{3.13}$$

Since $\|h(z,t)\| \le C_1$ for $z \in B$ and $t \in [0,\infty)$, it follows by Schwarz's lemma that

$$\|Dh(0,t)\| \le C_1, \quad t \in [0,\infty). \tag{3.14}$$

On the other hand, since $Dh(0,t) = (a'(t)/a(t))I$, we deduce in view of the previous inequality that $|a'(t)/a(t)| \le C_1$ for $t \ge 0$, and hence

$$\sup_{t\in[0,\infty)} \Re \frac{a'(t)}{a(t)} \le C_1 < \infty, \tag{3.15}$$

as claimed.

In view of the above relations, we obtain that

$$\Re\langle h^*(z,t^*), z\rangle \ge \frac{C}{C_1}\|z\|^2, \quad z \in B,\ t^* \ge 0. \tag{3.16}$$

Further, taking into account (3.5), we obtain

$$\begin{aligned} \|h^*(z,t^*)\| &\le \frac{1}{\Re(a'(t)/a(t))}\left[\left\|h(ze^{-i\theta(t)},t)e^{i\theta(t)}\right\| + \left\|\frac{d\theta(t)}{dt}z\right\|\right] \\ &\le \frac{1}{\Re(a'(t)/a(t))}\left[\left\|h(ze^{-i\theta(t)},t)\right\| + \left|\Im\frac{a'(t)}{a(t)}\right|\right] \le \frac{2C_1}{\inf_{t\ge 0}\Re(a'(t)/a(t))} \le \frac{2C_1}{C}. \end{aligned} \tag{3.17}$$

Therefore, we have proved that the mapping $h^*(z,t^*)$ satisfies conditions (iv) and (v) in Theorem 2.6.

Finally, since $L^*(z,0) = L(ze^{-i\theta(0)},0) = L(z,0),\ z \in B$, we conclude that $L(\cdot,0)$ extends to a quasiconformal homeomorphism F of $\mathbb{R}^{2n}$ onto itself such that $F|_B = L(\cdot,0)$, as desired. This completes the proof. □

We next consider the following class of mappings which satisfy the conditions (3.3) and (3.4). The proof of this result may be found in [16].

Remark 3.2. Let $q \in [0,1)$ and let $h = h(z,t) : B \times [0,\infty) \to \mathbb{C}^n$ be given by

$$h(z,t) = \left[I - E(z,t)\right]^{-1}\left[I + E(z,t)\right](z), \tag{3.18}$$

where the mapping $E(z,t)$ satisfies the following conditions.

(i) $E(z,t) \in \mathcal{L}(\mathbb{C}^n),\ z \in B,\ t \in [0,\infty)$.

(ii) $E(\cdot,t) : B \to \mathcal{L}(\mathbb{C}^n)$ is a holomorphic mapping.

(iii) $\|E(z,t)\| \le q$ for $z \in B$ and $t \ge 0$.

Then the mapping $h(z,t)$ satisfies the following inequalities:

$$\begin{aligned} \|z\|\frac{1-q}{1+q} &\le \|h(z,t)\| \le \|z\|\frac{1+q}{1-q}, \quad z \in B,\ t \ge 0; \\ \|z\|^2\frac{1-q}{1+q} &\le \Re\langle h(z,t), z\rangle \le \|z\|^2\frac{1+q}{1-q}, \quad z \in B,\ t \ge 0. \end{aligned} \tag{3.19}$$

4. Applications

In this section we obtain certain applications of Theorem 3.1. The main result of this paper is given in Theorem 4.1, which provides a general quasiconformal extension result in $\mathbb{C}^n$.

Theorem 4.1. *Let $q \in (0,1)$ and let $F = F(u,v) : B \times \mathbb{C}^n \to \mathbb{C}^n$ be a mapping which satisfies the conditions (P) in Definition 2.5. Assume that*

$$\left\| [D_v F(0,0)]^{-1} [D_u F(0,0)] \right\| \leq q, \tag{4.1}$$

$$\|G(z,z)\| \leq q, \quad z \in B \setminus \{0\}, \tag{4.2}$$

$$\left\| G\left(z, \frac{z}{\|z\|^2}\right) \right\| \leq q, \quad z \in B \setminus \{0\}, \tag{4.3}$$

where

$$G(u,v) = \frac{\langle u,v \rangle}{\|v\|^2} [D_v F(u,v)]^{-1} [D_u F(u,v)], \quad u \in B, \ v \in \mathbb{C}^n \setminus \{0\}. \tag{4.4}$$

Moreover, assume that there exist some constants $M > 0$, $K \geq 1$ and $\alpha \in [0,1)$ such that

$$\|D_v F(u,v)\| \leq \frac{M}{(1 - \|u\|)^{\alpha}}, \quad u \in B, \ v \in \mathbb{C}^n, \tag{4.5}$$

$$\|D_v F(u,v)\|^n \leq K \left| \det D_v F(u,v) \right|, \quad u \in B, \ v \in \mathbb{C}^n. \tag{4.6}$$

Then the mapping $f : B \to \mathbb{C}^n$, given by $f(z) = F(z,z)$, extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. We prove that the mapping $L : B \times [0,\infty) \to \mathbb{C}^n$ given by

$$L(z,t) = F\left(e^{-t} z, e^t z\right), \quad z \in B, \ t \geq 0, \tag{4.7}$$

satisfies the conditions of Theorem 3.1.

Indeed, it is obvious that $L(\cdot,t) \in \mathcal{H}(B)$, $L(0,t) = F(0,0) = 0$, $DL(0,t) = e^{-t} D_u F(0,0) + e^t D_v F(0,0) = a(t) I$, where $a(\cdot) \in C^1([0,\infty))$ and $a(0) = 1$. Since the mapping $F = F(u,v)$ is of class C^1 on $B \times \mathbb{C}^n$, it follows that $L(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B$. In view of (4.7), we obtain that

$$\begin{aligned} DL(z,t) &= e^{-t} D_u F\left(e^{-t} z, e^t z\right) + e^t D_v F\left(e^{-t} z, e^t z\right) \\ &= e^t D_v F\left(e^{-t} z, e^t z\right) \left\{ I + e^{-2t} \left[D_v F\left(e^{-t} z, e^t z\right) \right]^{-1} D_u F\left(e^{-t} z, e^t z\right) \right\} \\ &= e^t D_v F\left(e^{-t} z, e^t z\right) [I - E(z,t)], \end{aligned} \tag{4.8}$$

where for each fixed $(z,t) \in B \times [0,\infty)$, $E(z,t)$ is the linear operator defined by

$$E(z,t) = -e^{-2t} \left[D_v F\left(e^{-t} z, e^t z\right) \right]^{-1} \left[D_u F\left(e^{-t} z, e^t z\right) \right]. \tag{4.9}$$

It is easy to see that $E(z,t) = -G(e^{-t}z, e^t z)$ for $z \in B\setminus\{0\}$ and $t \geq 0$. Then

$$\|E(0,t)\| = e^{-2t}\| [D_v F(0,0)]^{-1} [D_u F(0,0)] \| \leq q, \quad t \geq 0, \tag{4.10}$$

by (4.1). Also

$$\|E(z,0)\| = \|G(z,z)\| \leq q, \quad z \in B\setminus\{0\}, \tag{4.11}$$

by (4.2). Moreover, in view of the weak maximum modulus theorem for holomorphic mappings and relation (4.3) (see also the proof of [15, Theorem 2]), we obtain that

$$\|E(z,t)\| \leq \max_{\|w\|=1} \|E(w,t)\| = \max_{\|w\|=1} \left\| G\left(e^{-t}w, \frac{e^{-t}w}{\|e^{-t}w\|^2} \right) \right\| \leq q, \quad z \in B\setminus\{0\},\ t > 0. \tag{4.12}$$

Hence, taking into account the above relations, we deduce that

$$\|E(z,t)\| \leq q, \quad z \in B,\ t \geq 0. \tag{4.13}$$

On the other hand, using elementary computations, it is not difficult to deduce that

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t) [I - E(z,t)]^{-1} [I + E(z,t)](z), \tag{4.14}$$

and thus $L(z,t)$ satisfies the Loewner differential equation (2.9) with

$$h(z,t) = [I - E(z,t)]^{-1} [I + E(z,t)](z), \quad z \in B,\ t \geq 0. \tag{4.15}$$

Also, in view of (4.13), and (3.19), we deduce that the mapping $h(z,t)$ satisfies relations (3.3) and (3.4) with

$$C = \frac{1-q}{1+q}, \qquad C_1 = \frac{1+q}{1-q}. \tag{4.16}$$

We now prove that $\lim_{t\to\infty}|a(t)| = \infty$. Indeed, since

$$a(t)I = DL(0,t) = e^t D_v F(0,0) [I - E(0,t)], \tag{4.17}$$

it follows that

$$a(t) [I - E(0,t)]^{-1} = e^t D_v F(0,0). \tag{4.18}$$

Further, since

$$\| [I - E(0,t)]^{-1} \| \leq (1 - \|E(0,t)\|)^{-1} \leq \frac{1}{1-q}, \quad t \geq 0, \tag{4.19}$$

we obtain in view of the above relations that

$$|a(t)| \geq (1-q) \|D_v F(0,0)\| e^t. \tag{4.20}$$

Thus $\lim_{t\to\infty}|a(t)| = \infty$, as desired.

Now, we prove that $L(\cdot,t)$ is K^*-quasiregular for $t \geq 0$, where K^* is a positive constant. Indeed, in view of (4.6), we obtain that

$$\begin{aligned}
\|DL(z,t)\|^n &\leq e^{nt}\|D_vF(e^{-t}z,e^tz)\|^n\|I-E(z,t)\|^n \\
&\leq e^{nt}(1+q)^n\|D_vF(e^{-t}z,e^tz)\|^n \\
&\leq e^{nt}(1+q)^nK\left|\det D_vF(e^{-t}z,e^tz)\right| \\
&= (1+q)^nK\frac{\left|\det DL(z,t)\right|}{\left|\det[I-E(z,t)]\right|} \\
&\leq \left(\frac{1+q}{1-q}\right)^nK\left|\det DL(z,t)\right|, \quad z\in B,\ t\in[0,\infty).
\end{aligned} \tag{4.21}$$

Hence $L(\cdot,t)$ is K^*-quasiregular for $t \geq 0$, where $K^* = K(1+q)^n/(1-q)^n$.

It remains to prove relations (3.1) and (3.2). Clearly, (3.2) is a direct consequence of condition (iv) in Definition 2.5. On the other hand, taking into account (4.5), we obtain

$$\begin{aligned}
\|DL(z,t)\| &\leq |a(t)|\cdot\left\|[I-E(0,t)]^{-1}\right\|\cdot\left\|[D_vF(0,0)]^{-1}\right\|\cdot\|D_vF(e^{-t}z,e^tz)\|\cdot\|I-E(z,t)\| \\
&\leq \frac{1+q}{1-q}|a(t)|\frac{M}{(1-\|z\|)^\alpha}\left\|[D_vF(0,0)]^{-1}\right\| = \frac{M^*|a(t)|}{(1-\|z\|)^\alpha}, \quad z\in B,\ t\geq 0.
\end{aligned} \tag{4.22}$$

Concluding the above arguments, we deduce that the mapping $L(z,t)$ satisfies the assumptions of Theorem 3.1, and thus $f(z) = L(z,0) = F(z,z)$ extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto $\mathbb{R}^{2n}$, as desired. This completes the proof. □

We next obtain some particular cases of Theorem 4.1. The following result, due to Pfaltzgraff [2], is the n-dimensional version of Becker's quasiconformal extension result [1].

Theorem 4.2. *Let $q \in [0,1)$ and let $f : B \to \mathbb{C}^n$ be a normalized quasiregular holomorphic mapping on B. If*

$$(1-\|z\|^2)\left\|[Df(z)]^{-1}D^2f(z)(z,\cdot)\right\| \leq q, \quad z\in B, \tag{4.23}$$

then f extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. Let $F : B\times\mathbb{C}^n \to \mathbb{C}^n$ be given by $F(u,v) = f(u) + Df(u)(v-u)$. Then F is of class C^1 on $B\times\mathbb{C}^n$ and $F(0,0) = 0$. Since $f(z) = F(z,z)$, it suffices to prove that the mapping F satisfies the assumptions of Theorem 4.1. First we prove that F satisfies conditions (P). Indeed, the mapping

$$F(e^{-t}z,e^tz) = f(e^{-t}z) + (e^t-e^{-t})Df(e^{-t}z)(z) \tag{4.24}$$

is holomorphic on B for $t \geq 0$. Also, since

$$D_vF(u,v) = Df(u), \qquad D_uF(u,v) = D^2f(u)(v-u,\cdot), \tag{4.25}$$

we deduce that $D_vF(u,v)$ is invertible for all $(u,v) \in B \times \mathbb{C}^n$, and $a(t) = e^t$ for $t \geq 0$. Further calculations yield that $G(z,z) \equiv 0$ and

$$G\left(z, \frac{z}{\|z\|^2}\right) = \left(1 - \|z\|^2\right)\left[Df(z)\right]^{-1} D^2 f(z)(z,\cdot), \quad z \in B\setminus\{0\}. \tag{4.26}$$

Hence, in view of (4.23), we deduce that relations (4.1), (4.2), and (4.3) hold.

It remains to prove relations (4.5) and (4.6). Since $D_vF(u,v) = Df(u)$, it follows by arguments similar to those in the proof of [2, Theorem 2.4] that relations (4.5) and (4.6) are fulfilled. This completes the proof. □

The second particular case of Theorem 4.1 is the n-dimensional version of Ahlfors' and Becker's quasiconformal extension result [8].

Theorem 4.3. *Let $f : B \to \mathbb{C}^n$ be a normalized quasiregular holomorphic mapping on B. If there exist some constants $q \in [0,1)$ and $c \in \mathbb{C}$, $|c| \leq q$, such that*

$$\left\| c\|z\|^2 I + \left(1 - \|z\|^2\right)\left[Df(z)\right]^{-1} D^2 f(z)(z,\cdot)\right\| \leq q, \quad z \in B, \tag{4.27}$$

then f extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. Let $F : B \times \mathbb{C}^n \to \mathbb{C}^n$ be given by

$$F(u,v) = f(u) + \frac{1}{1+c} Df(u)(v-u). \tag{4.28}$$

We next apply arguments similar to those in the proof of Theorem 4.2 to deduce that the mapping F satisfies the assumptions of Theorem 4.1. Indeed, since

$$\begin{gathered} D_vF(u,v) = \frac{1}{1+c} Df(u) \\ D_uF(u,v) = \frac{c}{1+c} Df(u) + \frac{1}{1+c} D^2 f(u)(v-u,\cdot), \end{gathered} \tag{4.29}$$

we obtain that $D_vF(u,v)$ is invertible for all $(u,v) \in B \times \mathbb{C}^n$, and

$$a(t) = \frac{e^{-t} + ce^t}{1+c}, \quad t \geq 0. \tag{4.30}$$

On the other hand, it is not difficult to see that

$$G(u,v) = \frac{\langle u,v\rangle}{\|v\|^2}\left[cI + \left[Df(u)\right]^{-1} D^2 f(u)(v-u,\cdot)\right]. \tag{4.31}$$

Hence $G(z,z) = cI$ for $z \in B$, and

$$G\left(z, \frac{z}{\|z\|^2}\right) = c\|z\|^2 I + \left(1 - \|z\|^2\right)\left[Df(z)\right]^{-1} D^2 f(z)(z,\cdot), \quad z \in B\setminus\{0\}. \tag{4.32}$$

Next, taking into account (4.27), we deduce that the relations (4.1), (4.2), and (4.3) hold.

Finally, using the fact that $D_vF(u,v) = Df(u)/(1+c)$, we obtain the relations (4.5) and (4.6), by using arguments similar to those in [5, 8]. The proof is now complete. □

The following result was obtained by Ren and Ma [18] (see also [6, 9]; compare with [19]).

Theorem 4.4. *Let $f, g : B \to \mathbb{C}^n$ be normalized holomorphic mappings such that g is quasiregular on B. Assume that there exists $q \in [0,1)$ such that*

$$\begin{gathered} \left\| [Dg(z)]^{-1} Df(z) - I \right\| \leq q, \quad z \in B, \\ \left\| \|z\|^2 \left\| \{ [Dg(z)]^{-1} Df(z) - I \} + (1 - \|z\|^2) [Dg(z)]^{-1} D^2 g(z)(z, \cdot) \right\| \leq q, \end{gathered} \tag{4.33}$$

for all $z \in B$. Then f extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. Let $F : B \times \mathbb{C}^n \to \mathbb{C}^n$ be given by

$$F(u, v) = f(u) + Dg(u)(v - u). \tag{4.34}$$

Since

$$\begin{gathered} D_v F(u, v) = Dg(u), \\ D_u F(u, v) = Df(u) - Dg(u) + D^2 g(u)(v - u, \cdot), \end{gathered} \tag{4.35}$$

it follows that $D_v F(u, v)$ is invertible for $(u, v) \in B \times \mathbb{C}^n$, and $a(t) = e^t$ for $t \geq 0$.

On the other hand, straightforward computations yield that

$$G(u, v) = \frac{\langle u, v \rangle}{\|v\|^2} \left[[Dg(u)]^{-1} Df(u) - I + [Dg(u)]^{-1} D^2 g(u)(v - u, \cdot) \right]. \tag{4.36}$$

The previous equality implies that

$$\begin{gathered} G(z, z) = [Dg(z)]^{-1} Df(z) - I, \quad z \in B, \\ G\left(z, \frac{z}{\|z\|^2} \right) = \|z\|^2 [Dg(z)]^{-1} Df(z) - I] + (1 - \|z\|^2) [Dg(z)]^{-1} D^2 g(z)(z, \cdot), \end{gathered} \tag{4.37}$$

for $z \in B \setminus \{0\}$. Next, taking into account (4.33), we deduce that the relations (4.1), (4.2), and (4.3) are fulfilled. Finally, since $D_u F(u, v) = Dg(u)$, we obtain the relations (4.5) and (4.6), by using arguments similar to those in [5, 9, 19]. This completes the proof. $\square$

5. The asymptotical case of Theorem 4.1

Let $F = F(u, v)$ be the mapping which satisfies the assumptions of Definition 2.5. In this section we prove that under certain assumptions the mapping $f(z) = F(z, z)$ can be extended to a quasiconformal homeomorphism of $\overline{\mathbb{R}^{2n}}$ onto itself. To this end, we need the following result due to the authors [14, Theorem 2.2] (cf. [6, Theorem 3.1]).

Lemma 5.1. *Let $a : [0, \eta] \to \mathbb{C}$ be a function of class C^1 such that $a(0) = 1$, $a(t) \neq 0$, and $\Re[a'(t)/a(t)] > 0$ for $t \in [0, \eta]$. Let $h = h(z, t) : B \times [0, \eta] \to \mathbb{C}^n$ be such that $h(\cdot, t) \in \mathcal{N}$, $Dh(0, t) = (a'(t)/a(t))I$ for $t \in [0, \eta]$, and $h(z, \cdot)$ is measurable on $[0, \eta]$ for $z \in B$. Also let $L(z, t) = a(t)z + \cdots$ be a mapping such that $L(\cdot, t) \in H(B)$, $L(0, t) = 0$, $DL(0, t) = a(t)I$, and $L(z, \cdot)$*

is absolutely continuous on $[0, \eta]$ *locally uniformly with respect to* $z \in B$. *Suppose that* $L(z, t)$ *satisfies the differential equation*

$$\frac{\partial L}{\partial t}(z, t) = DL(z, t)h(z, t), \quad \text{a.e. } t \in [0, \eta], \ \forall z \in B. \tag{5.1}$$

Moreover, assume that $L(\cdot, 0)$ *is continuous and injective on* $\overline{B}$. *Also assume that the following conditions hold.*

(i) *There exist some constants* $M > 0$ *and* $k \in [0, 1)$ *such that*

$$\|DL(z, t)\| \leq \frac{M|a(t)|}{(1 - \|z\|)^k}, \quad z \in B, \ t \in [0, \eta]. \tag{5.2}$$

(ii) *There exists a constant* $c_1 > 0$ *such that*

$$\Re\langle h(z, t), z\rangle \geq c_1\|z\|^2, \quad z \in B, \ t \in [0, \eta]. \tag{5.3}$$

(iii) *There exists a constant* $c_2 > 0$ *such that*

$$\|h(z, t)\| \leq c_2, \quad z \in B, \ t \in [0, \eta]. \tag{5.4}$$

(vi) *There exists a constant* $K > 0$ *such that* $f(\cdot, t)$ *is* K*-quasiregular for each* $t \in [0, \eta]$.

Then there exists a quasiconformal homeomorphism F *of* $\overline{\mathbb{R}^{2n}}$ *onto itself such that* $F|_B = L(\cdot, 0)$.

Taking into account Lemma 5.1, we may prove the following asymptotical case of Theorem 4.1:

Theorem 5.2. *Let* $q \in (0, 1)$ *and let* $F = F(u, v)$ *be a mapping which satisfies conditions* (P) *in Definition 2.5. Assume that* $F(z, z)$ *is continuous and injective on* $\overline{B}$. *Also, assume that*

$$\begin{gathered} \|[D_vF(0,0)]^{-1}[D_uF(0,0)]\| \leq q, \\ \|G(z, z)\| \leq q, \quad z \in B \setminus \{0\}, \end{gathered} \tag{5.5}$$

and there exists $r \in (0, 1)$ *such that*

$$\left\|G\left(z, \frac{z}{\|z\|^2}\right)\right\| \leq q, \quad r \leq \|z\| < 1, \tag{5.6}$$

where $G(u, v)$ *is the mapping given by* (4.4). *Moreover, assume that there exist some constants* $M > 0$, $K \geq 1$ *and* $\alpha \in [0, 1)$ *such that conditions* (4.5) *and* (4.6) *hold. Then the mapping* $f(z) = F(z, z)$ *extends to a quasiconformal homeomorphism of* $\overline{\mathbb{R}^{2n}}$ *onto itself.*

Proof. Let $\eta = -\ln r$ and let $L : B \times [0,\eta] \to \mathbb{C}^n$ be given by

$$L(z,t) = F\left(e^{-t}z, e^{t}z\right), \quad z \in B,\ t \in [0,\eta]. \tag{5.7}$$

We prove that $L(z,t)$ satisfies the assumptions of Lemma 5.1.

Indeed, the differentiability and the local absolute continuity properties of $L(z,t)$ are clear. As in the proof of Theorem 4.1, let $E(z,t)$ be the linear operator

$$E(z,t) = -e^{-2t}\left[D_v F\left(e^{-t}z, e^{t}z\right)\right]^{-1} D_u F\left(e^{-t}z, e^{t}z\right), \quad z \in B,\ t \ge 0. \tag{5.8}$$

Then $E(z,t) = -G(e^{-t}z, e^{t}z)$ for $z \in B \setminus \{0\}$ and $t \ge 0$. Hence

$$\left\|E(z,0)\right\| \le q, \quad z \in B, \tag{5.9}$$

by (5.5). Moreover, using the weak maximum modulus theorem for holomorphic mappings and condition (5.6), we obtain that

$$\left\|E(z,t)\right\| \le \max_{\|w\|=1}\left\|E(w,t)\right\| \le q, \quad z \in B,\ t \in (0,\eta]. \tag{5.10}$$

Therefore

$$\left\|E(z,t)\right\| \le q, \quad z \in B,\ t \in [0,\eta]. \tag{5.11}$$

On the other hand, if

$$h(z,t) = \left[I - E(z,t)\right]^{-1}\left[I + E(z,t)\right](z), \quad z \in B,\ t \in [0,\eta], \tag{5.12}$$

then

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t)h(z,t), \quad \text{a.e. } t \in [0,\eta],\ \forall z \in B. \tag{5.13}$$

Finally, it suffices to apply similar arguments as in the proof of Theorem 4.1 to deduce that the assumptions of Lemma 5.1 hold. □

We next obtain the following particular cases of Theorem 5.2. The first result is the asymptotical case of Theorem 4.2. This result was obtained by Hamada and Kohr [6]. In the case of one complex variable, see [20, Satz 4].

Corollary 5.3. *Let $f : \overline{B} \to \mathbb{C}^n$ be a normalized quasiregular holomorphic mapping on B and continuous and injective on $\overline{B}$. If*

$$\limsup_{\|z\|\to 1-0}\left(1 - \|z\|^2\right)\left\|\left[Df(z)\right]^{-1}D^2 f(z)(z,\cdot)\right\| < 1, \tag{5.14}$$

then f extends to a quasiconformal homeomorphism of $\overline{\mathbb{R}^{2n}}$ onto itself.

Proof. It suffices to apply arguments similar to those in the proof of Theorem 4.2 to show that the mapping $F(u,v) = f(u) + Df(u)(v-u)$ satisfies the assumptions of Theorem 5.2. □

Remark 5.4. In view of condition (5.14), we have (compare [3, Theorem 2.4])

$$\left(1-\|z\|^2\right)\left\|\left[Df(z)\right]^{-1}D^2f(z)(z,\cdot)\right\|\le q,\qquad r\le\|z\|<1,\tag{5.15}$$

for some $r\in(0,1)$ and $q\in[0,1)$.

The second result due to the authors [14] may be considered the asymptotical case of the n-dimensional version of Ahlfors' and Becker's quasiconformal extension result [20].

Corollary 5.5. *Let $f:\overline{B}\to\mathbb{C}^n$ be a normalized quasiregular holomorphic mapping on B and continuous and injective on $\overline{B}$. If there exist some constants $q\in[0,1)$, $c\in\mathbb{C}$, $|c|\le q$, and $r\in(0,1)$ such that*

$$\left\|c\|z\|^2I+\left(1-\|z\|^2\right)\left[Df(z)\right]^{-1}D^2f(z)(z,\cdot)\right\|\le q,\qquad r\le\|z\|<1,\tag{5.16}$$

then f extends to a quasiconformal homeomorphism of $\overline{\mathbb{R}^{2n}}$ onto itself.

Proof. It suffices to apply arguments similar to those in the proof of Theorem 4.3 to show that the mapping $F(u,v)=f(u)+(1/(1+c))Df(u)(v-u)$ satisfies the assumptions of Theorem 5.2. □

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