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Research Article

The Method of Subsuper Solutions for Weighted $p(r)$-Laplacian Equation Boundary Value Problems

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This paper investigates the existence of solutions for weighted $p(r)$-Laplacian ordinary boundary value problems. Our method is based on Leray-Schauder degree. As an application, we give the existence of weak solutions for $p(x)$-Laplacian partial differential equations.

1. Introduction

In this paper, we consider the existence of solutions for the following weighted $p(r)$-Laplacian ordinary equation with right-hand terms depending on the first-order derivative:

$$-\left(w(r)\left|u'\right|^{p(r)-2}u'\right)' + f\left(r,u,(w(r))^{1/(p(r)-1)}u'\right) = 0, \quad \forall r \in (T_1,T_2), \tag{P}$$

with one of the following boundary value conditions:

$$u(T_1) = c, \qquad u(T_2) = d, \tag{1.1}$$

$$g\left(u(T_1),(w(T_1))^{1/(p(T_1)-1)}u'(T_1)\right) = 0, \qquad u(T_2) = d, \tag{1.2}$$

$$g\left(u(T_1),(w(T_1))^{1/(p(T_1)-1)}u'(T_1)\right) = 0, \qquad h\left(u(T_2),(w(T_2))^{1/(p(T_2)-1)}u'(T_2)\right) = 0, \tag{1.3}$$

$$u(T_1) = u(T_2), \qquad w(T_1)\left|u'(T_1)\right|^{p(T_1)-2}u'(T_1) = w(T_2)\left|u'(T_2)\right|^{p(T_2)-2}u'(T_2), \tag{1.4}$$

where $p \in C([T_1,T_2],\mathbb{R})$ and $p(r) > 1$; $w \in C([T_1,T_2],\mathbb{R})$ satisfies $0 < w(r)$, $\forall r \in (T_1,T_2)$, and $(w(r))^{-1/(p(r)-1)} \in L^1(T_1,T_2)$; $-(w(r)|u'|^{p(r)-2}u')'$ is called the weighted $p(r)$-Laplacian; the notation $(w(T_1))^{1/(p(T_1)-1)}u'(T_1)$ means $\lim_{r\to T_1^+}(w(r))^{1/(p(r)-1)}u'(r)$ exists and

$$(w(T_1))^{1/(p(T_1)-1)}u'(T_1) := \lim_{r\to T_1^+}(w(r))^{1/(p(r)-1)}u'(r), \tag{1.5}$$

similarly

$$(w(T_2))^{1/(p(T_2)-1)}u'(T_2) := \lim_{r\to T_2^-}(w(r))^{1/(p(r)-1)}u'(r); \tag{1.6}$$

where $g(x,y)$ and $h(x,y)$ are continuous and increasing in y for any fixed x, respectively.

The study of differential equations and variational problems with nonstandard $p(r)$-growth conditions is a new and interesting topic. Many results have been obtained on these kinds of problem, for example, [1–18]. If $w(r)\equiv p(r)\equiv p$ (a constant), (P) is the well-known p-Laplacian problem. Because of the nonhomogeneity of $p(x)$-Laplacian, $p(x)$-Laplacian problems are more complicated than those of p-Laplacian, many methods and results for p-Laplacian problems are invalid for $p(x)$-Laplacian problems. For example,

(1) if $\Omega\subset\mathbb{R}^n$ is an open bounded domain, then the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega (1/p(x))|\nabla u|^{p(x)}dx}{\int_\Omega (1/p(x))|u|^{p(x)}dx} \tag{1.7}$$

is zero in general, and only under some special conditions $\lambda_{p(x)}>0$ (see [4]), but the fact that $\lambda_p>0$ is very important in the study of p-Laplacian problems. In [19], the author considers the existence and nonexistence of positive weak solution to the following quasilinear elliptic system:

$$\begin{gathered} -\Delta_p u = \lambda f(u,v) = \lambda u^\alpha v^\gamma \ \text{ in } \Omega, \\ -\Delta_q v = \lambda g(u,v) = \lambda u^\delta v^\beta \ \text{ in } \Omega, \\ u = v = 0 \ \text{ on } \partial\Omega, \end{gathered} \tag{S}$$

the first eigenfunction is used to constructing the subsolution of problem (S) successfully. On the $p(x)$-Laplacian problems, maybe $p(x)$-Laplacian does not have the first eigenvalue and the first eigenfunction. Because of the nonhomogeneity of $p(x)$-Laplacian, the first eigenfunction cannot be used to construct the subsolution of $p(x)$-Laplacian problems, even if the first eigenfunction of $p(x)$-Laplacian exists.On the existence of solutions for $p(x)$-Laplacian equations Dirichlet problems via subsuper solution methods, we refer to [13, 14];

(2) if $w(r)\equiv p(r)\equiv p$ (a constant) and $-\Delta_p u>0$, then u is concave, this property is used extensively in the study of one-dimensional p-Laplacian problems, but it is invalid for $-\Delta_{p(r)}$. It is another difference on $-\Delta_p$ and $-\Delta_{p(r)} := -(|u'|^{p(r)-2}u')'$;

(3) on the existence of solutions of the typical $p(r)$-Laplacian problem:

$$-\left(|u'|^{p(r)-2}u'\right)' = |u|^{q(r)-2}u + C, \quad r\in(0,1), \tag{1.8}$$

because of the nonhomogeneity of $p(t)$-Laplacian, when we use critical point theory to deal with the existence of solutions, we usually need the corresponding functional is coercive or satisfy Palais-Smale conditions. If $1\le \max_{r\in[0,1]} q(r) < \min_{r\in[0,1]} p(r)$, then the corresponding functional is coercive, if $\max_{r\in[0,1]} p(r) < \min_{r\in[0,1]} q(r)$, then the corresponding functional satisfies Palais-Smale conditions (see [3]). But if $\min_{r\in[0,1]} p(r) \le q(r) \le \max_{r\in[0,1]} p(r)$, one can see that the corresponding functional is neither coercive nor satisfying Palais-Smale conditions, the results on this case are rare.

There are many papers on the existence of solutions for p-Laplacian boundary value problems via subsuper solution method (see [20–24]). But results on the sub-super-solution method for $p(x)$-Laplacian equations and systems are rare. In this paper, when $p(r)$ is a general function, we establish several sub-super-solution theorems for the existence of solutions for weighted $p(r)$-Laplacian equation with Dirichlet, Robin, and Periodic boundary value conditions. Moreover, the case of $\min_{r\in[0,1]} p(r) \le q(r) \le \max_{r\in[0,1]} p(r)$ is discussed. Our results partially generalize the results of [13, 14, 20, 25].

Let $T_1 < T_2$ and $I = [T_1, T_2]$, the function $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to be Caratheodory, by this we mean the following:

(i) for almost every $t \in I$, the function $f(t,\cdot,\cdot)$ is continuous;

(ii) for each $(x,y) \in \mathbb{R}\times\mathbb{R}$, the function $f(\cdot,x,y)$ is measurable on I;

(iii) for each $\rho > 0$, there is a $\alpha_\rho \in L^1(I,\mathbb{R})$ such that, for almost every $t \in I$ and every $(x,y) \in \mathbb{R}\times\mathbb{R}$ with $|x| \le \rho$, $|y| \le \rho$, one has

$$\left| f(t,x,y) \right| \le \alpha_\rho(t). \tag{1.9}$$

We set $C = C(I,\mathbb{R})$, $C^1 = \{u \in C \mid u'$ is continuous in (T_1,T_2), $\lim_{r\to T_1^+} w(r)|u'|^{p(r)-2}u'(r)$ and $\lim_{r\to T_2^-} w(r)|u'|^{p(r)-2}u'(r)$ exist$\}$. Denote $\|u\|_0 = \sup_{r\in(T_1,T_2)} |u(r)|$ and $\|u\|_1 = \|u\|_0 + \|(w(r))^{1/(p(r)-1)}u'\|_0$. The spaces C and C^1 will be equipped with the norm $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively.

We say a function $u : I \to \mathbb{R}$ is a solution of (P), if $u \in C^1$ and $w(r)|u'|^{p(r)-2}u'(r)$ is absolutely continuous and satisfies (P) almost every on I.

Functions $\alpha, \beta \in C^1$ are called subsolution and supersolution of (P), if $|\alpha'|^{p(r)-2}\alpha'(r)$ and $|\beta'|^{p(r)-2}\beta'(r)$ are absolutely continuous and satisfy

$$\begin{aligned} &-\left(w(r)\left|\alpha'\right|^{p(r)-2}\alpha'\right)' + f\left(r,\alpha,(w(r))^{1/(p(r)-1)}\alpha'\right) \le 0, \quad \text{a.e. on } I, \\ &-\left(w(r)\left|\beta'\right|^{p(r)-2}\beta'\right)' + f\left(r,\beta,(w(r))^{1/(p(r)-1)}\beta'\right) \ge 0, \quad \text{a.e. on } I. \end{aligned} \tag{1.10}$$

Throughout this paper, we assume that $\alpha \le \beta$ are subsolution and supersolution, respectively. Denote

$$\begin{aligned} &\Omega_0 = \left\{(t,x) \mid t \in I,\ x \in \left[\alpha(t),\beta(t)\right]\right\}, \\ &\Omega_1 = \left\{(t,x,y) \mid t \in I,\ x \in \left[\alpha(t),\beta(t)\right],\ y \in \mathbb{R}\right\}. \end{aligned} \tag{1.11}$$

We also assume that

(H_1) $|f(t,x,y)| \le A_1(t,x)K_1(t,x,y) + A_2(t,x)K_2(t,x,y)$, for all $(t,x,y) \in \Omega_1$, where $A_i(t,x)$ $(i = 1,2)$ are positive value and continuous on Ω_0, $K_i(t,x,y)$ $(i = 1,2)$ are positive value and continuous on Ω_1.

(H_2) There exist positive numbers M_1 and M_2 such that $K_1(t,x,y) \le |y|\phi(|y|)$, $K_2(t,x,y) \le M_1\phi(|y|)$, for $|y| \ge M_2$, where $\phi \in C([1,+\infty),[1,+\infty))$ is increasing and satisfies $\int_1^{+\infty} (1/\phi(y^{1/(p^- -1)}))dy = \infty$, where $p^- = \min_{r\in I} p(r)$.

Our main results are as the following theorem.

Theorem 1.1. *If f is Caratheodory and satisfies (H_1) and (H_2), α and β satisfy $\alpha(T_1) \leq c \leq \beta(T_1)$, $\alpha(T_2) \leq d \leq \beta(T_2)$, then* (P) *with* (1.1) *possesses a solution.*

Theorem 1.2. *If f is Caratheodory and satisfies (H_1) and (H_2), α and β satisfy $\alpha(T_2) \leq d \leq \beta(T_2)$, and*

$$g\left(\alpha(T_1), (w(T_1))^{1/(p(T_1)-1)}\alpha'(T_1)\right) \geq 0 \geq g\left(\beta(T_1), (w(T_1))^{1/(p(T_1)-1)}\beta'(T_1)\right), \tag{1.12}$$

then (P) *with* (1.2) *possesses a solution.*

Theorem 1.3. *If f is Caratheodory and satisfies (H_1) and (H_2), α and β satisfy*

$$\begin{aligned} &g\left(\alpha(T_1), (w(T_1))^{1/(p(T_1)-1)}\alpha'(T_1)\right) \geq 0 \geq g\left(\beta(T_1), (w(T_1))^{1/(p(T_1)-1)}\beta'(T_1)\right), \\ &h\left(\alpha(T_2), (w(T_2))^{1/(p(T_2)-1)}\alpha'(T_2)\right) \leq 0 \leq h\left(\beta(T_2), (w(T_2))^{1/(p(T_2)-1)}\beta'(T_2)\right), \end{aligned} \tag{1.13}$$

then (P) *with* (1.3) *possesses a solution.*

Theorem 1.4. *If f is Caratheodory and satisfies (H_1) and (H_2), α and β satisfy*

$$\begin{gathered} \alpha(T_1) = \alpha(T_2) < \beta(T_1) = \beta(T_2), \\ w(T_1)\left|\alpha'(T_1)\right|^{p(T_1)-2}\alpha'(T_1) \geq w(T_2)\left|\alpha'(T_2)\right|^{p(T_2)-2}\alpha'(T_2), \\ w(T_1)\left|\beta'(T_1)\right|^{p(T_1)-2}\beta'(T_1) \leq w(T_2)\left|\beta'(T_2)\right|^{p(T_2)-2}\beta'(T_2), \end{gathered} \tag{1.14}$$

then (P) *with* (1.4) *possesses a solution.*

As an application, we consider the existence of weak solutions for the following $p(x)$-Laplacian partial differential equation:

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + f\left(x, u, |x|^{(n-1)/(p(x)-1)}|\nabla u|\right) = 0, \quad \forall x \in \Omega, \tag{1.15}$$

where Ω is a bounded symmetric domain in $\mathbb{R}^n$, $p \in C(\overline{\Omega}; \mathbb{R})$ is radially symmetric. We will write $p(x) = p(|x|) = p(r)$, and $p(r)$ satisfies $1 < p(r) \in C$, $f \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is radially symmetric with respect to x, namely, $f(x, u, v) = f(|x|, u, v) = f(r, u, v)$, and f satisfies the Caratheodory condition.

2. Preliminary

Denote $\varphi(r, x) = |x|^{p(r)-2}x$, $\forall (r, x) \in I \times \mathbb{R}$. Obviously, φ has the following properties.

Lemma 2.1. *φ is a continuous function and satisfies*

(i) *for any $r \in [T_1, T_2]$, $\varphi(r, \cdot)$ is strictly increasing;*

(ii) *$\varphi(r, \cdot)$ is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ for any fixed $r \in I$.*

For any fixed $r \in I$, denote $\varphi^{-1}(r,\cdot)$ as

$$\varphi^{-1}(r,x) = |x|^{(2-p(r))/(p(r)-1)}x, \quad \text{for } x \in \mathbb{R} \setminus \{0\},\ \varphi^{-1}(r,0) = 0. \tag{2.1}$$

It is clear that $\varphi^{-1}(r,\cdot)$ is continuous and send bounded sets into bounded sets.

Let us now consider the simple problem

$$\left(w(r)\varphi(r,u'(r))\right)' = f(r), \tag{2.2}$$

with boundary value condition (1.1), where $f \in L^1$. If u is a solution of (2.2) with (1.1), by integrating (2.2) from T_1 to r, we find that

$$w(r)\varphi(r,u'(r)) = w(T_1)\varphi(T_1,u'(T_1)) + \int_{T_1}^{r} f(t)\,dt. \tag{2.3}$$

Denote

$$F(f)(r) = \int_{T_1}^{r} f(t)\,dt, \qquad a = w(T_1)\varphi(T_1,u'(T_1)), \tag{2.4}$$

then

$$u(r) = u(T_1) + \int_{T_1}^{r} \varphi^{-1}\left[r,(w(r))^{-1}(a+F(f)(r))\right]dr. \tag{2.5}$$

The boundary conditions imply that

$$\int_{T_1}^{T_2} \varphi^{-1}\left[r,(w(r))^{-1}(a+F(f)(r))\right]dr = d-c. \tag{2.6}$$

For fixed $h \in C$, we denote

$$\Lambda_h(a) = \int_{T_1}^{T_2} \varphi^{-1}\left[r,(w(r))^{-1}(a+h(r))\right]dr + c - d. \tag{2.7}$$

We have the following lemma.

Lemma 2.2. *The function Λ_h has the following properties.* (i) *For any fixed $h \in C$, the equation*

$$\Lambda_h(a) = 0 \tag{2.8}$$

has a unique solution $\widetilde{a}(h) \in \mathbb{R}$.

(ii) *The function $\widetilde{a} : C \to \mathbb{R}$, defined in (i), is continuous and sends bounded sets to bounded sets.*

Proof. (i) Obviously, for any fixed $h \in C$, $\Lambda_h(\cdot)$ is continuous and strictly increasing, then, if (2.8) has a solution, it is unique.

Since $(w(r))^{-1/(p(r)-1)} \in L^1(T_1,T_2)$ and $h \in C$, it is easy to see that

$$\lim_{a\to+\infty} \Lambda_h(a) = +\infty, \qquad \lim_{a\to-\infty} \Lambda_h(a) = -\infty. \tag{2.9}$$

It means the existence of solutions of $\Lambda_h(a) = 0$.

In this way, we define a function $\widetilde{a}(h) : C[T_1, T_2] \to \mathbb{R}$, which satisfies

$$\int_{T_1}^{T_2} \varphi^{-1}\left[r, (w(r))^{-1}\left(\widetilde{a}(h) + h(r)\right)\right] dr = 0. \tag{2.10}$$

(ii) We claim that

$$\left|\widetilde{a}(h)\right| \le \left\{ \frac{|c-d|}{\int_{T_1}^{T_2} \varphi^{-1}\left[r, (w(r))^{-1}\right] dr} + 1 \right\}^{p^+ + 1} + \|h\|_0, \quad \forall h \in C. \tag{2.11}$$

If it is false. Without loss of generality, we may assume that there are some $h \in C$ such that

$$\widetilde{a}(h) > \left\{ \frac{|c-d|}{\int_{T_1}^{T_2} \varphi^{-1}\left[r, (w(r))^{-1}\right] dr} + 1 \right\}^{p^+ + 1} + \|h\|_0, \tag{2.12}$$

then

$$\begin{gathered}
\widetilde{a}(h) + h > \left\{ \frac{|c-d|}{\int_{T_1}^{T_2} \varphi^{-1}\left[r, (w(r))^{-1}\right] dr} + 1 \right\}^{p^+ + 1}, \\
\begin{aligned}
&\int_{T_1}^{T_2} \varphi^{-1}\left[r, (w(r))^{-1}\left(\widetilde{a}(h) + h(r)\right)\right] dr + d - c \\
&\quad > \left\{ \frac{|c-d|}{\int_{T_1}^{T_2} \varphi^{-1}\left[r, (w(r))^{-1}\right] dr} + 1 \right\} \int_{T_1}^{T_2} \varphi^{-1}\left[r, (w(r))^{-1}\right] dr + d - c \\
&\quad = |c-d| + \int_{T_1}^{T_2} \varphi^{-1}\left[r, (w(r))^{-1}\right] dr + d - c \\
&\quad > 0.
\end{aligned}
\end{gathered} \tag{2.13}$$

It is a contradiction. Thus, (2.11) is valid. It mens that $\widetilde{a}$ sends bounded sets to bounded sets.

Finally, to show the continuity of $\widetilde{a}$, let $\{u_n\}$ be a convergent sequence in C and $u_n \to u$, as $n \to +\infty$. Obviously, $\{\widetilde{a}(u_n)\}$ is a bounded sequence, then it contains a convergent subsequence $\{\widetilde{a}(u_{n_j})\}$. Let $\widetilde{a}(u_{n_j}) \to a_0$ as $j \to +\infty$. Since

$$\int_{T_1}^{T_2} \varphi^{-1}\left[r, (w(r))^{-1}\left(\widetilde{a}(u_{n_j}) + u_{n_j}(r)\right)\right] dr = 0, \tag{2.14}$$

letting $j \to +\infty$, we have

$$\int_{T_1}^{T_2} \varphi^{-1}\left[r, (w(r))^{-1}\left(a_0 + u(r)\right)\right] dr = 0, \tag{2.15}$$

from (i), we get $a_0 = \widetilde{a}(u)$, it means $\widetilde{a}$ is continuous.

This completes the proof. $\square$

Now, we define $a : L^1 \to \mathbb{R}$ is defined by

$$a(h) = \widetilde{a}(F(h)). \tag{2.16}$$

It is clear that a is a continuous function which send bounded sets of L^1 into bounded sets of $\mathbb{R}$, and hence it is a complete continuous mapping.

We continue now with our argument previous to Lemma 2.2. By solving for u' in (2.3) and integrating, we find

$$u(r) = u(T_1) + F\{\varphi^{-1}[r, (w(r))^{-1}(a(f) + F(f)(r))]\}(r). \tag{2.17}$$

Let us define

$$K(h)(t) = F\{\varphi^{-1}[r, (w(r))^{-1}(a(h) + F(h))]\}(t), \quad \forall t \in [T_1, T_2]. \tag{2.18}$$

We denote by $N_f(u) : C^1 \times [T_1, T_2] \to L^1$, the Nemytsky operator associated to f defined by

$$N_f(u)(r) = f(r, u(r), (w(r))^{1/(p(r)-1)} u'(r)), \quad \text{a.e. on } I. \tag{2.19}$$

It is easy to see the following lemma.

Lemma 2.3. *u is a solution of* (P) *with boundary value condition* (1.1) *if and only if u is a solution of the following abstract equation:*

$$u = c + K(N_f(u)). \tag{2.20}$$

Lemma 2.4. *The operator K is continuous and sends equi-integrable sets in L^1 into relatively compact sets in C^1.*

Proof. It is easy to check that $K(h)(t) \in C^1$. Since $(w(r))^{-1/(p(r)-1)} \in L^1$, and

$$(w(t))^{1/(p(t)-1)} K(h)'(t) = \varphi^{-1}[t, (a(h) + F(h))], \quad \forall t \in [T_1, T_2], \tag{2.21}$$

it is easy to check that K is a continuous operator from L^1 to C^1.

Let now U be an equi-integrable set in L^1, then there exists $\rho \in L^1$, such that

$$|u(t)| \le \rho(t) \quad \text{a.e. in } I, \text{ for any } u \in U. \tag{2.22}$$

We want to show that $\overline{K(U)} \subset C^1$ is a compact set.

Let $\{u_n\}$ be a sequence in $K(U)$, then there exist a sequence $\{h_n\} \in U$ such that $u_n = K(h_n)$. For $t_1, t_2 \in I$, we have that

$$|F(h_n)(t_1) - F(h_n)(t_2)| \le \left| \int_{t_1}^{t_2} \rho(t) dt \right|. \tag{2.23}$$

Hence, the sequence $\{F(h_n)\}$ is uniformly bounded and equicontinuous, then there exists a subsequence of $\{F(h_n)\}$ which is convergent in C, and we name the same. Since the operator $\widetilde{a}$ is bounded and continuous, we can choose a subsequence of $\{a(h_n) + F(h_n)\}$ (which we still denote $\{a(h_n) + F(h_n)\}$) that is convergent in C, then

$$w(t)\varphi(t, (K(h_n))'(t)) = a(h_n) + F(h_n) \tag{2.24}$$

is convergent in C. Since

$$K(h_n)(t) = F\{(w(r))^{-1/(p(r)-1)} \varphi^{-1}[r, (a(h_n) + F(h_n)]\}(t), \quad \forall t \in [T_1, T_2], \tag{2.25}$$

according to the continuous of φ^{-1} and the integrability of $(w(r))^{-1/(p(r)-1)}$ in L^1, then $K(h_n)$ is convergent in C. Then, we can conclude that $\{u_n\}$ convergent in C^1. □

Lemma 2.5. *Let $\alpha, \beta \in C^1$ be subsolution and supersolution of* (P)*, respectively, which satisfies $\alpha(t) \le \beta(t)$ for any $t \in [T_1, T_2]$, then there exists a positive constant L such that, for any solution x of* (P) *with* (1.1) *whichsatisfies $\alpha(t) \le x(t) \le \beta(t)$,one has $\|(w(t))^{1/(p(t)-1)} x'\|_0 \le L$.*

Proof. We denote

$$\mu_0 = \int_{T_1}^{T_2} [A_1(t, x(t)) + A_2(t, x(t))] dt, \qquad a_0 = \max\left\{ (w(r))^{1/(p(r)-1)} \mid r \in [T_1, T_2] \right\},$$
$$\sigma = \max\{\beta(s) - \alpha(t) \mid t, s \in [T_1, T_2]\},$$
$$\gamma = \max\left\{ (w(t))^{1/(p(t)-1)} A_1(t, x) \mid (t, x) \in \Omega_0 \right\}, \tag{2.26}$$

then there exists a $t_0 \in (T_1, T_2)$ such that

$$\left| (w(t_0))^{1/(p(t_0)-1)} x'(t_0) \right| \le a_0 |x'(t_0)| \le a_0 \frac{\sigma}{T_2 - T_1}. \tag{2.27}$$

From (H_2), there exist positive numbers σ_1 and N_1 such that

$$N_1 \ge \sigma_1 \ge \max_{r \in I} \left(M_2 + a_0 \frac{\sigma}{T_2 - T_1} + 1 \right)^{p(r)},$$
$$\int_{\sigma_1}^{N_1} \frac{1}{\phi(y^{1/(p(r)-1)})} dy > \gamma\sigma + M_1\mu_0, \quad \text{for } r \in [T_1, T_2] \text{ uniformly.} \tag{2.28}$$

Assume that our conclusion is not true, combining (2.27), then there exists $[t_1, t_2] \subset [T_1, T_2]$ such that $(w(r))^{1/(p(r)-1)} x'$ keeps the same sign on $[t_1, t_2]$, and

$$w(t_1) |x'|^{p(t_1)-2} x'(t_1) = \sigma_1, \qquad w(t_2) |x'|^{p(t_2)-2} x'(t_2) = N_1, \tag{2.29}$$

or inversely

$$w(t_1) |x'|^{p(t_1)-2} x'(t_1) = -\sigma_1, \qquad w(t_2) |x'|^{p(t_2)-2} x'(t_2) = -N_1. \tag{2.30}$$

For simplicity, we assume that the former appears. Hence,

$$\begin{aligned} \gamma\sigma + M_1\mu_0 &< \left| \int_{\sigma_1}^{N_1} \frac{1}{\phi(y^{1/(p(r)-1)})} dy \right| \\ &= \left| \int_{t_1}^{t_2} \frac{\left(w(r) |x'|^{p(r)-1} \right)'}{\phi\left(\left(w(r) |x'|^{p(r)-1} \right)^{1/(p(r)-1)} \right)} dr \right| \\ &= \int_{t_1}^{t_2} \left| \frac{f\left(r, x, (w(r))^{1/(p(r)-1)} x' \right)}{\phi\left((w(r))^{1/(p(r)-1)} |x'| \right)} \right| dr \\ &\le \int_{t_1}^{t_2} (w(r))^{1/(p(r)-1)} A_1(r, x(r)) |x'| dr + M_1\mu_0 \\ &\le \gamma\sigma + M_1\mu_0, \end{aligned} \tag{2.31}$$

which is impossible. The proof is completed. □

Let us consider the auxiliary SBVP of the form

$$\left(w(r)\left|u'\right|^{p(r)-2}u'\right)' = f\left(r, R(r,u), R_1\left[(w(r))^{1/(p(r)-1)}u'\right]\right) + R_2(r,u) \overset{\text{def}}{=} \widetilde{f}(r,u), \quad r \in (T_1, T_2), \tag{2.32}$$

where

$$R(t,u) = \begin{cases} \beta(t), & u(t) > \beta(t), \\ u, & \alpha(t) \le u(t) \le \beta(t), \\ \alpha(t), & u(t) < \alpha(t), \end{cases}$$
$$R_1[y] = \begin{cases} L_1, & y > L_1, \\ y, & |y| \le L_1, \\ -L_1, & y < -L_1, \end{cases} \tag{2.33}$$

where

$$L_1 = 1 + \max\left\{L, \sup_{r\in(T_1,T_2)}\left|(w(r))^{1/(p(r)-1)}\beta'(r)\right|, \sup_{r\in(T_1,T_2)}\left|(w(r))^{1/(p(r)-1)}\alpha'(r)\right|\right\}, \tag{2.34}$$

where L is defined in Lemma 2.5, and

$$R_2(t,u) = \begin{cases} e(t,u)\dfrac{u-\beta(t)}{1+u^2} & u(t) > \beta(t), \\ 0, & \alpha(t) \le u(t) \le \beta(t), \\ e(t,u)\dfrac{u-\alpha(t)}{1+u^2} & u(t) < \alpha(t), \end{cases} \tag{2.35}$$

where $e(t,u) = 1 + A_1(t, R(t,u)) + A_2(t, R(t,u))$.

Lemma 2.6. *Let the conditions of Lemma 2.5 hold, and let $u(t)$ be any solution of SBVP with* (1.1) *satisfies $\alpha(T_1) \le c \le \beta(T_1)$ and $\alpha(T_2) \le d \le \beta(T_2)$, then $\alpha(t) \le u(t) \le \beta(t)$, for any $t \in [T_1, T_2]$.*

Proof. We will only prove that $u(t) \le \beta(t)$ for any $t \in [T_1, T_2]$. The argument of the case of $\alpha(t) \le u(t)$ for any $t \in [T_1, T_2]$ is similar.

Assume that $u(t) > \beta(t)$ for some $t \in (T_1, T_2)$, then there exist a $t_0 \in (T_1, T_2)$ and a positive number δ such that $u(t_0) = \beta(t_0) + \delta$, $u(t) \le \beta(t) + \delta$, for any $t \in [T_1, T_2]$. Hence,

$$(w(t_0))^{1/(p(t_0)-1)}u'(t_0) = (w(t_0))^{1/(p(t_0)-1)}\beta'(t_0). \tag{2.36}$$

There exists a positive number η such that $u(t) > \beta(t)$, for any $t \in J := (t_0 - \eta, t_0 + \eta) \subset [T_1, T_2]$. From the definition of $\beta, u,$ and $\widetilde{f}$ we conclude that

$$\left(w(r)\left|\beta'\right|^{p(r)-2}\beta'\right)' \le f\left(r, \beta, (w(r))^{1/(p(r)-1)}\beta'\right) = \widetilde{f}(r,\beta) < \widetilde{f}(r,u) \text{ on } \left[t_0 - \eta_1, t_0 + \eta_1\right], \tag{2.37}$$

where $\eta_1 \in (0,\eta)$ is small enough. For any $r \in (t_0, t_0+\eta_1]$, we have

$$\int_{t_0}^{r} \left(w(r)\left|\beta'\right|^{p(r)-2}\beta'\right)' dr < \int_{t_0}^{r} \widetilde{f}(r,u)dr = \int_{t_0}^{r} \left(w(r)\left|u'\right|^{p(r)-2}u'\right)' dr. \tag{2.38}$$

From (2.36) and (2.38), we have

$$\left|\beta'\right|^{p(r)-2}\beta' < \left|u'\right|^{p(r)-2}u' \text{ on } \left(t_0,\, t_0+\eta_1\right], \tag{2.39}$$

it means that

$$(\beta+\delta)' < u' \text{ on } \left(t_0,\, t_0+\eta_1\right]. \tag{2.40}$$

It is a contradiction to the definition of t_0, so $u(t) \le \beta(t)$, for any $t \in [T_1, T_2]$. □

3. Proofs of main results

In this section, we will deal with the proofs of main results.

Proof of Theorem 1.1. From Lemmas 2.5 and 2.6, we only need to prove the existence of solutions for SBVP with (1.1). Obviously, u is a solution of SBVP with (1.1) if and only if u is a solution of

$$u = \Phi_{\widetilde{f}}(u) := c + K\left(N_{\widetilde{f}}(u)\right). \tag{3.1}$$

We set

$$C^1_{c,d} = \left\{u \in C^1 \mid u\left(T_1\right) = c,\, u\left(T_2\right) = d\right\}. \tag{3.2}$$

Obviously, $N_{\widetilde{f}}(u)$ sends C^1 into equi-integrable sets in L^1. Similar to the proof of Lemma 2.4, we can conclude that K sends equi-integrable sets in L^1 into relatively compact sets in C^1, then $\Phi_{\widetilde{f}}(u)$ is compact continuous.

Obviously, for any $u \in C^1$, we have $\Phi_{\widetilde{f}}(u) \in C^1_{c,d}$, and $\Phi_{\widetilde{f}}(C^1)$ is bounded. By virtue of Schauder fixed point theorem, $\Phi_{\widetilde{f}}(u)$ has at least one fixed point u in $C^1_{c,d}$. Then, u is a solution of SBVP with (1.1). This completes the proof. □

Proof of Theorem 1.2. Let d with $\alpha(T_2) \le d \le \beta(T_2)$ be fixed. According to Theorem 1.1, (P) with the following boundary value condition:

$$u_1\left(T_1\right) = \alpha\left(T_1\right), \qquad u_1\left(T_2\right) = d, \tag{3.3}$$

possesses a solution u_1 such that

$$\alpha(t) \le u_1(t) \le \beta(t), \quad \forall t \in \left[T_1, T_2\right]. \tag{3.4}$$

Since $\lim_{r\to T_1^+} w(r)\left|u_1'\right|^{p(r)-2}u_1'(r)$ exists, we have

$$\begin{aligned} u_1(r) - u_1\left(T_1\right) &= \int_{T_1}^{r} (w(t))^{-1/(p(t)-1)}\left[(w(t))^{1/(p(t)-1)}u_1'(t)\right]dt \\ &= \left(w\left(T_1\right)\right)^{1/(p(T_1)-1)}u_1'\left(T_1\right)\int_{T_1}^{r} (w(t))^{-1/(p(t)-1)}(1+o(1))dt. \end{aligned} \tag{3.5}$$

Similarly,

$$\alpha(r)-\alpha(T_1)=(w(T_1))^{1/(p(T_1)-1)}\alpha'(T_1)\int_{T_1}^{r}(w(t))^{-1/(p(t)-1)}(1+o(1))dt. \tag{3.6}$$

Obviously

$$0\le\lim_{r\to T_1^+}\frac{u_1(r)-\alpha(r)}{\int_{T_1}^{r}(w(t))^{-1/(p(t)-1)}dt}=(w(T_1))^{1/(p(T_1)-1)}u_1'(T_1)-(w(T_1))^{1/(p(T_1)-1)}\alpha'(T_1), \tag{3.7}$$

then, we can conclude that

$$(w(T_1))^{1/(p(T_1)-1)}u_1'(T_1)\ge(w(T_1))^{1/(p(T_1)-1)}\alpha'(T_1). \tag{3.8}$$

Since $u_1(T_1)=\alpha(T_1)$, and $g(x,y)$ is increasing in y, we have

$$g\big(u_1(T_1),(w(T_1))^{1/(p(T_1)-1)}u_1'(T_1)\big)\ge g\big(\alpha(T_1),(w(T_1))^{1/(p(T_1)-1)}\alpha'(T_1)\big)\ge 0. \tag{3.9}$$

We may assume that $g(u_1(T_1),(w(T_1))^{1/(p(T_1)-1)}u_1'(T_1))>0$, or we get a solution for (P) with (1.2).

Since u_1 is a solution of (P), it is also a subsolution of (P). Similarly, (P) with boundary value condition

$$v_1(T_1)=\beta(T_1),\qquad v_1(T_2)=d, \tag{3.10}$$

possesses a solution v_1 such that

$$u_1(t)\le v_1(t)\le\beta(t),\quad\forall t\in[T_1,T_2], \tag{3.11}$$

which satisfies

$$(w(T_1))^{1/(p(T_1)-1)}v_1'(T_1)\le(w(T_1))^{1/(p(T_1)-1)}\beta'(T_1), \tag{3.12}$$

then

$$g\big(v_1(T_1),(w(T_1))^{1/(p(T_1)-1)}v_1'(T_1)\big)\le g\big(\beta(T_1),(w(T_1))^{1/(p(T_1)-1)}\beta'(T_1)\big)\le 0. \tag{3.13}$$

Obviously, $u_1(t)$ and $v_1(t)$ are subsolution and supersolution of (P) with (1.2), respectively. According to Theorem 1.1, (P) with boundary value condition

$$x(T_1)=\frac{u_1(T_1)+v_1(T_1)}{2},\qquad x(T_2)=d, \tag{3.14}$$

possesses a solution x such that

$$u_1(t)\le x(t)\le v_1(t),\quad\forall t\in[T_1,T_2]. \tag{3.15}$$

We may assume that $g(x(T_1),(w(T_1))^{1/(p(T_1)-1)}x'(T_1))\neq 0$, or we get a solution for (P) with (1.2).

If $g(x(T_1),(w(T_1))^{1/(p(T_1)-1)}x'(T_1)) > 0$, then denote $u_2(t) = x(t)$ and $v_2(t) = v_1(t)$; if $g(x(T_1),(w(T_1))^{1/(p(T_1)-1)}x'(T_1)) < 0$, then denote $v_2(t) = x(t)$ and $u_2(t) = u_1(t)$. It is easy to see that $u_2(t)$ and $v_2(t)$ both are solutions of (P) and satisfy

$$\begin{gathered}
g\left(u_2(T_1),(w(T_1))^{1/(p(T_1)-1)}u_2'(T_1)\right) > 0 > g\left(v_2(T_1),(w(T_1))^{1/(p(T_1)-1)}v_2'(T_1)\right), \\
u_2(t) \le v_2(t), \quad \forall t \in [T_1,T_2], \qquad [u_2(t),v_2(t)] \subseteq [u_1(t),v_1(t)], \quad \forall t \in [T_1,T_2], \\
u_2(T_2) = d = v_2(T_2), \\
v_2(T_1) - u_2(T_1) = \frac{v_1(T_1)-u_1(T_1)}{2}.
\end{gathered} \tag{3.16}$$

Repeated the step, we get two sequences $\{u_n\}$ and $\{v_n\}$, all are solutions of (P), and satisfy

$$g\left(u_n(T_1),(w(T_1))^{1/(p(T_1)-1)}u_n'(T_1)\right) > 0 > g\left(v_n(T_1),(w(T_1))^{1/(p(T_1)-1)}v_n'(T_1)\right), \tag{3.17}$$

$$u_n(t) \le v_n(t), \quad \forall t \in [T_1,T_2], \qquad [u_{n+1}(t),v_{n+1}(t)] \subseteq [u_n(t),v_n(t)], \quad \forall t \in [T_1,T_2], \tag{3.18}$$

$$u_n(T_2) = d = v_n(T_2), \tag{3.19}$$

$$v_{n+1}(T_1) - u_{n+1}(T_1) = \frac{v_n(T_1)-u_n(T_1)}{2}. \tag{3.20}$$

According to Lemma 2.5, $\{u_n(t)\}$ and $\{v_n(t)\}$ both are bounded in C^1, then $\{(w(T_1))^{1/(p(T_1)-1)}u_n'(T_1)\}$ is a bounded set and has a convergent subsequence. Note that $\{u_n(t)\}$ are solutions of (P) and satisfy

$$w(r)\varphi(r,u_n'(r)) = a_n + F(N_f(u_n))(r), \tag{3.21}$$

where

$$F(N_f(u_n))(r) = \int_{T_1}^{r} N_f(u_n)dt, \qquad a_n = w(T_1)\varphi(T_1,u_n'(T_1)). \tag{3.22}$$

Similar to the proof of Lemma 2.4, $\{u_n(t)\}$ possesses a convergent subsequence $\{u_{n_i}(t)\}$ in C^1, and then $\{a_n\}$ is bounded. From [2], we can see that $\{u_n(t)\}$ and $\{v_n(t)\}$ have uniform $C^{1,\alpha}$ regularity. We may assume that $u_{n_i}(t) \to u(t)$ in C^1 and $v_{n_j}(t) \to v(t)$ in C^1.

It is easy to see that $u(t) \le v(t)$ both are solutions of (P). From the definition of $\{u_n(t)\}$ and $\{v_n(t)\}$, we can see that

$$u(T_2) = d = v(T_2). \tag{3.23}$$

Combining (3.18) and (3.20), we have

$$\begin{gathered}
u(t) \le v(t), \quad \forall t \in [T_1,T_2], \\
u(T_1) = \lim_{j\to\infty} u_{n_i}(T_1) = \lim_{j\to\infty} v_{n_i}(T_1) = v(T_1).
\end{gathered} \tag{3.24}$$

Similar to (3.7), we have

$$(w(T_1))^{1/(p(T_1)-1)}u'(T_1) \le (w(T_1))^{1/(p(T_1)-1)}v'(T_1). \tag{3.25}$$

From (3.17) and the continuity of g, we can see that

$$g(u(T_1),(w(T_1))^{1/(p(T_1)-1)}u'(T_1)) \ge 0 \ge g(v(T_1),(w(T_1))^{1/(p(T_1)-1)}v'(T_1)). \tag{3.26}$$

From (3.25), (3.26), and the increasing property of $g(x,y)$ with respect to y, we have

$$g(u(T_1),(w(T_1))^{1/(p(T_1)-1)}u'(T_1)) = 0 = g(v(T_1),(w(T_1))^{1/(p(T_1)-1)}v'(T_1)). \tag{3.27}$$

Thus, u and v both are solutions of (P) with (1.2). This completes the proof. □

Proof of Theorem 1.3. According to Theorem 1.2, (P) possesses a solution u_1 such that

$$\begin{gathered} g(u_1(T_1),(w(T_1))^{1/(p(T_1)-1)}u_1'(T_1)) = 0, \\ u_1(T_2) = \alpha(T_2), \\ \alpha(t) \le u_1(t) \le \beta(t), \quad \forall t \in [T_1,T_2]. \end{gathered} \tag{3.28}$$

Similar to the proof of (3.7), we have

$$(w(T_2))^{1/(p(T_2)-1)}u_1'(T_2) \le (w(T_2))^{1/(p(T_2)-1)}\alpha'(T_2). \tag{3.29}$$

Obviously, $h(u_1(T_2),(w(T_2))^{1/(p(T_2)-1)}u_1'(T_2)) \le 0$. We may assume that

$$h(u_1(T_2),(w(T_2))^{1/(p(T_2)-1)}u_1'(T_2)) < 0, \tag{3.30}$$

or we get a solution for (P) with (1.3), then u_1 is a subsolution of (P) with (1.3).

According to Theorem 1.2, (P) possesses a solution v_1 such that

$$\begin{gathered} g(v_1(T_1),(w(T_1))^{1/(p(T_1)-1)}v_1'(T_1)) = 0, \\ v_1(T_2) = \beta(T_2), \\ u_1(t) \le v_1(t) \le \beta(t), \quad \forall t \in [T_1,T_2]. \end{gathered} \tag{3.31}$$

Similarly, $h(v_1(T_2),(w(T_2))^{1/(p(T_2)-1)}v_1'(T_2)) \ge 0$. We may assume that

$$h(v_1(T_2),(w(T_2))^{1/(p(T_2)-1)}v_1'(T_2)) > 0, \tag{3.32}$$

or we get a solution for (P) with (1.3), then v_1 is a supersolution of (P) with (1.3).

According to Theorem 1.2, (P) possesses a solution x such that

$$\begin{gathered} g(x(T_1),(w(T_1))^{1/(p(T_1)-1)}x'(T_1)) = 0, \qquad x(T_2) = \frac{u_1(T_2)+v_1(T_2)}{2}, \\ u_1(t) \le x(t) \le v_1(t), \quad \forall t \in [T_1,T_2]. \end{gathered} \tag{3.33}$$

We may assume that $h(x(T_2),(w(T_2))^{1/(p(T_2)-1)}x'(T_2)) \neq 0$, or we get a solution for (P) with (1.3). If $h(x(T_2),(w(T_2))^{1/(p(T_2)-1)}x'(T_2)) > 0$, then denote $v_2(t) = x(t)$ and $u_2(t) = u_1(t)$, if $h(x(T_2),(w(T_2))^{1/(p(T_2)-1)}x'(T_2))<0$, then denote $v_2(t)=v_1(t)$ and $u_2(t)=x(t)$. It is easy to see that $u_2(t)$ and $v_2(t)$ both are solutions of (P) and satisfy

$$\begin{gathered} h\left(u_2(T_2),(w(T_2))^{1/(p(T_2)-1)}u_2'(T_2)\right)<0<h\left(v_2(T_2),(w(T_2))^{1/(p(T_2)-1)}v_2'(T_2)\right), \\ u_2(t)\le v_2(t),\quad \forall t\in[T_1,T_2],\qquad [u_2(t),v_2(t)]\subseteq[u_1(t),v_1(t)],\quad \forall t\in[T_1,T_2], \\ v_2(T_2)-u_2(T_2)=\frac{v_1(T_2)-u_1(T_2)}{2}. \end{gathered} \tag{3.34}$$

Repeating the step, similar to the proof of Theorem 1.2, we get two sequences $\{u_n\}$ and $\{v_n\}$, all are solutions of (P), and satisfy

$$\begin{gathered} g\left(u_n(T_1),(w(T_1))^{1/(p(T_1)-1)}u_n'(T_1)\right)=0=g\left(v_n(T_1),(w(T_1))^{1/(p(T_1)-1)}v_n'(T_1)\right), \\ h\left(u_n(T_2),(w(T_2))^{1/(p(T_2)-1)}u_n'(T_2)\right)<0<h\left(v_n(T_2),(w(T_2))^{1/(p(T_2)-1)}v_n'(T_2)\right), \\ u_n(t)\le v_n(t),\quad \forall t\in[T_1,T_2],\qquad [u_{n+1}(t),v_{n+1}(t)]\subseteq[u_n(t),v_n(t)],\quad \forall t\in[T_1,T_2]. \\ v_{n+1}(T_2)-u_{n+1}(T_2)=\frac{v_n(T_2)-u_n(T_2)}{2}. \end{gathered} \tag{3.35}$$

Similar to the proof of Theorem 1.2, $\{u_n(t)\}$ and $\{v_n(t)\}$ possess convergent subsequence $\{u_{n_i}(t)\}$ and $\{v_{n_j}(t)\}$ in C^1, respectively. We may assume that $u_{n_i}(t)\to u(t)$ in C^1, and similar $v_{n_j}(t)\to v(t)$ in C^1. It is easy to see that $u(t)\le v(t)$ both are solutions of (P) with (1.3). This completes the proof. □

Proof of Theorem 1.4. According to Theorem 1.1, (P) possesses solution u_1 which satisfies

$$u_1(T_1)=\alpha(T_1),\qquad u_1(T_2)=\alpha(T_2),\qquad \alpha(t)\le u_1(t)\le\beta(t),\quad t\in[T_1,T_2]. \tag{3.36}$$

We may assume that $w(T_1)\varphi(T_1,u_1'(T_1))\ne w(T_2)\varphi(T_2,u_1'(T_2))$, or we get a solution for (P) with (1.4), then $w(T_1)\varphi(T_1,u_1'(T_1))>w(T_2)\varphi(T_2,u_1'(T_2))$, and u_1 is a subsolution of (P). According to Theorem 1.1, (P) possesses solutions v_1 which satisfies

$$v_1(T_1)=\beta(T_1),\qquad v_1(T_2)=\beta(T_2),\qquad u_1(t)\le v_1(t)\le\beta(t),\quad t\in[T_1,T_2]. \tag{3.37}$$

We may assume that $w(T_1)\varphi(T_1,v_1'(T_1))\ne w(T_2)\varphi(T_2,v_1'(T_2))$, or we get a solution for (P) with (1.4), then $w(T_1)\varphi(T_1,v_1'(T_1))<w(T_2)\varphi(T_2,v_1'(T_2))$, and v_1 is a supersolution of (P). According to Theorem 1.1, (P) possesses solutions x and satisfies

$$x(T_1)=\frac{u_1(T_1)+v_1(T_1)}{2}=x(T_2),\qquad u_1(t)\le x(t)\le v_1(t),\quad t\in[T_1,T_2]. \tag{3.38}$$

Similar to the proof of Theorem 1.2, we obtain u and v that are solutions of (P), which satisfy

$$u(t)\le v(t),\quad t\in[T_1,T_2], \tag{3.39}$$

$$u(T_1)=u(T_2)=v(T_1)=v(T_2), \tag{3.40}$$

$$w(T_1)\varphi(T_1,u'(T_1))\ge w(T_2)\varphi(T_2,u'(T_2)), \tag{3.41}$$

$$w(T_1)\varphi(T_1,v'(T_1))\le w(T_2)\varphi(T_2,v'(T_2)). \tag{3.42}$$

From (3.39) and (3.40), we have

$$\begin{aligned} w(T_1)\varphi(T_1,u'(T_1)) &\le w(T_1)\varphi(T_1,v'(T_1)),\\ w(T_2)\varphi(T_2,u'(T_2)) &\ge w(T_2)\varphi(T_2,v'(T_2)). \end{aligned} \tag{3.43}$$

From (3.41), (3.42), and (3.43), we can conclude that (P) with (1.4) possesses a solution. This completes the proof. □

On the case of $\min_{r\in[-R,R]} p(r) \le q(r) \le \max_{r\in[-R,R]} p(r)$, we consider

$$\begin{gathered} -\left(|u'|^{p(r)-2}u'\right)' = C|u|^{q(r)-2}u + e(r) \quad r \in (-R,R),\\ u(-R) = u(R) = 0, \end{gathered} \tag{I}$$

where $q(r), e(r) \in C([-R,R],\mathbb{R}^+)$, $\min_{r\in[-R,R]} p(r) \le q(r) \le \max_{r\in[-R,R]} p(r)$, C is a positive constant. Denote

$$p^+ = \max_{r\in[-R,R]} p(r), \qquad p^- = \min_{r\in[-R,R]} p(r). \tag{3.44}$$

We have the following corollary.

Corollary 3.1. *If $p \in C(\mathbb{R},(1,+\infty))$ is even, R satisfies*

$$R \le \left[1 + C + \max_{r\in[-R,R]} e(r)\right]^{-(p^+-1)/(p^+(p^--1))}, \tag{3.45}$$

then (I) *possesses at least a nontrivial solution.*

Proof. It is easy to see that $\alpha \equiv 0$ is a subsolution of (I). Denote

$$\beta(r) = 1 - \int_0^r |\mu s|^{1/(p(s)-1)-1}\mu s\,ds, \tag{3.46}$$

where μ is a positive constant satisfying $\beta(R) = 0$. Since p is even, then $\beta(-R) = 0$. It is easy to see that $0 \le \beta(r) \le 1$, $\forall r \in [-R,R]$, and

$$\begin{aligned} -\left(|\beta'|^{p(r)-2}\beta'\right)' = \mu &= \left(\int_0^R |s|^{1/(p(s)-1)}ds\right)^{1-p(\xi)} \ge \left(\int_0^R |s|^{1/(p^+-1)}ds\right)^{1-p(\xi)}\\ &\ge \left(\int_0^R |s|^{1/(p^+-1)}ds\right)^{1-p^-} \ge 1 + C + \max_{r\in[-R,R]} e(r) \ge C|\beta|^{q(r)-2}\beta + e(r), \end{aligned} \tag{3.47}$$

where $\xi \in [-R,R]$. Then, β is a supersolution of (I). From Theorem 1.1, one can see that (I) possesses at least a nontrivial solution. □

4. Applications in PDE

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. In this section, we always denote

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \qquad p^- = \min_{x\in\overline{\Omega}} p(x). \tag{4.1}$$

Let us now consider (1.15) with one of the following boundary value conditions:

$$u|_{\partial\Omega} = 0, \tag{4.2}$$

$$\nabla u = 0, \quad \forall x \in \partial\Omega. \tag{4.3}$$

If u is a radial solution of (1.15), then it can be transformed into

$$-\left(r^{n-1}\left|u'\right|^{p(r)-2}u'\right)' + r^{n-1}f\left(r, u, |r|^{(n-1)/(p(r)-1)}\left|u'\right|\right) = 0, \quad r \in (T_1, T_2), \text{ where } T_1 \geq 0, \tag{4.4}$$

and the boundary value condition will be transformed into (1.1), (1.2), or (1.3), respectively.

Theorem 4.1. *If* (4.4) *has subsolution and supersolution α and β, respectively, satisfying $\alpha(t) \leq \beta(t)$ for any $t \in [T_1, T_2]$, and f is continuous and satisfies (H_1)-(H_2), in each of the following cases:*

(i) *$0 < T_1 < T_2$, $\Omega = \{x \in \mathbb{R}^n \mid T_1 < |x| < T_2\}$, $\alpha(T_1) \leq 0 \leq \beta(T_1)$, and $\alpha(T_2) \leq 0 \leq \beta(T_2)$;*

(ii) *$0 = T_1 < T_2$, $\Omega = \{x \in \mathbb{R}^n \mid T_1 < |x| < T_2\} = B(0; T_2) \setminus \{0\}$, and $p^- > n$; $\alpha(T_1) \leq 0 \leq \beta(T_1)$, $\alpha(T_2) \leq 0 \leq \beta(T_2)$;*

(iii) *$0 = T_1 < T_2$, $\Omega = \{x \in \mathbb{R}^n \mid |x| < T_2\} = B(0; T_2)$, and $p^- > n$; $(w(T_1))^{1/(p(T_1)-1)}\alpha'(T_1) \geq 0 \geq (w(T_1))^{1/(p(T_1)-1)}\beta'(T_1)$, $\alpha(T_2) \leq 0 \leq \beta(T_2)$;*

then (1.15) *with* (4.2) *has at least one weak radially symmetric solution u.*

Proof. Notice that $(r^{n-1})^{-1/(p(r)-1)} \in L^1(0, T_2)$ and satisfies $0 < r^{n-1}$, $\forall r \in (0, T_2)$. We can conclude the existence of solutions for (4.4) with (1.1), (1.2), or (1.3), from Theorems 1.1, 1.2, and 1.3. If $\lim_{r\to 0} r^{n-1}|u'|^{p(r)-2}u'(r) = 0$, notice that

$$\begin{aligned} \left|\left|u'\right|^{p(r)-2}u'(r)\right| &\leq r^{1-n}\int_0^r t^{n-1}\left|f\left(t, u, |t|^{(n-1)/(p(t)-1)}\left|u'\right|\right)\right|dt \\ &\leq \int_0^r \left|f\left(t, u, |t|^{(n-1)/(p(t)-1)}\left|u'\right|\right)\right|dt \longrightarrow 0 \quad (\text{as } r \longrightarrow 0), \end{aligned} \tag{4.5}$$

then we have $u'(0) = 0$. This completes the proof. □

Similarly, we have the following theorem.

Theorem 4.2. *If* (4.4) *has subsolution and supersolution α and β, respectively, satisfying $\alpha(t) \leq \beta(t)$ for any $t \in [T_1, T_2]$, and*

$$\begin{aligned} &\left(w(T_1)\right)^{1/(p(T_1)-1)}\alpha'(T_1) \geq 0 \geq \left(w(T_1)\right)^{1/(p(T_1)-1)}\beta'(T_1), \\ &\left(w(T_2)\right)^{1/(p(T_2)-1)}\alpha'(T_2) \leq 0 \leq \left(w(T_2)\right)^{1/(p(T_2)-1)}\beta'(T_2), \end{aligned} \tag{4.6}$$

and f is continuous and satisfies (H_1)-(H_2), *in each of the following cases:*

(i) $0 < T_1 < T_2$; $\Omega = \{x \in \mathbb{R}^n \mid T_1 < |x| < T_2\}$;

(ii) $0 = T_1 < T_2$; $\Omega = \{x \in \mathbb{R}^n \mid T_1 < |x| < T_2\} = B(0;T_2) \setminus \{0\}$ *or* $\Omega = B(0;T_2)$; $p \in C^1(\overline{\Omega};\mathbb{R})$ *and* $p^- > n$;

then (1.15) *with* (4.3) *has at least one weak radially symmetric solution* u.

On the case of $p^- \le q(x) \le p^+$, we consider

$$\begin{gathered}-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = C|u-1|^{q(x)-2}u + e(x), \quad x \in \Omega,\\ u(x) = 0, \quad x \in \partial\Omega,\end{gathered} \tag{II}$$

where $\Omega = \{x \in \mathbb{R}^n \mid 0 < |x| < R\}$, $q(x), e(x) \in C(\overline{\Omega}, \mathbb{R}^+)$, $2 \le n < p^- \le q(x) \le p^+$, C is a positive constant.

We have the following corollary.

Corollary 4.3. *If $p \in C(\mathbb{R}^n, (1,+\infty))$ is radial, and R satisfies*

$$R \le \min\left\{1, \left[\left(1 - \frac{1}{2(p^- - 1)}\right)^{p^+ - 1} \frac{(n - 3/2)}{1 + C + \max_{x\in\overline{\Omega}} e(x)}\right]^{1/(p^- - 3/2)}\right\}, \tag{4.7}$$

then (II) *possesses at least a nontrivial solution.*

Proof. It is easy to see that $\alpha \equiv 0$ is a subsolution of (II). Denote

$$\beta(r) = 1 - \int_0^r \left|\mu s^{-1/2}\right|^{1/(p(s)-1)} ds, \tag{4.8}$$

where μ is a positive constant satisfying $\beta(R) = 0$. It is easy to see that $0 \le \beta(r) \le 1$, $\forall r \in [0,R]$, and

$$\begin{aligned}-\left(r^{n-1}\left|\beta'\right|^{p(r)-2}\beta'\right)' &= \mu\left(n - \frac{3}{2}\right)r^{n-5/2} = \left(\int_0^R |s|^{-1/2(p(s)-1)} ds\right)^{1-p(\xi)}\left(n - \frac{3}{2}\right)r^{n-5/2}\\ &\ge \left(\int_0^R |s|^{-1/2(p^- -1)} ds\right)^{1-p(\xi)}\left(n - \frac{3}{2}\right)r^{n-1}\\ &\ge \left(R^{-1/2(p^- -1)+1}\right)^{1-p^-}\left(1 - \frac{1}{2(p^- - 1)}\right)^{p^+ - 1}\left(n - \frac{3}{2}\right)r^{n-1}\\ &\ge r^{n-1}\left[1 + C + \max_{x\in\overline{\Omega}} e(x)\right] \ge r^{n-1}\left[C|\beta - 1|^{q(x)-2}\beta + e(x)\right],\end{aligned} \tag{4.9}$$

where $\xi \in \overline{\Omega}$. Then, β is a supersolution of (II). From Theorem 4.1, one can see that (II) possesses at least a nontrivial solution. $\square$

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