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Research Article

New Results of a Class of Two-Neuron Networks with Time-Varying Delays

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With the help of the continuation theorem of the coincidence degree, a priori estimates, and differential inequalities, we make a further investigation of a class of planar systems, which is generalization of some existing neural networks under a time-varying environment. Without assuming the smoothness, monotonicity, and boundedness of the activation functions, a set of sufficient conditions is given for checking the existence of periodic solution and global exponential stability for such neural networks. The obtained results extend and improve some earlier publications.

1. Introduction

Neural networks are complex and large-scale nonlinear dynamics, while the dynamics of the delayed neural network are even richer and more complicated [1]. To obtain a deep and clear understanding of the dynamics of neural networks, one of the usual ways is to investigate the delayed neural network models with two neurons, which can be described by differential systems (see [2–8]). It is hoped that, through discussing the dynamics of two-neuron networks, we can get some light for our understanding about the large networks. In [7], Táboas considered the system of delay differential equations

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + \alpha f_1\big(x_1(t-\tau), x_2(t-\tau)\big),\\ \dot{x}_2(t) &= -x_2(t) + \alpha f_2\big(x_1(t-\tau), x_2(t-\tau)\big), \end{aligned} \tag{1.1}$$

which arises as a model for a network of two saturating amplifiers (or neurons) with delayed outputs, where $\alpha > 0$ is a constant, f_1 and f_2 are bounded C^3 functions on R^2 satisfying

$$\frac{\partial f_1}{\partial x_2}(0,0) \neq 0, \qquad \frac{\partial f_2}{\partial x_1}(0,0) \neq 0, \tag{1.2}$$

and the negative feedback conditions: $x_2 f_1(x_1, x_2) > 0$, $x_2 \neq 0$; $x_1 f_2(x_1, x_2) < 0$, $x_1 \neq 0$. Táboas showed that there is an $\alpha_0 > 0$ such that for $\alpha > \alpha_0$, there exists a nonconstant periodic solution with period greater than 4. Further study on the global existence of periodic solutions to system (1.1) can be found in [2, 3]. All together there is only one delay appearing in both equations. Ruan and Wei [6] investigated the existence of nonconstant periodic solutions of the following planar system with two delays

$$\begin{aligned} \dot{x}_1(t) &= -a_0 x_1(t) + a_1 f_1\big(x_1(t-\tau_1), x_2(t-\tau_2)\big), \\ \dot{x}_2(t) &= -b_0 x_2(t) + b_1 f_2\big(x_1(t-\tau_1), x_2(t-\tau_2)\big), \end{aligned} \tag{1.3}$$

where $a_0 > 0$, $b_0 > 0$, a_1 and b_1 are constants, the functions f_1 and f_2 satisfy $f_j \in C^3(R^2)$, $f_j(0,0) = 0$, $(\partial f_j/\partial x_j)(0,0) = 0$, $j = 1,2$; $x_2 f_1(x_1, x_2) \neq 0$ for $x_2 \neq 0$; $x_1 f_2(x_1, x_2) \neq 0$ for $x_1 \neq 0$; $(\partial f_1/\partial x_2)(0,0) \neq 0$, $(\partial f_2/\partial x_1)(0,0) \neq 0$.

Recently, Chen and Wu [9] considered the following system:

$$\begin{aligned} \dot{x}(t) &= -\mu_1 x(t) + F\big(y(t-\tau_1)\big) + I_1, \\ \dot{y}(t) &= -\mu_2 y(t) - G\big(x(t-\tau_2)\big) + I_2, \end{aligned} \tag{1.4}$$

where μ_1 and μ_2 are positive constants, τ_1 and τ_2 are nonnegative constants with $\tau := \tau_1 + \tau_2 > 0$, I_1 and I_2 are constants, and F and G are bounded C^1-functions with $\dot{F}(t) > 0$, and $\dot{G}(t) > 0$ for $t \in R$. By discussing a two-dimensional unstable manifold of the transformation of the system (1.4), they got some results about slowly oscillating periodic solutions.

However, delays considered in all above systems (1.1)–(1.4) are constants. It is well known that the delays in artificial neural networks are usually time-varying, and they sometime vary violently with time due to the finite switching speed of amplifiers and faults in the electrical circuit. They slow down the transmission rate and tend to introduce some degree of instability in circuits. Therefore, fast response must be required in practical artificial neural-network designs. As pointed out by Gopalsamy and Sariyasa [10, 11], it would be of great interest to study the neural networks in periodic environments. On the other hand, drop the assumptions of continuous first derivative, monotonicity, and boundedness for the activation might be better candidates for some purposes, (see [12]). Motivated by above mentioned, in this paper, we continue to consider the following planar system:

$$\begin{aligned} \dot{x}_1(t) &= -a_1(t) x_1(t) + b_1(t) f_1\big(x_1(t-\tau_{11}(t)), x_2(t-\tau_{12}(t))\big) + I_1(t), \\ \dot{x}_2(t) &= -a_2(t) x_2(t) + b_2(t) f_2\big(x_1(t-\tau_{21}(t)), x_2(t-\tau_{22}(t))\big) + I_2(t), \end{aligned} \tag{1.5}$$

where $a_i(t) \in C(R, (0,\infty))$, $b_i(t), I_i(t) \in C(R, R)$, $i = 1,2$, are periodic with a common period $\omega(>0)$, $f_i(\cdot,\cdot) \in C(R^2, R)$, $\tau_{ij}(t) \in C(R, [0,\infty))$, $i,j = 1,2$ being ω-periodic.

Under the help of the continuation theorem of the coincidence degree, a priori estimates, and differential inequalities, we make a further investigation of system (1.5). Without assuming the smoothness, monotonicity, and boundedness of the activation functions, a family of sufficient conditions are given for checking the existence of periodic solution and global exponential stability for such neural networks. Our results extend and improve some earlier publications. The remainder of this article is organized as follows. In Section 2, the basic notations and assumptions are introduced. After giving the criteria for checking the existence of periodic solution and global exponential stability for the neuron networks in Section 3, one illustrative example and simulations are given in Section 4. We also conclude this paper in Section 5.

2. Preliminaries

In this section, we state some notations, definitions, and lemmas. Assume that nonlinear system (1.5) is supplemented with initial values of the type

$$u_i(t) = \phi_i(t), \quad -\tau \le t \le 0, \quad \tau = \max_{1\le i,j\le 2}\{\tau_{ij}(t)\}, \tag{2.1}$$

in which $\phi_i(t),\ i = 1,2$ are continuous functions. Set $\phi = (\phi_1, \phi_2, \phi_3)^T$, if $x^*(t) = (x_1^*(t), x_2^*(t))^T$ is a solution of system (1.5), then we denote

$$\|\phi - x^*\| = \sum_{i=1}^{2} \sup_{-\tau\le t\le 0} \left|\phi_i(t) - x_i^*(t)\right|. \tag{2.2}$$

Definition 2.1. The solution $x^*(t) = (x_1^*(t), x_2^*(t))^T$ is said to be globally exponentially stable, if there exist $\lambda > 0$ and $m \ge 1$ such that for any solution $x(t) = (x_1(t), x_2(t))^T$ of (1.5), one has

$$\left|x_i(t) - x_i^*(t)\right| \le m\left\|\phi - x^*\right\| e^{-\lambda t} \quad \text{for } t \ge 0, \tag{2.3}$$

where λ is called to be globally exponentially convergent rate.

To establish the main results of the model (1.5), some of the following assumptions will be applied:

(H_1) $|f_i(x_1, x_2)| \le \alpha_i|x_1| + \beta_i|x_2| + M_i$ for all $(x_1, x_2)^T \in R^2,\ i = 1,2$, where $\alpha_i \ge 0,\ \beta_i \ge 0,\ M_i > 0$ are constants;

(H_2) there exist constants $\alpha_i \ge 0,\ \beta_i \ge 0$, such that $|f_j(x_1, x_2) - f_j(y_1, y_2)| \le \alpha_i|x_1 - y_1| + \beta_i|x_2 - y_2|$ for any $(x_1, x_2)^T \in R^2,\ (y_1, y_2)^T \in R^2$.

For $V \in C((a, +\infty), R)$, one denotes

$$\begin{aligned} D^- V(t) &= \limsup_{h\to 0^-} \frac{V(t+h) - V(t)}{h}, \\ D_- V(t) &= \liminf_{h\to 0^-} \frac{V(t+h) - V(t)}{h}. \end{aligned} \tag{2.4}$$

To obtain the existence of the periodic solution of (1.5), we will introduce some results from Gaines and Mawhin [13].

Consider an abstract equation in a Branch space X In this section, we use the coincidence degree theory to obtain the existence of an ω-periodic solution to (1.5). For the sake of convenience, we briefly summarize the theory as below.

Let X and Z be normed spaces, $L : \operatorname{Dom} L \subset X \mapsto Z$ be a linear mapping, and $N : X \mapsto Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \mapsto X$ and $Q : Z \mapsto Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$. It follows that $L|\operatorname{Dom} L \cap \operatorname{Ker} P : (I-P)X \mapsto \operatorname{Im} L$ is invertible. We denote the inverse of this map by K_p. If Ω is a bounded open subset of X, the mapping N is called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \mapsto X$ is compact. Because $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \mapsto \operatorname{Ker} L$.

Let $\Omega \subset R^n$ be open and bounded, $f \in C^1(\Omega, R^n) \cap C(\overline{\Omega}, R^n)$ and $y \in R^n \setminus f(\partial\Omega \cup S_f)$, that is, y is a regular value of f. Here, $S_f = \{x \in \Omega : J_f(x) = 0\}$, the critical set of f, and J_f is the Jacobian of f at x. Then, the degree $\deg\{f, \Omega, y\}$ is defined by

$$\deg\{f, \Omega, y\} = \sum_{x \in f^{-1}(y)} \operatorname{sgn} J_f(x), \tag{2.5}$$

with the agreement that the above sum is zero if $f^{-1}(y) = \varnothing$. For more details about the degree theory, one refers to the book of Deimling [14].

Now, with the above notation, we are ready to state the continuation theorem.

Lemma 2.2 (Continuation theorem [13, page 40]). *Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega}$. Suppose that*

(a) *for each $\lambda \in (0,1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*

(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$ and*

$$\deg\{JQN,\ \Omega \cap \operatorname{Ker} L,\ 0\} \neq 0. \tag{2.6}$$

Then, the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$. For more details about degree theory, we refer to the book by Deimling [14].

For the simplicity of presentation, in the remaining part of this paper, we also introduce the following notation:

$$a_i = \min\{a_i(t)\}, \qquad b_{ij} = \max\{|b_{ij}(t)|\}, \qquad I_i = \max\{|I_i(t)|\},$$
$$D = \begin{vmatrix} a_1 - b_1\alpha_1 & -b_1\beta_1 \\ -b_2\alpha_2 & a_2 - b_2\beta_2 \end{vmatrix}, \tag{2.7}$$
$$D_1 = \begin{vmatrix} I_1 + b_1 M_1 & -b_1\beta_1 \\ I_2 + b_2 M_2 & a_2 - b_2\beta_2 \end{vmatrix}, \qquad D_2 = \begin{vmatrix} a_1 - b_1\alpha_1 & I_1 + b_1 M_1 \\ -b_2\alpha_2 & I_2 + b_2 M_2 \end{vmatrix}.$$

3. Main results

Theorem 3.1. *Suppose* (H_1) *holds and* $D_1/D > 0$, $D_2/D > 0$, *then system* (1.5) *has a* ω*-periodic solution.*

Proof. Take $X = \{u(t) = (x_1(t), x_2(t))^T \in C(R, R^2) : u(t) = u(t+\omega) \text{ for } t \in R\}$ and denote

$$|x_i| = \max_{t\in[0,\omega]} |x_i(t)|, \quad i = 1,2; \quad \|u\|_0 = \max_{i=1,2} |x_i|. \tag{3.1}$$

Equipped with the norm $\|\cdot\|_0$, X is a Banach space. For any $u(t) \in X$, because of the periodicity, it is easy to check that

$$t \longmapsto \begin{pmatrix} -a_1(t)x(t) + b_1(t)f_1\big(x(t-\tau_{11}(t)), y(t-\tau_{12}(t))\big) + I_1(t) \\ -a_2(t)y(t) + b_2(t)f_2\big(x(t-\tau_{21}(t)), y(t-\tau_{22}(t))\big) + I_2(t) \end{pmatrix} \in X. \tag{3.2}$$

Let

$$\begin{gathered} L : \operatorname{Dom} L = \{u \in X : u \in C^1(R, R^2)\} \ni u \longmapsto u' \in X, \\ P : X \ni u \longmapsto \overline{u} \in X, \qquad Q : X \ni x \longmapsto \overline{x} \in X, \end{gathered} \tag{3.3}$$

where for any $\gamma = (\gamma_1, \gamma_2)^T \in R^2$, we identify it as the constant function in X with the value vector $\gamma = (\gamma_1, \gamma_2)^T$. Define $N : X \mapsto X$ given by

$$(Nu)(t) = \begin{pmatrix} -a_1(t)x_1(t) + b_1(t)f_1\big(x_1(t-\tau_{11}(t)), x_2(t-\tau_{12}(t))\big) + I_1(t) \\ -a_2(t)x_2(t) + b_2(t)f_2\big(x_1(t-\tau_{21}(t)), x_2(t-\tau_{22}(t))\big) + I_2(t) \end{pmatrix} \in X. \tag{3.4}$$

Then, system (1.5) can be reduced to the operator equation $Lu = Nu$. It is easy to see that

$$\begin{gathered} \operatorname{Ker} L = R^2, \\ \operatorname{Im} L = \{x \in X : \overline{x} = 0\}, \quad \text{which is closed in } X, \\ \dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 2 < \infty, \end{gathered} \tag{3.5}$$

and P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q). \tag{3.6}$$

It follows that L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \operatorname{Im} L \mapsto \operatorname{Ker} P \cap \operatorname{Dom} L$ is given by

$$(K_p(u))(t) = \begin{pmatrix} \displaystyle\int_0^t x_1(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^s x_1(v)\,dv\,ds \\ \displaystyle\int_0^t x_2(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^s x_2(v)\,dv\,ds \end{pmatrix}. \tag{3.7}$$

Thus,

$$(QNu)(t) = \begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\{-a_1(s)x_1(s) + b_1(s)f_1(x_1(s-\tau_{11}(s)), x_2(s-\tau_{12}(s))) + I_1(s)\}ds \\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\{-a_2(s)x_2(s) + b_2(s)f_2(x_1(s-\tau_{21}(s)), x_2(s-\tau_{22}(s))) + I_2(s)\}ds \end{pmatrix},$$

$$\begin{aligned}
&(K_p(I-Q)Nu)(t) \\
&= \begin{pmatrix} \displaystyle\int_0^{t}\{-a_1(s)x_1(s) + b_1(s)f_1(x_1(s-\tau_{11}(s)), x_2(s-\tau_{12}(s))) + I_1(s)\}ds \\ \displaystyle\int_0^{t}\{-a_2(s)x_2(s) + b_2(s)f_2(x_1(s-\tau_{21}(s)), x_2(s-\tau_{22}(s))) + I_2(s)\}ds \end{pmatrix} \\
&\quad - \begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\int_0^{s}\{-a_1(v)x_1(v) + b_1(v)f_1(x_1(v-\tau_{11}(v)), x_2(v-\tau_{12}(v))) + I_1(v)\}dv\,ds \\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\int_0^{s}\{-a_2(v)x_2(v) + b_2(v)f_2(x_1(v-\tau_{21}(v)), x_2(v-\tau_{22}(v))) + I_2(v)\}dv\,ds \end{pmatrix} \\
&\quad + \begin{pmatrix} \left(\dfrac{1}{2} - \dfrac{t}{\omega}\right)\displaystyle\int_0^{\omega}\{-a_1(s)x_1(s) + b_1(s)f_1(x_1(s-\tau_{11}(s)), x_2(s-\tau_{12}(s))) + I_1(s)\}ds \\ \left(\dfrac{1}{2} - \dfrac{t}{\omega}\right)\displaystyle\int_0^{\omega}\{-a_2(s)x_2(s) + b_2(s)f_2(x_1(s-\tau_{21}(s)), x_2(s-\tau_{22}(s))) + I_2(s)\}ds \end{pmatrix}.
\end{aligned} \tag{3.8}$$

Clearly, QN and $K_p(I-Q)N$ are continuous. For any bounded open subset $\Omega \subset X$, $QN(\overline{\Omega})$ is obviously bounded. Moreover, applying the Arzela-Ascoli theorem, one can easily show that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact. Note that $K_p(I-Q)N$ is a compact operator and $QN(\overline{\Omega})$ is bounded, therefore, N is L-compact on $\overline{\Omega}$ with any bounded open subset $\Omega \subset X$. Since $\operatorname{Im} Q = \operatorname{Ker} L$, we take the isomorphism J of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$ to be the identity mapping. Corresponding to equation $Lu = \lambda Nu$, $\lambda \in (0,1)$, we have

$$\begin{aligned}
\dot{x}_1(t) &= \lambda\{-a_1(t)x_1(t) + b_1(t)f_1(x_1(t-\tau_{11}(t)), x_2(t-\tau_{12}(t))) + I_1(t)\}, \\
\dot{x}_2(t) &= \lambda\{-a_2(t)x_2(t) + b_2(t)f_2(x_1(t-\tau_{21}(t)), x_2(t-\tau_{22}(t))) + I_2(t)\}.
\end{aligned} \tag{3.9}$$

Now we reach the position to search for an appropriate open bounded subset Ω for the application of the Lemma 2.2. Assume that $u = u(t) \in X$ is a solution of system (3.9) for some $\lambda \in (0,1)$. Then, the components $x_i(t)$ $(i = 1,2)$ of $u(t)$ are continuously differentiable. Thus, there exists $t_i \in [0,\omega]$ such that $|x_i(t_i)| = \max_{t\in[0,\omega]}|x_i(t)|$. Hence, $\dot{x}_i(t_i) = 0$. This implies

$$\begin{aligned}
|a_1(t_1)x_1(t_1)| &= |b_1(t_1)f_1(x_1(t_1-\tau_1(t_1)), x_2(t_1-\tau_2(t_1))) + I_1(t_1)|, \\
|a_2(t_2)x_2(t_2)| &= |b_2(t_2)f_2(x_1(t_2-\tau_3(t_2)), x_2(t_2-\tau_4(t_2))) + I_1(t_2)|.
\end{aligned} \tag{3.10}$$

Since

$$\left|f_i(x_1,x_2)\right| \le \alpha_i\left|x_1\right| + \beta_i\left|x_2\right| + M_i \quad \text{for } i = 1,2, \tag{3.11}$$

we get

$$\begin{aligned}\left|x_1(t_1)\right| &\le \frac{\alpha_1 b_1\left|x_1(t_1-\tau_{11}(t_1))\right|}{a_1} + \frac{\beta_1 b_1\left|x_2(t_1-\tau_{12}(t_1))\right|}{a_1} + \frac{b_1M_1+I_1}{a_1},\\ \left|x_2(t_2)\right| &\le \frac{\alpha_2 b_2\left|x_1(t_2-\tau_{21}(t_2))\right|}{a_2} + \frac{\beta_2 b_2\left|x_2(t_2-\tau_{22}(t_2))\right|}{a_2} + \frac{b_2M_2+I_2}{a_2}.\end{aligned} \tag{3.12}$$

Set $k_1 = D_1/D$, $k_2 = D_2/D$, we find that

$$\begin{aligned}k_1 &= \frac{\alpha_1 b_1}{a_1}k_1 + \frac{\beta_1 b_1}{a_1}k_2 + \frac{b_1M_1+I_1}{a_1},\\ k_2 &= \frac{\alpha_2 b_2}{a_2}k_1 + \frac{\beta_2 b_2}{a_2}k_2 + \frac{b_2M_2+I_2}{a_2}.\end{aligned} \tag{3.13}$$

Now, we choose a constant number $d > 1$ and take

$$\Omega = \left\{(x_1,x_2)^T \in R^2;\ \left|x_i\right| < dk_i \text{ for } i = 1,2\right\}, \tag{3.14}$$

where $k_1 = D_1/D > 0$, $k_2 = D_2/D > 0$. We will show that Ω satisfies all the requirements given in Lemma 2.2. In fact, we will prove that if $x_i(t-\tau_{ij}(t)) \in \Omega$ then $x_i(t) \in \Omega$ for $i,j = 1,2,\ t \ge 0$. Therefore, it means that $u = u(t)$ is uniformly bounded with respect to λ when the initial value function belongs to Ω. It follows from (3.12) and (3.13) that

$$\begin{aligned}\left|x_i\right| &= \frac{\alpha_i b_i}{a_i}\left|x_1(t_i-\tau_{i1}(t))\right| + \frac{\beta_i b_i}{a_i}\left|x_1(t_i-\tau_{i2}(t))\right| + \frac{b_iM_i+I_i}{a_i}\\ &\le \frac{\alpha_i b_i}{a_i}dk_1 + \frac{\beta_i b_i}{a_i}dk_2 + \frac{b_iM_i+I_i}{a_i} < d\left(\frac{\alpha_i b_i}{a_i}k_1 + \frac{\beta_i b_i}{a_i}k_2 + \frac{b_iM_i+I_i}{a_i}\right),\end{aligned} \tag{3.15}$$

$i = 1,2$. Therefore,

$$\left|x_i\right|_0 < dk_i \quad \text{for } i = 1,2. \tag{3.16}$$

Clearly, dk_i, $i = 1,2$ are independent of λ. It is easy to see that there are no $\lambda \in (0,1)$ and $u \in \partial\Omega$ such that $Lu = \lambda Nu$. If $u = (x_1,x_2)^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2$, then u is a constant vector in R^2 with $|x_i| = dk_i$ for $i = 1,2$. Note that $QNu = JQNu$, we have

$$QNu = \begin{pmatrix} -x_1\dfrac{1}{\omega}\displaystyle\int_0^{\omega} a_1(t)\mathrm{d}t + f_1(x_1,x_2)\dfrac{1}{\omega}\displaystyle\int_0^{\omega} b_1(t)\mathrm{d}t + \dfrac{1}{\omega}\displaystyle\int_0^{\omega} I_1(t)\mathrm{d}t \\ -x_2\dfrac{1}{\omega}\displaystyle\int_0^{\omega} a_2(t)\mathrm{d}t + f_2(x_1,x_2)\dfrac{1}{\omega}\displaystyle\int_0^{\omega} b_2(t)\mathrm{d}t + \dfrac{1}{\omega}\displaystyle\int_0^{\omega} I_2(t)\mathrm{d}t \end{pmatrix}. \tag{3.17}$$

We claim that

$$\left|(QNu)_i\right| > 0 \quad u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^3,\ i = 1,2. \tag{3.18}$$

Contrarily, suppose that there exists some i such that $|(QNu)_i| = 0$, that is,

$$x_i \frac{1}{\omega}\int_0^\omega a_i(t)\mathrm{d}t = f_i(x_1, x_2)\frac{1}{\omega}\int_0^\omega b_i(t)\mathrm{d}t + \frac{1}{\omega}\int_0^\omega I_i(t)\mathrm{d}t. \tag{3.19}$$

So, we have

$$\begin{aligned}
dk_i = |x_i| &\le \frac{1}{a_i}\left[f_i(x_1, x_2)\frac{1}{\omega}\int_0^\omega b_i(t)\mathrm{d}t + \frac{1}{\omega}\int_0^\omega I_i(t)\mathrm{d}t\right] \\
&\le \frac{\alpha_i b_i}{a_i} dk_1 + \frac{\beta_i b_i}{a_i} dk_2 + \frac{b_i M_i + I_i}{a_i} \\
&< d\left[\frac{\alpha_i b_i}{a_i} k_1 + \frac{\beta_i b_i}{a_i} k_2 + \frac{b_i M_i + I_i}{a_i}\right] = dk_i,
\end{aligned} \tag{3.20}$$

this is a contradiction. Therefore, (3.17) holds, and hence,

$$QNu \neq 0, \quad \text{for any } u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2. \tag{3.21}$$

Now, consider the homotopy $F : (\overline{\Omega} \cap \operatorname{Ker} L) \times [0,1] \to \overline{\Omega} \cap \operatorname{Ker} L$, defined by

$$F(u, \mu) = -\mu \operatorname{diag}\left(\frac{1}{\omega}\int_0^\omega a_1(t)\mathrm{d}t, \frac{1}{\omega}\int_0^\omega a_2(t)\mathrm{d}t\right)u + (1-\mu)QNu, \tag{3.22}$$

where $u \in \overline{\Omega} \cap \operatorname{Ker} L = \overline{\Omega} \cap R^2$ and $\mu \in [0,1]$. When $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2$ and $\mu \in [0,1]$, $u = (x_1, x_2)^T$ is a constant vector in R^2 with $|x_i| = dk_i$ for $i = 1,2$. Thus

$$\|F(u,\mu)\|_0 = \max_{i=1,2}\left| - x_i \frac{1}{\omega}\int_0^\omega a_i(t)\mathrm{d}t + (1-\mu) f_i(x_1, x_2)\frac{1}{\omega}\int_0^\omega b_i(t)\mathrm{d}t\right|. \tag{3.23}$$

We claim that

$$F(u,\mu) \neq 0 \quad (u,\mu) \in (\partial\Omega \cap \operatorname{Ker} L) \times [0,1]. \tag{3.24}$$

Contrarily, suppose that $\|F(u,\mu)\|_0 = 0$, then,

$$x_i \frac{1}{\omega}\int_0^\omega a_i(t)\mathrm{d}t = (1-\mu) f_i(x_1, x_2)\frac{1}{\omega}\int_0^\omega b_i(t)\mathrm{d}t, \quad \text{for } i = 1,2. \tag{3.25}$$

Thus,

$$dk_i = |x_i| \le \frac{(1-\mu)}{a_i} b_i f_i(x_1, x_2) < d\left[\frac{\alpha_i b_i}{a_i} k_1 + \frac{\beta_i b_i}{a_i} k_2 + \frac{b_i M_i + I_i}{a_i}\right] = dk_i. \tag{3.26}$$

This is impossible. Thus, (3.22) holds. From the property of invariance under a homotopy, it follows that

$$\begin{aligned}
&\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \\
&\quad = \deg\{F(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0\} \\
&\quad = \deg\{F(\cdot, 1), \Omega \cap \operatorname{Ker} L, 0\} \\
&\quad = \operatorname{sgn}\begin{vmatrix} -\dfrac{1}{\omega}\displaystyle\int_0^\omega a_1(t)\mathrm{d}t & 0 \\ 0 & -\dfrac{1}{\omega}\displaystyle\int_0^\omega a_2(t)\mathrm{d}t \end{vmatrix} \\
&\quad = \operatorname{sgn}\left\{\frac{1}{\omega}\int_0^\omega a_1(t)\mathrm{d}t \frac{1}{\omega}\int_0^\omega a_2(t)\mathrm{d}t\right\} \ne 0.
\end{aligned} \tag{3.27}$$

We have shown that Ω satisfies all the assumptions of Lemma 2.2. Hence, $Lu = Nu$ has at least one ω-periodic solution on $\operatorname{Dom} L \cap \overline{\Omega}$. This completes the proof. □

Corollary 3.2. *Suppose that there exist positive constants M_i such that $|f_i(x_1, x_2)| \le M_i$ for $i = 1, 2$. Then system* (1.5) *has at least an ω-periodic solution.*

Proof. As $|f_i(x_1, x_2)| \le M_i$ $(i = 1, 2,)$ implies that $\alpha_i, \beta_i = 0$, hence, the conditions in Theorem 2.1 are all satisfied. □

Remark 3.3. From Corollary 3.2, we can find that the condition $\overline{a_i} > \overline{b_i}$ in [4, Theorem 2.1] can be dropped out. Therefore, Theorem 3.1 is greatly generalized results to [4, Theorem 2.1]. Furthermore, we should point out that the Theorem 3.1 is different from the the existing work in [5] when without assuming the boundedness, monotonicity, and differentiability of activation functions. In fact, the explicit presence of ω in Theorem 2.1 (see [5]) may impose a very strict constraint on the coefficients of (1.5) (e.g., when ω is very large or small). Since our results are presented independent of ω, it is more convenient to design a neural network with delays.

Theorem 3.4. *Suppose that f_i, $i = 1, 2$ satisfy the hypotheses* (H_2)*. If $D_1/D > 0$, $D_2/D > 0$, then system* (1.5) *has exactly one periodic solution $\widehat{x}(t)$. Moreover, it is globally exponentially stable.*

Proof. From (H_2), we can conclude (H_1) is true. It is obvious that all the hypotheses in Theorem 2.1 hold with $M_i = |f_i(0, 0)|$, $i = 1, 2$. Thus, system (1.5) has at least one periodic solution, say $\widehat{x}(t)$. Let $x(t)$ be an arbitrary solution of system (1.5). For $t \ge 0$, a direct calculation of the upper left derivative of $|\widehat{x}_i(t) - x_i(t)|$ along the solutions of system (1.5) leads to

$$\begin{aligned}
D^+ \left|\widehat{x}_i(t) - x_i(t)\right| &\\
&= D^- \left\{ \operatorname{sgn}\left(\widehat{x}_i(t) - x_i(t)\right)\left(\widehat{x}_i(t) - x_i(t)\right) \right\} \\
&\le -a_i(t)\left|\widehat{x}_1(t) - x_1(t)\right| + \left|b_i(t)\right| \\
&\quad \times \left\{ \left| f_i\left(\widehat{x}_1(t-\tau_{i1}(t)), \widehat{x}_2(t-\tau_{i2}(t))\right) - f_i\left(x_1(t-\tau_{i1}(t)), x_2(t-\tau_{i2}(t))\right) \right| \right\} \\
&\le -a_i(t)\left|\widehat{x}_1(t) - x_1(t)\right| + \left|b_i(t)\right| \\
&\quad \times \left\{ \alpha_i \left|\widehat{x}_1(t-\tau_{i1}(t)) - x_1(t-\tau_{i1}(t))\right| + \beta_i \left|\widehat{x}_2(t-\tau_{i2}(t)) - x_2(t-\tau_{i2}(t))\right| \right\},
\end{aligned} \tag{3.28}$$

where D^- denotes the upper left derivative, $i = 1, 2$. Let $z_i(t) = |\widehat{x}_i(t) - x_i(t)|$. Then, (3.27) can be transformed into

$$D^- z_i(t) \le -a_i z_i(t) + b_i \alpha_i \sup_{t-\tau\le s\le t} z_1(s) + b_i \beta_i \sup_{t-\tau\le s\le t} z_2(s). \tag{3.29}$$

From $D_1/D > 0$, $D_2/D > 0$ and (3.13), we have

$$\begin{aligned}
k_1 &> \frac{b_1\alpha_1}{a_1} k_1 + \frac{b_1\beta_1}{a_1} k_2, \\
k_2 &> \frac{b_2\alpha_2}{a_2} k_1 + \frac{b_2\beta_2}{a_2} k_2.
\end{aligned} \tag{3.30}$$

Then, there exists a constant $\sigma > 0$ such that

$$\begin{aligned}
-k_1 a_1 + k_1 b_1 \alpha_1 + k_2 b_1 \beta_1 &< \sigma, \\
-k_2 a_2 + k_1 b_2 \alpha_2 + k_2 b_2 \beta_2 &< \sigma.
\end{aligned} \tag{3.31}$$

Thus, we can choose a constant $0 < \lambda \ll 1$ such that

$$\lambda k_i + \left[-k_i a_i + \left(k_1 b_i \alpha_i + k_2 b_i \beta_i\right) e^{\lambda\tau} \right] < 0, \quad i = 1, 2. \tag{3.32}$$

Now, we choose a constant $N \gg 1$ such that

$$N k_i e^{-\lambda t} > 1, \quad \forall t \in [-\tau, 0], \ \ i = 1, 2. \tag{3.33}$$

Set

$$Y_i(t) = N k_i \left[\sup_{-\tau\le s\le 0} z_1(s) + \sup_{-\tau\le s\le 0} z_2(s) + \varepsilon \right] e^{-\lambda t}, \quad i = 1, 2, \tag{3.34}$$

for $\forall\varepsilon > 0$. From (3.32) and (3.34), we obtain

$$
\begin{aligned}
D_{-}Y_i(t) &> \left[-k_i a_i + \left(k_1 b_i \alpha_i + k_2 b_i \beta_i\right)e^{\lambda\tau}\right] N \left[\sum_{j=1}^{2} \sup_{-\tau\le s\le 0} z_j(s) + \varepsilon\right] e^{-\lambda t} \\
&= -a_i Y_i(t) + b_i\alpha_i \sup_{t-\tau\le s\le t} Y_1(s) + b_i\beta_i \sup_{t-\tau\le s\le t} Y_2(s).
\end{aligned}
\tag{3.35}
$$

In view of (3.33) and (3.34), we have

$$
Y_i(t) > \sum_{j=1}^{2} \sup_{-\tau\le s\le 0} z_j(s) + \varepsilon > z_i(t), \quad \forall t \in [-\tau, 0]. \tag{3.36}
$$

We claim that

$$
z_i(t) < Y_i(t), \quad \forall t > 0,\ i = 1, 2. \tag{3.37}
$$

Contrarily, there must exist $i \in \{1, 2\}$ and $t_i > 0$ such that

$$
z_i(t_i) = Y_i(t_i), \quad z_i(t) < Y_i(t) \quad \forall t \in [-\tau, t_i),\ i = 1, 2. \tag{3.38}
$$

It follows that

$$
\begin{aligned}
0 &\le D^{-}\left(z_i(t_i) - Y_i(t_i)\right) \\
&\le \limsup_{h\to 0^-} \frac{z_i(t_i + h) - z_i(t_i)}{h} - \liminf_{h\to 0^-} \frac{Y_i(t_i + h) - Y_i(t_i)}{h} \\
&= D^{-}z_i(t_i) - D_{-}Y_i(t_i).
\end{aligned}
\tag{3.39}
$$

From (3.29), (3.35), and (3.38), we obtain

$$
D^{-}z_i(t_i) \le -a_i z_i(t_i) + b_i\alpha_i \sup_{t_i-\tau\le s\le t_i} z_1(s) + b_i\beta_i \sup_{t_i-\tau\le s\le t_i} z_2(s) < D_{-}Y_i(t_i), \tag{3.40}
$$

which contradicts (3.39). Hence, (3.37) holds. Letting $\varepsilon \to 0^+$ and $m = \max_{1\le i\le 2}\{N k_i + 1\}$, from (3.34) and (3.37), we have

$$
\begin{aligned}
\left|x_i(t) - \hat{x}_i(t)\right| = z_i(t) &\le N k_i \left[\sum_{j=1}^{2} \sup_{-\tau\le s\le 0} z_j(s) + \varepsilon\right] e^{-\lambda t} \\
&\le m\left\|\phi - \hat{x}\right\| e^{-\lambda t}, \quad \text{for } t \ge 0,\ i = 1, 2.
\end{aligned}
\tag{3.41}
$$

This completes the proof. □

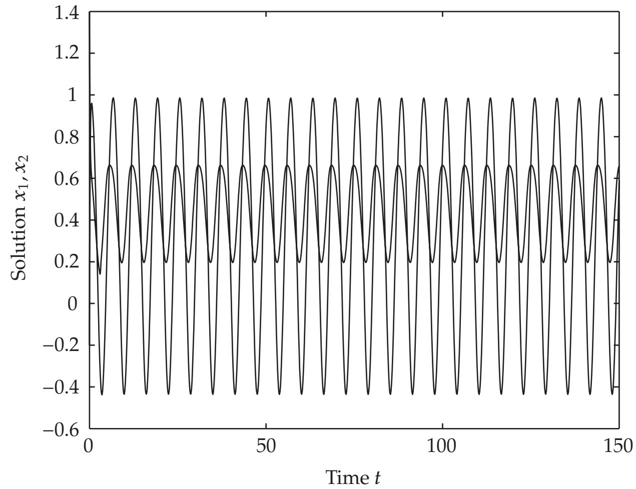

Figure 1: Numerical solution $x_1(t)$, $x_2(t)$ of system (4.1), where $\tau_{11}(t) = 5$, $\tau_{12}(t) = 2$, $\tau_{21}(t) = 3$, $\tau_{22}(t) = 6$, $x_1(s) = -0.2$, $x_2(s) = 1.4$ for $s \in [-6, 0]$.

Remark 3.5. From Theorem 3.4, it is easy to see that our results independent of ω and τ, we have dropped out the condition (b) of Theorem 3.1 in [4] and the condition: $\overline{a_i} > (\alpha_i + \beta_i)\overline{|b_i|}e^{a_i^* \tau}$ of Theorem 3.1 in [5]. Therefore, it is generalized the corresponding results in [4, 5].

4. Example and numerical simulation

In this section, we will give a example to illustrate our results. Using the method of numerical simulation in [15], we will find that the theoretical conclusions are in excellent agreement with the numerically observed behavior.

Example 4.1. Consider a two-neuron networks with delays as follows:

$$\begin{aligned}
\dot{x}_1(t) &= -a_1(t)x_1(t) + b_1(t)f_1\big(x_1(t-\tau_{11}(t)), x_2(t-\tau_{12}(t))\big) + I_1(t), \\
\dot{x}_2(t) &= -a_2(t)x_2(t) + b_2(t)f_2\big(x_1(t-\tau_{21}(t)), x_2(t-\tau_{22}(t))\big) + I_2(t),
\end{aligned} \tag{4.1}$$

where $a_1(t) = 49/8 + \cos t$, $a_2(t) = 15/2 + \sin t$, $b_1(t) = 1 + 2\cos t$, $b_2(t) = 1 - \sin t$, $I_1(t) = \sin t$, $I_2(t) = 2\cos t$, $f_1(x_1, x_2) = \cos(1/4)\, x_1 + (1/8)|x_1| + \sin x_2 - (1/2)|x_2| + 1$, $f_2(x_1, x_2) = \sin x_1 - (1/4)\, x_1 + \cos(1/2)x_2 + (1/4)|x_2| + 2$, and $0 \le \tau_{ij}(t) < +\infty$ are any 2π-periodic continuous functions for $i, j = 1, 2$.

Obviously, the requirements of smoothness, monotonicity, and boundedness on the activation functions are relaxed in our model. By some simple calculations, we obtain $\alpha_1 = 3/8$, $\alpha_2 = 5/4$, $\beta_1 = 3/2$, $\beta_2 = 3/4$, $D = 35/4 > 0$, $D_1 = 47 > 0$, $D_2 = 34 > 0$. Therefore, $D_1/D > 0$, $D_2/D > 0$. Applying Theorem 3.4, there exists a unique 2π-periodic solution for (4.1) and it is globally exponentially stable. These conclusions are verified by the following numerical simulations in Figures 1 and 2.

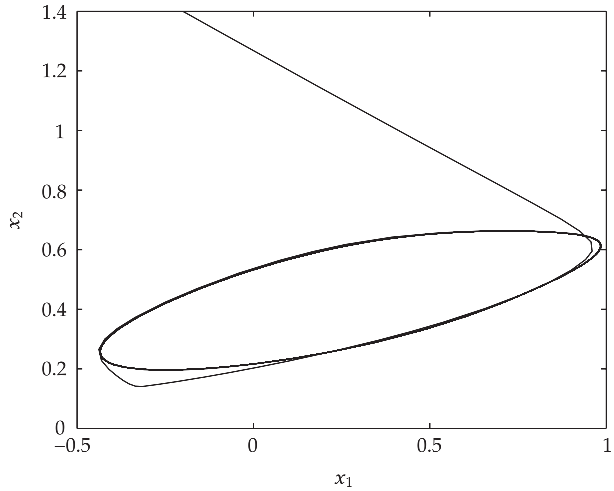

Figure 2: Phase trajectories of system (4.1).

5. Conclusion

In this paper, we have derived sufficient algebraic conditions in terms of the parameters of the connection and activation functions for periodic solutions and global exponential stability of a two-neuron networks with time-varying delays. The obtained results extend and improve some earlier publications, which are all independent of the delays and the period (ω) and may be highly important significance in some applied fields.

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