Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 598632, 13 pages
doi:10.1155/2008/598632

Research Article

Finite-Step Relaxed Hybrid Steepest-Descent Methods for Variational Inequalities

Yen-Cherng Lin

Department of Occupational Safety and Health, General Education Center, China Medical University, Taichung 404, Taiwan

Correspondence should be addressed to Yen-Cherng Lin, yclin@mail.cmu.edu.tw

Received 22 August 2007; Revised 2 January 2008; Accepted 13 March 2008

Recommended by Jong Kim

The classical variational inequality problem with a Lipschitzian and strongly monotone operator on a nonempty closed convex subset in a real Hilbert space was studied. A new finite-step relaxed hybrid steepest-descent method for this class of variational inequalities was introduced. Strong convergence of this method was established under suitable assumptions imposed on the algorithm parameters.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H, and let $F : C \to H$ be an operator. The classical variational inequality problem: find $u^* \in C$ such that

$$\mathrm{VI}(F,C) \quad \langle F(u^*), v - u^* \rangle \geq 0, \quad \forall v \in C, \tag{1.1}$$

was initially studied by Kinderlehrer and Stampacchia [1]. It is also known that the $\mathrm{VI}(F,C)$ is equivalent to the fixed-point equation

$$u^* = P_C(u^* - \mu F(u^*)), \tag{1.2}$$

where P_C is the (nearest point) projection from H onto C, that is, $P_C x = \operatorname{argmin}_{y \in C} \|x - y\|$ for each $x \in H$ and where $\mu > 0$ is an arbitrarily fixed constant. If F is strongly monotone and Lipschitzian on C and $\mu > 0$ is small enough, then the mapping determined by the right-hand side of this equation is a contraction. Hence the Banach contraction principle guarantees that the Picard iterates converge in norm to the unique solution of the $\mathrm{VI}(F,C)$. Such a method is called the projection method. However, Zeng and Yao [2] point out that the fixed-point equation involves the projection P_C which may not be easy to compute due to the complexity of the convex set C. To reduce the complexity problem probably caused by the projection P_C, a class of hybrid steepest-descent methods for solving $\mathrm{VI}(F,C)$ has been introduced and studied recently by many authors (see, e.g., [3, 4]). Zeng and Yao [2] have established the method of two-step relaxed hybrid steepest-descent for variational inequalities. A natural arising problem is whether there exists a general relaxed hybrid steepest-descent algorithm that is more than two steps for finding approximate solutions of $\mathrm{VI}(F,C)$ or not. Motivated and inspired by the recent research work in this direction, we introduce the following finite step relaxed hybrid steepest-descent algorithm for finding approximate solutions of $\mathrm{VI}(F,C)$ and aim to unify the convergence results of this kind of methods.

Algorithm 1.1. *Let* $\{\alpha_n^{(k)}\} \subset [0,1)$, $\{\lambda_n^{(k)}\} \subset (0,1)$, *for* $k = 1,2,\ldots,m$, *and take fixed numbers* $t^{(k)} \in (0,2\eta/\kappa^2)$, $k = 1,2,\ldots,m$. *Starting with arbitrarily chosen initial points* $u_0^{(1)} \in H$, *compute the sequences* $\{u_n^{(k)}\}$ *such that*

$$
\begin{aligned}
u_{n+1}^{(1)} &= \alpha_n^{(1)} u_n^{(1)} + \left(1-\alpha_n^{(1)}\right)\left[Tu_n^{(2)} - \lambda_{n+1}^{(1)} t^{(1)} F\left(Tu_n^{(2)}\right)\right],\\
u_n^{(2)} &= \alpha_n^{(2)} u_n^{(1)} + \left(1-\alpha_n^{(2)}\right)\left[Tu_n^{(3)} - \lambda_{n+1}^{(2)} t^{(2)} F\left(Tu_n^{(3)}\right)\right],\\
u_n^{(3)} &= \alpha_n^{(3)} u_n^{(1)} + \left(1-\alpha_n^{(3)}\right)\left[Tu_n^{(4)} - \lambda_{n+1}^{(3)} t^{(3)} F\left(Tu_n^{(4)}\right)\right],\\
&\;\;\vdots\\
u_n^{(m)} &= \alpha_n^{(m)} u_n^{(1)} + \left(1-\alpha_n^{(m)}\right)\left[Tu_n^{(1)} - \lambda_{n+1}^{(m)} t^{(m)} F\left(Tu_n^{(1)}\right)\right].
\end{aligned}
\tag{1.3}
$$

We will prove a strong convergence result for Algorithm 1.1 under suitable restrictions imposed on the parameters.

2. Preliminaries

The following lemmas will be used for proving the main result of the paper in next section.

Lemma 2.1 (see [5]). *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers satisfying the inequality*

$$
s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\tau_n + \gamma_n, \quad \forall n \ge 0, \tag{2.1}
$$

where $\{\alpha_n\}$, $\{\tau_n\}$, *and* $\{\gamma_n\}$ *satisfy the following conditions:*

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, *or equivalently*, $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$;

(ii) $\limsup_{n\to\infty}\tau_n \le 0$;

(iii) $\{\gamma_n\} \subset [0,\infty)$, $\sum_{n=0}^{\infty}\gamma_n < \infty$.

Then $\lim_{n\to\infty}s_n = 0$.

Lemma 2.2 (see [6]). *Demiclosedness principle: assume that T is a nonexpansive self-mapping on a nonempty closed convex subset C of a Hilbert space H. If T has a fixed point, then $I-T$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some $y \in H$, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

The following lemma is an immediate consequence of an inner product.

Lemma 2.3. *In a real Hilbert space H, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,y \in H. \tag{2.2}$$

Lemma 2.4. *Let C be a nonempty closed convex subset of H. For any $x,y \in H$ and $z \in C$, the following statements hold:*

(i) $\langle P_C x - x, z - P_C x\rangle \ge 0$;

(ii) $\|P_C x - P_C y\|^2 \le \|x-y\|^2 - \|P_C x - x + y - P_C y\|^2$.

3. Convergence theorem

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \to H$ be an operator such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on C; that is, F satisfies the conditions

$$\begin{gathered} \|Fx - Fy\| \le \kappa\|x-y\|, \quad \forall x,y \in C, \\ \langle Fx - Fy,\ x-y\rangle \ge \eta\|x-y\|^2, \quad \forall x,y \in C, \end{gathered} \tag{3.1}$$

respectively. Since F is η-strongly monotone, the variational inequality problem $\mathrm{VI}(F,C)$ has a unique solution $u^* \in C$ (see, e.g., [7]).

Assume that $T : H \to H$ is a nonexpansive mapping with the fixed points set $\mathrm{Fix}(T) = C$. Note that obviously $\mathrm{Fix}(P_C) = C$. For any given numbers $\lambda \in (0,1)$ and $\mu \in (0, 2\eta/\kappa^2)$, we define the mapping $T_\mu^\lambda : H \to H$ by

$$T_\mu^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in H.$$

Lemma 3.1 (see [3]). *Let T_μ^λ be a contraction provided that $0<\lambda<1$ and $0<\mu<2\eta/\kappa^2$. Indeed,*

$$\left\|T_\mu^\lambda x - T_\mu^\lambda y\right\| \le (1-\lambda\tau)\|x-y\|, \quad \forall x,y \in H, \tag{3.2}$$

where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)} \in (0,1)$.

We now state and prove the main result of this paper.

Theorem 3.2. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \to H$ be an operator such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on C. Assume that $T : H \to H$ is a nonexpansive mapping with the fixed points set $\mathrm{Fix}(T) = C$, the real sequences $\{\alpha_n^{(k)}\}$, $\{\lambda_n^{(k)}\}$, for $k = 1,2,\ldots,m$, in Algorithm 1.1 satisfy the following conditions:*

(i) $\sum_{n=1}^{\infty} |\alpha_n^{(k)} - \alpha_{n-1}^{(k)}| < \infty$, *for* $k = 1,2,\ldots,m$;

(ii) $\lim_{n\to\infty} \alpha_n^{(1)} = 0$ *and* $\lim_{n\to\infty} \alpha_n^{(k)} = 1$, *for* $k = 2,3,\ldots,m$;

(iii) $\lim_{n\to\infty}\lambda_n^{(1)} = 0$, $\lim_{n\to\infty}(\lambda_n^{(1)}/\lambda_{n+1}^{(1)}) = 1$, $\sum_{n=1}^{\infty}\lambda_n^{(1)} = \infty$;

(iv) $\lambda_n^{(1)} \geq \max\{\lambda_n^{(k)} : k = 2,3,\ldots,m\}$, *for all* $n \geq 1$.

Then the sequences $\{u_n^{(k)}\}$ *generated by Algorithm 1.1 converge strongly to* u^* *which is the unique solution of the* VI(F,C).

Proof. Since F is η-strongly monotone, by [7], the VI(F,C) has the unique solution $u^* \in C$. Next we divide the rest of the proof into several steps. □

Step 1. Let $\{u_n^{(k)}\}$ is bounded for each $k = 1,2,\ldots,m$. Indeed, let us denote that $T_t^{\lambda}u^* = Tu^* - \lambda tF(Tu^*)$, then we have

$$\begin{aligned}\left\|u_{n+1}^{(1)} - u^*\right\| &= \left\|\alpha_n^{(1)}u_n^{(1)} + \left(1-\alpha_n^{(1)}\right)T_{t^{(1)}}^{\lambda_{n+1}^{(1)}}u_n^{(2)} - u^*\right\| \\ &\leq \alpha_n^{(1)}\left\|u_n^{(1)} - u^*\right\| + \left(1-\alpha_n^{(1)}\right)\left\|T_{t^{(1)}}^{\lambda_{n+1}^{(1)}}u_n^{(2)} - u^*\right\| \\ &\leq \alpha_n^{(1)}\left\|u_n^{(1)} - u^*\right\| + \left(1-\alpha_n^{(1)}\right)\left[\left\|T_{t^{(1)}}^{\lambda_{n+1}^{(1)}}u_n^{(2)} - T_{t^{(1)}}^{\lambda_{n+1}^{(1)}}u^*\right\| + \left\|T_{t^{(1)}}^{\lambda_{n+1}^{(1)}}u^* - u^*\right\|\right] \\ &\leq \alpha_n^{(1)}\left\|u_n^{(1)} - u^*\right\| + \left(1-\alpha_n^{(1)}\right)\left[\left(1-\lambda_{n+1}^{(1)}\tau^{(1)}\right)\left\|u_n^{(2)} - u^*\right\| + \lambda_{n+1}^{(1)}t^{(1)}\left\|F(u^*)\right\|\right],\end{aligned} \tag{3.3}$$

where $\tau^{(1)} = 1 - \sqrt{1 - t^{(1)}(2\eta - t^{(1)}\kappa^2)} \in (0,1)$. Moreover, we also have

$$\begin{aligned}\left\|u_n^{(2)} - u^*\right\| &= \left\|\alpha_n^{(2)}u_n^{(1)} + \left(1-\alpha_n^{(2)}\right)T_{t^{(2)}}^{\lambda_{n+1}^{(2)}}u_n^{(3)} - u^*\right\| \\ &\leq \alpha_n^{(2)}\left\|u_n^{(1)} - u^*\right\| + \left(1-\alpha_n^{(2)}\right)\left[\left\|T_{t^{(2)}}^{\lambda_{n+1}^{(2)}}u_n^{(3)} - T_{t^{(2)}}^{\lambda_{n+1}^{(2)}}u^*\right\| + \left\|T_{t^{(2)}}^{\lambda_{n+1}^{(2)}}u^* - u^*\right\|\right] \\ &\leq \alpha_n^{(2)}\left\|u_n^{(1)} - u^*\right\| + \left(1-\alpha_n^{(2)}\right)\left[\left(1-\lambda_{n+1}^{(2)}\tau^{(2)}\right)\left\|u_n^{(3)} - u^*\right\| + \lambda_{n+1}^{(2)}t^{(2)}\left\|F(u^*)\right\|\right] \\ &\leq \alpha_n^{(2)}\left\|u_n^{(1)} - u^*\right\| + \left(1-\alpha_n^{(2)}\right)\left\|u_n^{(3)} - u^*\right\| + \left(1-\alpha_n^{(2)}\right)\lambda_{n+1}^{(2)}t^{(2)}\left\|F(u^*)\right\|,\end{aligned} \tag{3.4}$$

where $\tau^{(2)} = 1 - \sqrt{1 - t^{(2)}(2\eta - t^{(2)}\kappa^2)} \in (0,1)$, and for $k = 2,3,\ldots,m-1$,

$$\begin{aligned}\left\|u_n^{(k)} - u^*\right\| &= \left\|\alpha_n^{(k)}u_n^{(1)} + \left(1-\alpha_n^{(k)}\right)T_{t^{(k)}}^{\lambda_{n+1}^{(k)}}u_n^{(k+1)} - u^*\right\| \\ &\leq \alpha_n^{(k)}\left\|u_n^{(1)} - u^*\right\| + \left(1-\alpha_n^{(k)}\right)\left[\left\|T_{t^{(k)}}^{\lambda_{n+1}^{(k)}}u_n^{(k+1)} - T_{t^{(k)}}^{\lambda_{n+1}^{(k)}}u^*\right\| + \left\|T_{t^{(k)}}^{\lambda_{n+1}^{(k)}}u^* - u^*\right\|\right] \\ &\leq \alpha_n^{(k)}\left\|u_n^{(1)} - u^*\right\| + \left(1-\alpha_n^{(k)}\right)\left[\left(1-\lambda_{n+1}^{(k)}\tau^{(k)}\right)\left\|u_n^{(k+1)} - u^*\right\| + \lambda_{n+1}^{(k)}t^{(k)}\left\|F(u^*)\right\|\right] \\ &\leq \alpha_n^{(k)}\left\|u_n^{(1)} - u^*\right\| + \left(1-\alpha_n^{(k)}\right)\left\|u_n^{(k+1)} - u^*\right\| + \left(1-\alpha_n^{(k)}\right)\lambda_{n+1}^{(k)}t^{(k)}\left\|F(u^*)\right\|,\end{aligned} \tag{3.5}$$

where $\tau^{(k)} = 1 - \sqrt{1 - t^{(k)}(2\eta - t^{(k)}\kappa^2)} \in (0,1)$, and

$$\begin{aligned}
\|u_n^{(m)} - u^*\| &= \|\alpha_n^{(m)} u_n^{(1)} + (1-\alpha_n^{(m)}) T_{t^{(m)}}^{\lambda_{n+1}^{(m)}} u_n^{(1)} - u^*\| \\
&\le \alpha_n^{(m)} \|u_n^{(1)} - u^*\| + (1-\alpha_n^{(m)}) \left[\|T_{t^{(m)}}^{\lambda_{n+1}^{(m)}} u_n^{(1)} - T_{t^{(m)}}^{\lambda_{n+1}^{(m)}} u^*\| + \|T_{t^{(m)}}^{\lambda_{n+1}^{(m)}} u^* - u^*\| \right] \\
&\le \alpha_n^{(m)} \|u_n^{(1)} - u^*\| + (1-\alpha_n^{(m)}) \left[(1-\lambda_{n+1}^{(m)}\tau^{(m)}) \|u_n^{(1)} - u^*\| + \lambda_{n+1}^{(m)} t^{(m)} \|F(u^*)\| \right] \\
&\le \|u_n^{(1)} - u^*\| + \lambda_{n+1}^{(m)} t^{(m)} \|F(u^*)\|,
\end{aligned} \tag{3.6}$$

where $\tau^{(m)} = 1 - \sqrt{1 - t^{(m)}(2\eta - t^{(m)}\kappa^2)} \in (0,1)$.

Thus we obtain

$$\begin{aligned}
&\|u_n^{(m-1)} - u^*\| \\
&\quad\le \alpha_n^{(m-1)} \|u_n^{(1)} - u^*\| + (1-\alpha_n^{(m-1)}) \|u_n^{(m)} - u^*\| + \lambda_{n+1}^{(m-1)} t^{(m-1)} \|F(u^*)\| \\
&\quad\le \alpha_n^{(m-1)} \|u_n^{(1)} - u^*\| + (1-\alpha_n^{(m-1)}) \left[\|u_n^{(1)} - u^*\| + \lambda_{n+1}^{(m)} t^{(m)} \|F(u^*)\| \right] + \lambda_{n+1}^{(m-1)} t^{(m-1)} \|F(u^*)\|] \\
&\quad\le \|u_n^{(1)} - u^*\| + (1-\alpha_n^{(m-1)}) \max\{\lambda_{n+1}^{(m)}, \lambda_{n+1}^{(m-1)}\} (t^{(m)} + t^{(m-1)}) \|F(u^*)\|, \\
&\|u_n^{(k)} - u^*\| \le \|u_n^{(1)} - u^*\| + (1-\alpha_n^{(k)}) \max_{k\le j\le m} \{\lambda_{n+1}^{(j)}\} \left(\sum_{j=k}^{m} t^{(j)} \right) \|F(u^*)\|,
\end{aligned} \tag{3.7}$$

for $k = 2, 3, \ldots, m-1$. In particular,

$$\|u_n^{(2)} - u^*\| \le \|u_n^{(1)} - u^*\| + (1-\alpha_n^{(2)}) \max_{2\le j\le m} \{\lambda_{n+1}^{(j)}\} \left(\sum_{j=2}^{m} t^{(j)} \right) \|F(u^*)\|. \tag{3.8}$$

Hence, substituting (3.8) in (3.3) and by condition (iv), we obtain

$$\begin{aligned}
&\|u_{n+1}^{(1)} - u^*\| \\
&\quad\le \alpha_n^{(1)} \|u_n^{(1)} - u^*\| + (1-\alpha_n^{(1)}) \left[(1-\lambda_{n+1}^{(1)}\tau^{(1)}) \|u_2^{(2)} - u^*\| + \lambda_{n+1}^{(1)} t^{(1)} \|F(u^*)\| \right] \\
&\quad\le \alpha_n^{(1)} \|u_n^{(1)} - u^*\| + (1-\alpha_n^{(1)}) \Bigg[(1-\lambda_{n+1}^{(1)}\tau^{(1)}) \|u_n^{(1)} - u^*\| \\
&\qquad\qquad + \max_{2\le j\le m} \{\lambda_{n+1}^{(j)}\} \sum_{j=2}^{m} t^{(j)} \|F(u^*)\| + \lambda_{n+1}^{(1)} t^{(1)} \|F(u^*)\| \Bigg] \\
&\quad\le \alpha_n^{(1)} \|u_n^{(1)} - u^*\| + (1-\alpha_n^{(1)}) \left[(1-\lambda_{n+1}^{(1)}\tau^{(1)}) \|u_n^{(1)} - u^*\| + \max_{1\le j\le m} \{\lambda_{n+1}^{(j)}\} \sum_{j=1}^{m} t^{(j)} \|F(u^*)\| \right].
\end{aligned} \tag{3.9}$$

By induction, it is easy to see that

$$\|u_n^{(1)} - u^*\| \le \widetilde{M}, \quad \forall n \ge 0, \tag{3.10}$$

where $\widetilde{M} = \max\{\|u_0^{(1)} - u^*\|, ((\sum_{j=1}^{m} t^{(j)})/\tau^{(1)})\|F(u^*)\|\}$. Indeed, for $n = 0$, from (3.9) we obtain

$$\begin{aligned}\|u_1^{(1)} - u^*\| &\leq \alpha_0^{(1)}\|u_0^{(1)} - u^*\| + (1-\alpha_0^{(1)})\left[(1-\lambda_1^{(1)}\tau^{(1)})\|u_0^{(1)} - u^*\| + \max_{1\leq j\leq m}\lambda_1^{(j)}\left(\sum_{j=1}^{m} t^{(j)}\right)\|F(u^*)\|\right] \\ &\leq \alpha_0^{(1)}\widetilde{M} + (1-\alpha_0^{(1)})\left[(1-\lambda_1^{(1)}\tau^{(1)})\widetilde{M} + \lambda_1^{(1)}\tau^{(1)}\widetilde{M}\right] = \widetilde{M}.\end{aligned} \tag{3.11}$$

Suppose that $\|u_n^{(1)} - u^*\| \leq \widetilde{M}$, for $n \geq 1$. We want to claim that $\|u_{n+1}^{(1)} - u^*\| \leq \widetilde{M}$. Indeed,

$$\begin{aligned}\|u_{n+1}^{(1)} - u^*\| &\leq \alpha_n^{(1)}\|u_n^{(1)} - u^*\| + (1-\alpha_n^{(1)})\left[(1-\lambda_{n+1}^{(1)}\tau^{(1)})\|u_n^{(1)} - u^*\| + \lambda_{n+1}^{(1)}\sum_{j=1}^{m} t^{(j)}\|F(u^*)\|\right] \\ &\leq \alpha_n^{(1)}\widetilde{M} + (1-\alpha_n^{(1)})\left[(1-\lambda_{n+1}^{(1)}\tau^{(1)})\widetilde{M} + \lambda_{n+1}^{(1)}\tau^{(1)}\widetilde{M}\right] = \widetilde{M}.\end{aligned} \tag{3.12}$$

Therefore, we have $\|u_n^{(1)} - u^*\| \leq \widetilde{M}$, for all $n \geq 0$, and $\|u_n^{(m)} - u^*\| \leq \widetilde{M} + \lambda_{n+1}^{(m)}\tau^{(1)}\widetilde{M} \leq (1+\tau^{(1)})\widetilde{M}$, for all $n \geq 0$. In this case, from (3.8), it follows that

$$\|u_n^{(k)} - u^*\| \leq \widetilde{M} + \max_{k\leq j\leq m}\{\lambda_{n+1}^{(j)}\}\tau^{(1)}\widetilde{M} \leq (1+\tau^{(1)})\widetilde{M}, \quad \forall n \geq 0,\ \forall k = 2,3,\ldots,m-1. \tag{3.13}$$

Step 2. Let $\|u_{n+1}^{(1)} - Tu_n^{(1)}\| \to 0, n \to \infty$. Indeed by Step 1, $\{u_n^{(k)}\}$ is bounded for $1 \leq k \leq m$ and so are $\{Tu_n^{(k)}\}$ and $\{F(Tu_n^{(k)})\}$ for $1 \leq k \leq m$. Thus from the conditions that $\lim_{n\to\infty}\alpha_n^{(1)} = 0$, $\lim_{n\to\infty}\alpha_n^{(k)} = 1$, for $k = 2,3,\ldots,m$ and $\lim_{n\to\infty}\lambda_n^{(1)} = 0$, we have, for $k = 2,\ldots,m$,

$$\begin{aligned}&\|u_n^{(k)} - u_n^{(1)}\| \\ &\quad = \|\alpha_n^{(k)}u_n^{(1)} + (1-\alpha_n^{(k)})(Tu_n^{(k+1)} - \lambda_{n+1}^{(k)}t^{(k)}F(Tu_n^{(k+1)})) - u_n^{(1)}\| \\ &\quad = \|-(1-\alpha_n^{(k)})u_n^{(1)} + (1-\alpha_n^{(k)})(Tu_n^{(k+1)} - \lambda_{n+1}^{(k)}t^{(k)}F(Tu_n^{(k+1)}))\| \\ &\quad \leq (1-\alpha_n^{(k)})\|u_n^{(1)}\| + (1-\alpha_n^{(k)})(\|Tu_n^{(k+1)}\| + \lambda_{n+1}^{(k)}t^{(k)}\|F(Tu_n^{(k+1)})\|) \longrightarrow 0\end{aligned} \tag{3.14}$$

and so

$$\begin{aligned}&\|u_{n+1}^{(1)} - Tu_n^{(1)}\| \\ &\quad = \|\alpha_n^{(1)}u_n^{(1)} + (1-\alpha_n^{(1)})(Tu_n^{(2)} - \lambda_{n+1}^{(1)}t^{(1)}F(Tu_n^{(2)})) - Tu_n^{(1)}\| \\ &\quad = \|\alpha_n^{(1)}(u_n^{(1)} - Tu_n^{(1)}) + (1-\alpha_n^{(1)})((Tu_n^{(2)} - Tu_1^{(1)}) - \lambda_{n+1}^{(1)}t^{(1)}F(Tu_n^{(2)}))\| \\ &\quad \leq \alpha_n^{(1)}\|u_n^{(1)} - Tu_n^{(1)}\| + (1-\alpha_n^{(1)})\|Tu_n^{(2)} - Tu_n^{(1)}\| + (1-\alpha_n^{(1)})\lambda_{n+1}^{(1)}t^{(1)}\|F(Tu_n^{(2)})\| \\ &\quad \leq \alpha_n^{(1)}\|u_n^{(1)} - Tu_n^{(1)}\| + \|u_n^{(2)} - u_n^{(1)}\| + \lambda_{n+1}^{(1)}t^{(1)}\|F(Tu_n^{(2)})\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.15}$$

Step 3. Let $\|u_{n+1}^{(1)} - u_n^{(1)}\| \to 0,$ as $n \to \infty$. Indeed, we observe that

$$
\begin{aligned}
&\|u_n^{(m)} - u_{n-1}^{(m)}\| \\
&= \|\alpha_n^{(m)} u_n^{(1)} - \alpha_{n-1}^{(m)} u_{n-1}^{(1)} + (1-\alpha_n^{(m)}) T_{t^{(m)}}^{\lambda_{n+1}^{(m)}} u_n^{(1)} - (1-\alpha_{n-1}^{(m)}) T_{t^{(m)}}^{\lambda_n^{(m)}} u_{n-1}^{(1)}\| \\
&\le \alpha_n^{(m)} \|u_n^{(1)} - u_{n-1}^{(1)}\| + |\alpha_n^{(m)} - \alpha_{n-1}^{(m)}| \cdot \|u_{n-1}^{(1)}\| + (1-\alpha_n^{(m)}) \|T_{t^{(m)}}^{\lambda_{n+1}^{(m)}} u_n^{(1)} - T_{t^{(m)}}^{\lambda_{n+1}^{(m)}} u_{n-1}^{(1)}\| \\
&\quad + \|(1-\alpha_n^{(m)}) T_{t^{(m)}}^{\lambda_{n+1}^{(m)}} u_{n-1}^{(1)} - (1-\alpha_{n-1}^{(m)}) T_{t^{(m)}}^{\lambda_n^{(m)}} u_{n-1}^{(1)}\| \\
&\le \alpha_n^{(m)} \|u_n^{(1)} - u_{n-1}^{(1)}\| + |\alpha_n^{(m)} - \alpha_{n-1}^{(m)}| \cdot \|u_{n-1}^{(1)}\| + (1-\alpha_n^{(m)})(1-\lambda_{n+1}^{(m)}\tau^{(m)}) \|u_n^{(1)} - u_{n-1}^{(1)}\| \\
&\quad + |\alpha_n^{(m)} - \alpha_{n-1}^{(m)}| \cdot \|T u_{n-1}^{(1)}\| + |(1-\alpha_n^{(m)})\lambda_{n+1}^{(m)} - (1-\alpha_{n-1}^{(m)})\lambda_n^{(m)}| \cdot t^{(m)} \|F(T u_{n-1}^{(1)})\| \\
&= (1-(1-\alpha_n^{(m)})\lambda_{n+1}^{(m)}\tau^{(m)}) \|u_n^{(1)} - u_{n-1}^{(1)}\| \\
&\quad + |(1-\alpha_n^{(m)})\lambda_{n+1}^{(m)} - (1-\alpha_{n-1}^{(m)})\lambda_n^{(m)}| \cdot t^{(m)} \|F(T u_{n-1}^{(1)})\| + |\alpha_n^{(m)} - \alpha_{n-1}^{(m)}| \cdot (\|u_{n-1}^{(1)}\| + \|T u_{n-1}^{(1)}\|) \\
&\le \|u_n^{(1)} - u_{n-1}^{(1)}\| + |(1-\alpha_n^{(m)})\lambda_{n+1}^{(m)} - (1-\alpha_{n-1}^{(m)})\lambda_n^{(m)}| \cdot t^{(m)} \|F(T u_{n-1}^{(1)})\| \\
&\quad + |\alpha_n^{(m)} - \alpha_{n-1}^{(m)}| \cdot (\|u_{n-1}^{(1)}\| + \|T u_{n-1}^{(1)}\|),
\end{aligned}
\tag{3.16}
$$

and, for $2 \le k \le m-1$,

$$
\begin{aligned}
&\|u_n^{(k)} - u_{n-1}^{(k)}\| \\
&= \|\alpha_n^{(k)} u_n^{(1)} - \alpha_{n-1}^{(k)} u_{n-1}^{(1)} + (1-\alpha_n^{(k)}) T_{t^{(k)}}^{\lambda_{n+1}^{(k)}} u_n^{(k+1)} - (1-\alpha_{n-1}^{(k)}) T_{t^{(k)}}^{\lambda_n^{(k)}} u_{n-1}^{(k+1)}\| \\
&\le \alpha_n^{(k)} \|u_n^{(1)} - u_{n-1}^{(1)}\| + |\alpha_n^{(k)} - \alpha_{n-1}^{(k)}| \cdot \|u_{n-1}^{(1)}\| + (1-\alpha_n^{(k)}) \|T_{t^{(k)}}^{\lambda_{n+1}^{(k)}} u_n^{(k+1)} - T_{t^{(k)}}^{\lambda_{n+1}^{(k)}} u_{n-1}^{(k+1)}\| \\
&\quad + \|(1-\alpha_n^{(k)}) T_{t^{(k)}}^{\lambda_{n+1}^{(k)}} u_{n-1}^{(k+1)} - (1-\alpha_{n-1}^{(k)}) T_{t^{(k)}}^{\lambda_n^{(k)}} u_{n-1}^{(k+1)}\| \\
&\le \alpha_n^{(k)} \|u_n^{(1)} - u_{n-1}^{(1)}\| + |\alpha_n^{(k)} - \alpha_{n-1}^{(k)}| \cdot \|u_{n-1}^{(1)}\| + (1-\alpha_n^{(k)})(1-\lambda_{n+1}^{(k)}\tau^{(k)}) \|u_n^{(k+1)} - u_{n-1}^{(k+1)}\| \\
&\quad + |\alpha_n^{(k)} - \alpha_{n-1}^{(k)}| \cdot \|T u_{n-1}^{(k+1)}\| + |(1-\alpha_n^{(k)})\lambda_{n+1}^{(k)} - (1-\alpha_{n-1}^{(k)})\lambda_n^{(k)}| \cdot t^{(k)} \|F(T u_{n-1}^{(k+1)})\| \\
&= \alpha_n^{(k)} \|u_n^{(1)} - u_{n-1}^{(1)}\| + |\alpha_n^{(k)} - \alpha_{n-1}^{(k)}| \cdot (\|u_{n-1}^{(1)}\| + \|T u_{n-1}^{(k+1)}\|) \\
&\quad + (1-\alpha_n^{(k)})(1-\lambda_{n+1}^{(k)}\tau^{(k)}) \|u_n^{(k+1)} - u_{n-1}^{(k+1)}\| + |(1-\alpha_n^{(k)})\lambda_{n+1}^{(k)} - (1-\alpha_{n-1}^{(k)})\lambda_n^{(k)}| \cdot t^{(k)} \|F(T u_{n-1}^{(k+1)})\|,
\end{aligned}
\tag{3.17}
$$

$$\begin{aligned}
\|u_n^{(m)} - u_{n-1}^{(m)}\| &\le \|u_n^{(1)} - u_{n-1}^{(1)}\| + |\alpha_n^{(m)} - \alpha_{n-1}^{(m)}| \cdot (\|u_{n-1}^{(1)}\| \\
&\quad + \|Tu_{n-1}^{(1)}\|) + |(1-\alpha_n^{(m)})\lambda_{n+1}^{(m)} - (1-\alpha_{n-1}^{(m)})\lambda_n^{(m)}| t^{(m)} \|F(Tu_{n-1}^{(1)})\|, \\
\|u_n^{(m-1)} - u_{n-1}^{(m-1)}\| &\le \|u_n^{(1)} - u_{n-1}^{(1)}\| + |(1-\alpha_n^{(m)})\lambda_{n+1}^{(m)} - (1-\alpha_{n-1}^{(m)})\lambda_n^{(m)}| |t^{(m)} \|FTu_{n-1}^{(1)}\| \\
&\quad + |(1-\alpha_n^{(m-1)})\lambda_{n+1}^{(m-1)} - (1-\alpha_{n-1}^{(m-1)})\lambda_n^{(m-1)}| t^{(m-1)} \|FTu_{n-1}^{(m)}\| \\
&\quad + |\alpha_n^{(m-1)} - \alpha_{n-1}^{(m-1)}| (\|u_{n-1}^{(1)}\| + \|Tu_{n-1}^{(m)}\|) \\
&\quad + |\alpha_n^{(m)} - \alpha_{n-1}^{(m)}| (\|u_{n-1}^{(1)}\| + \|Tu_{n-1}^{(1)}\|), \\
&\quad \cdots
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
\|u_n^{(2)} - u_{n-1}^{(2)}\| &\le \|u_n^{(1)} - u_{n-1}^{(1)}\| + \sum_{k=2}^{m-1} |(1-\alpha_n^{(k)})\lambda_{n+1}^{(k)} - (1-\alpha_{n-1}^{(k)})\lambda_n^{(k)}| t^{(k)} \|FTu_{n-1}^{(k+1)}\| \\
&\quad + \sum_{k=2}^{m-1} |\alpha_n^{(k)} - \alpha_{n-1}^{(k)}| (\|(u_{n-1}^{(1)}\| + \|Tu_{n-1}^{(k+1)}\|) + |\alpha_n^{(m)} - \alpha_{n-1}^{(m)}| (\|u_{n-1}^{(1)}\| + \|Tu_{n-1}^{(1)}\|).
\end{aligned} \tag{3.19}$$

Hence it follows from the above inequalities (3.17)–(3.19) that

$$\begin{aligned}
&\|u_{n+1}^{(1)} - u_n^{(1)}\| \\
&= \|\alpha_n^{(1)} u_n^{(1)} - \alpha_{n-1}^{(1)} u_{n-1}^{(1)} + (1-\alpha_n^{(1)}) T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u_n^{(2)} - (1-\alpha_{n-1}^{(1)}) T_{t^{(1)}}^{\lambda_n^{(1)}} u_{n-1}^{(2)}\| \\
&\le \alpha_n^{(1)} \|u_n^{(1)} - u_{n-1}^{(1)}\| + |\alpha_n^{(1)} - \alpha_{n-1}^{(1)}| \cdot \|u_{n-1}^{(1)}\| + (1-\alpha_n^{(1)}) \|T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u_n^{(2)} - T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u_{n-1}^{(2)}\| \\
&\quad + \|(1-\alpha_n^{(1)}) T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u_{n-1}^{(2)} - (1-\alpha_{n-1}^{(1)}) T_{t^{(1)}}^{\lambda_n^{(1)}} u_{n-1}^{(2)}\| \\
&\le \alpha_n^{(1)} \|u_n^{(1)} - u_{n-1}^{(1)}\| + |\alpha_n^{(1)} - \alpha_{n-1}^{(1)}| \cdot \|u_{n-1}^{(1)}\| + (1-\alpha_n^{(1)})(1-\lambda_{n+1}^{(1)}\tau^{(1)}) \|u_n^{(2)} - u_{n-1}^{(2)}\| \\
&\quad + |\alpha_n^{(1)} - \alpha_{n-1}^{(1)}| \cdot \|Tu_{n-1}^{(2)}\| + |(1-\alpha_n^{(1)})\lambda_{n+1}^{(1)} - (1-\alpha_{n-1}^{(1)})\lambda_n^{(1)}| \cdot t^{(1)} \|F(Tu_{n-1}^{(2)})\| \\
&= \alpha_n^{(1)} \|u_n^{(1)} - u_{n-1}^{(1)}\| + |\alpha_n^{(1)} - \alpha_{n-1}^{(k)}| \cdot (\|u_{n-1}^{(1)}\| + \|Tu_{n-1}^{(k+1)}\|) \\
&\quad + (1-\alpha_n^{(1)})(1-\lambda_{n+1}^{(1)}\tau^{(1)}) \|u_n^{(2)} - u_{n-1}^{(2)}\| \\
&\quad + |(1-\alpha_n^{(1)})\lambda_{n+1}^{(1)} - (1-\alpha_{n-1}^{(1)})\lambda_n^{(1)}| \cdot t^{(1)} \|F(Tu_{n-1}^{(2)})\|.
\end{aligned} \tag{3.20}$$

Let us substitute (3.19) into (3.20), then we have

$$
\begin{aligned}
&\|u_{n+1}^{(1)}-u_n^{(1)}\|\\
&\le \left(\alpha_n^{(1)}+(1-\alpha_n^{(1)})(1-\lambda_{n+1}^{(1)}\tau^{(1)})\right)\|u_n^{(1)}-u_{n-1}^{(1)}\|+\sum_{k=1}^{m-1}|\alpha_n^{(k)}-\alpha_{n-1}^{(k)}|\cdot\left(\|u_{n-1}^{(1)}\|+\|Tu_{n-1}^{(k+1)}\|\right)\\
&\quad+\sum_{k=1}^{m-1}\left|(1-\alpha_n^{(k)})\lambda_{n+1}^{(k)}-(1-\alpha_{n-1}^{(k)})\lambda_n^{(k)}\right|t^{(k)}\cdot\|FTu_{n-1}^{(k+1)}\|\\
&\quad+(1-\alpha_n^{(m)})\lambda_{n+1}^{(m)}-(1-\alpha_{n-1}^{(m)})\lambda_n^{(m)}t^{(m)}\|FTu_{n-1}^{(1)}\|+|\alpha_n^{(m)}-\alpha_{n-1}^{(m)}|\left(\|u_{n-1}^{(1)}\|+\|Tu_{n-1}^{(1)}\|\right)\\
&=\left(1-(1-\alpha_n^{(1)})\lambda_{n+1}^{(1)}\tau^{(1)}\right)\|u_n^{(1)}-u_{n-1}^{(1)}\|+(1-\alpha_n^{(1)})\lambda_{n+1}^{(1)}\tau^{(1)}\nu_n+\delta_n,
\end{aligned}
\tag{3.21}
$$

where

$$
\begin{aligned}
\delta_n&=\sum_{k=1}^{m-1}|\alpha_n^{(k)}-\alpha_{n-1}^{(k)}|\cdot\left(\|u_{n-1}^{(1)}\|+\|Tu_{n-1}^{(k+1)}\|\right)+|\alpha_n^{(m)}-\alpha_{n-1}^{(m)}|\cdot\left(\|u_{n-1}^{(1)}\|+\|Tu_{n-1}^{(1)}\|\right),\\
\nu_n&=\frac{1}{(1-\alpha_n^{(1)})\lambda_{n+1}^{(1)}\tau^{(1)}}\Bigg(\left|(1-\alpha_n^{(m)})\lambda_{n+1}^{(m)}-(1-\alpha_{n-1}^{(m)})\lambda_n^{(m)}\right|t^{(m)}\|FTu_{n-1}^{(1)}\|\\
&\qquad+\sum_{k=1}^{m-1}\left|(1-\alpha_n^{(k)})\lambda_{n+1}^{(k)}-(1-\alpha_{n-1}^{(k)})\lambda_n^{(k)}\right|t^{(k)}\|FTu_{n-1}^{(k+1)}\|\Bigg).
\end{aligned}
\tag{3.22}
$$

We put

$$
\begin{aligned}
\xi&=\sup\left\{\|u_n^{(1)}\|:n\ge 0\right\}+\sup\left\{\|Tu_n^{(k)}\|:n\ge 0,\ k=1,2,\dots,m\right\}\\
&\quad+\sup\left\{\|FTu_n^{(k)}\|:n\ge 0,\ k=1,2,\dots,m\right\},\\
M&=\|u^*\|+\left(\sum_{k=1}^{m}t^{(k)}\right)\|F(u^*)\|+\xi.
\end{aligned}
\tag{3.23}
$$

Then $\delta_n\le 2M\sum_{k=1}^{m}|\alpha_n^{(k)}-\alpha_{n-1}^{(k)}|\to 0$, as $n\to\infty$, and

$$
\begin{aligned}
\nu_n&\le\left(\sum_{k=1}^{m}t^{(k)}\right)M\frac{1}{(1-\alpha_n^{(1)})\lambda_{n+1}^{(1)}\tau^{(1)}}\\
&\quad\times\left(\sum_{k=2}^{m}\left|(1-\alpha_n^{(k)})\lambda_{n+1}^{(k)}-(1-\alpha_{n-1}^{(k)})\lambda_n^{(k)}\right|+\left|(1-\alpha_n^{(1)})\lambda_{n+1}^{(1)}-(1-\alpha_{n-1}^{(1)})\lambda_n^{(1)}\right|\right).
\end{aligned}
\tag{3.24}
$$

From (ii)–(iv), we obtain $v_n \to 0$ as $n \to \infty$. Furthermore, from (i), $\sum_{n=1}^{\infty}\delta_n < \infty$. By Lemma 2.1, we deduce that $\|u_{n+1}^{(1)} - u_n^{(1)}\| \to 0$ as $n \to \infty$.

Step 4. Let $\|u_n^{(1)} - Tu_n^{(1)}\| \to 0$ as $n \to \infty$. From Steps 2 and 3, we have

$$\|u_n^{(1)} - Tu_n^{(1)}\| \le \|u_{n+1}^{(1)} - u_n^{(1)}\| + \|u_{n+1}^{(1)} - Tu_n^{(1)}\| \longrightarrow 0 \tag{3.25}$$

as $n \to \infty$.

Step 5. Let $\limsup_{n\to\infty}\langle -F(u^*), Tu_n^{(k)} - u^*\rangle \le 0$, for $k = 2, 3, \dots, m$. Let $\{Tu_{n_i}^{(1)}\}$ be a subsequence of $\{Tu_n^{(1)}\}$ such that

$$\limsup_{n\to\infty}\left\langle -F(u^*), Tu_n^{(1)} - u^*\right\rangle = \lim_{i\to\infty}\left\langle -F(u^*), Tu_{n_i}^{(1)} - u^*\right\rangle. \tag{3.26}$$

Without loss of generality, we assume that $Tu_{n_i}^{(1)} \to \tilde{u}^{(1)}$ weakly for some $\tilde{u}^{(1)} \in H$. By Step 4, we derive $u_{n_i}^{(1)} \to \tilde{u}^{(1)}$ weakly. But by Lemma 2.2 and Step 4, we have $\tilde{u}^{(1)} \in \mathrm{Fix}(T) = C$. Since u^* is the unique solution of the VI(F, C), we obtain

$$\limsup_{n\to\infty}\left\langle -F(u^*), Tu_n^{(1)} - u^*\right\rangle = \left\langle -F(u^*), \tilde{u}^{(1)} - u^*\right\rangle \le 0. \tag{3.27}$$

From the proof of Step 2,

$$\|Tu_n^{(k)} - Tu_n^{(1)}\| \le \|u_n^{(2)} - u_n^{(1)}\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{3.28}$$

for $k = 2, 3, \dots, m$. Then

$$\begin{aligned}
\limsup_{n\to\infty}\left\langle -F(u^*), Tu_n^{(k)} - u^*\right\rangle &= \limsup_{n\to\infty}\left[\left\langle -F(u^*), Tu_n^{(k)} - Tu_n^{(1)}\right\rangle + \left\langle -F(u^*), Tu_n^{(1)} - u^*\right\rangle\right] \\
&\le \limsup_{n\to\infty}\left\langle -F(u^*), Tu_n^{(k)} - Tu_n^{(1)}\right\rangle + \limsup_{n\to\infty}\left\langle -F(u^*), Tu_n^{(1)} - u^*\right\rangle \\
&= \limsup_{n\to\infty}\left\langle -F(u^*), Tu_n^{(1)} - u^*\right\rangle \\
&\le 0,
\end{aligned} \tag{3.29}$$

for $k = 2, 3, \dots, m$.

Step 6. Let $u_n^{(1)} \to u^*$ in norm and so does $\{u_n^{(k)}\}$ for $k = 2, 3, \ldots, m$. Indeed using Lemma 2.3 and (3.7) we get

$$
\begin{aligned}
&\|u_{n+1}^{(1)} - u^*\|^2 \\
&= \|\alpha_n^{(1)}(u_n^{(1)} - u^*) + (1-\alpha_n^{(1)})(T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u_n^{(2)} - u^*)\|^2 \\
&\le \alpha_n^{(1)}\|u_n^{(1)} - u^*\|^2 + (1-\alpha_n^{(1)})\|T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u_n^{(2)} - u^*\|^2 \\
&= \alpha_n^{(1)}\|u_n^{(1)} - u^*\|^2 + (1-\alpha_n^{(1)})\|(T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u_n^{(2)} - T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u^*) + (T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u^* - u^*)\|^2 \\
&\le \alpha_n^{(1)}\|u_n^{(1)} - u^*\|^2 + (1-\alpha_n^{(1)})[\|T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u_n^{(2)} - T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u^*\|^2 + 2\langle T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u^* - u^*, T_{t^{(1)}}^{\lambda_{n+1}^{(1)}} u_n^{(2)} - u^*\rangle] \\
&\le \alpha_n^{(1)}\|u_n^{(1)} - u^*\|^2 + (1-\alpha_n^{(1)})(1-\lambda_{n+1}^{(1)}\tau^{(1)})^2\|u_n^{(2)} - u^*\|^2 \\
&\quad + 2t^{(1)}\lambda_{n+1}^{(1)}\langle -F(u^*), Tu_n^{(2)} - \lambda_{n+1}^{(1)} t^{(1)} F(Tu_n^{(2)}) - u^*\rangle \\
&\le \alpha_n^{(1)}\|u_n^{(1)} - u^*\|^2 + (1-\alpha_n^{(1)})(1-\lambda_{n+1}^{(1)}\tau^{(1)})^2\left[\|u_n^{(1)} - u^*\| + (1-\alpha_n^{(2)})\max_{j=2}^{m}\{\lambda_{n+1}^{(j)}\}\left(\sum_{j=2}^{m} t^{(j)}\right)\|F(u^*)\|\right]^2 \\
&\quad + 2(1-\alpha_n^{(1)})\lambda_{n+1}^{(1)} t^{(1)}\langle -F(u^*), Tu_n^{(2)} - \lambda_{n+1}^{(1)} t^{(1)} F(Tu_n^{(2)}) - u^*\rangle \\
&\le (\alpha_n^{(1)} + (1-\alpha_n^{(1)})(1-\lambda_{n+1}^{(1)}\tau^{(1)}))\|u_n^{(1)} - u^*\|^2 \\
&\quad + 2(1-\alpha_n^{(1)})(1-\alpha_n^{(2)})(1-\lambda_{n+1}^{(1)}\tau^{(1)})^2\max_{2\le j\le m}\{\lambda_{n+1}^{(j)}\}\left(\sum_{j=2}^{m} t^{(j)}\right)\|F(u^*)\|\|u_n^{(1)} - u^*\| \\
&\quad + (1-\alpha_n^{(1)})(1-\alpha_n^{(2)})^2(1-\lambda_{n+1}^{(1)}\tau^{(1)})^2\left(\max_{2\le j\le m}\{\lambda_{n+1}^{(j)}\}\right)^2\left(\sum_{j=2}^{m} t^{(j)}\right)^2\|F(u^*)\|^2 \\
&\quad + 2t^{(1)}\lambda_{n+1}^{(1)}\langle -F(u^*), Tu_n^{(2)} - u^* - \lambda_{n+1}^{(1)} t^{(1)} F(Tu_n^{(2)})\rangle \\
&\le (1-(1-\alpha_n^{(1)})\lambda_{n+1}^{(1)}\tau^{(1)})\|u_n^{(1)} - u^*\|^2 + 2(1-\alpha_n^{(1)})(1-\alpha_n^{(2)})(1-\lambda_{n+1}^{(1)}\tau^{(1)})^2\max_{2\le j\le m}\{\lambda_{n+1}^{(j)}\}\left(\sum_{j=2}^{m} t^{(j)}\right)M^2 \\
&\quad + (1-\alpha_n^{(1)})(1-\alpha_n^{(2)})^2(1-\lambda_{n+1}^{(1)}\tau^{(1)})^2\left(\max_{2\le j\le m}\{\lambda_{n+1}^{(j)}\}\right)^2\left(\sum_{j=2}^{m} t^{(j)}\right)^2 M^2 \\
&\quad + 2\lambda_{n+1}^{(1)} t^{(1)}\langle -F(u^*), Tu_n^{(2)} - u^* - \lambda_{n+1}^{(1)} t^{(1)} F(Tu_n^{(2)})\rangle \\
&\le (1-(1-\alpha_n^{(1)})\lambda_{n+1}^{(1)}\tau^{(1)})\|u_n^{(1)} - u^*\|^2 + (1-\alpha_n^{(1)})\lambda_{n+1}^{(1)}\tau^{(1)}
\end{aligned}
$$

$$\times\left[\frac{2t^{(1)}\langle -F(u^*), Tu_n^{(2)} - u^* - \lambda_{n+1}^{(1)}t^{(1)}F(Tu_n^{(2)})\rangle}{\tau^{(1)}(1-\alpha_n^{(1)})}\right.$$
$$+\frac{2(1-\alpha_n^{(2)})(1-\lambda_{n+1}^{(1)}\tau^{(1)})^2\max_{2\le j\le m}\{\lambda_{n+1}^{(j)}\}(\sum_{j=2}^m t^{(j)})M^2}{\tau^{(1)}\lambda_{n+1}^{(1)}}$$
$$\left.+\frac{(1-\alpha_n^{(2)})^2(1-\lambda_{n+1}^{(1)}\tau^{(1)})^2(\max_{2\le j\le m}\{\lambda_{n+1}^{(j)}\})^2(\sum_{j=2}^m t^{(j)})^2M^2}{\tau^{(1)}\lambda_{n+1}^{(1)}}\right]$$
$$\le (1-(1-\alpha_n^{(1)})\lambda_{n+1}^{(1)}\tau^{(1)})\|u_n^{(1)}-u^*\|^2 + (1-\alpha_n^{(1)})\lambda_{n+1}^{(1)}\tau^{(1)}$$
$$\times\left[\frac{2t^{(1)}\langle -F(u^*), Tu_n^{(2)} - u^* - \lambda_{n+1}^{(1)}t^{(1)}F(Tu_n^{(2)})\rangle}{\tau^{(1)}(1-\alpha_n^{(1)})} + \frac{2}{\tau^{(1)}}(1-\alpha_n^{(2)})(1-\lambda_{n+1}^{(1)}\tau^{(1)})^2\left(\sum_{j=2}^m t^{(j)}\right)M^2\right.$$
$$\left.+\frac{1}{\tau^{(1)}}(1-\alpha_n^{(2)})^2(1-\lambda_{n+1}^{(1)}\tau^{(1)})^2\left(\max_{2\le j\le m}\{\lambda_{n+1}^{(j)}\}\right)\left(\sum_{j=2}^m t^{(j)}\right)^2M^2\right]. \tag{3.30}$$

From (ii), (iii), and Step 5, we have $\lim_{n\to\infty}\alpha_n^{(1)} = \lim_{n\to\infty}\lambda_n^{(k)} = 0$, for $k = 1,2,\ldots,m$ and $\lim_{n\to\infty}\alpha_n^{(k)} = 1$, for $k = 2,\ldots,m$, $\limsup_{n\to\infty}\langle -F(u^*), Tu_n^{(2)} - u^*\rangle \le 0$, and $\{F(Tu_n^{(2)})\}$ is bounded; by Lemma 2.4, we conclude that

$$\limsup_{n\to\infty}\left(\frac{2t^{(1)}\langle -F(u^*), Tu_n^{(2)} - u^* - \lambda_{n+1}^{(1)}t^{(1)}F(Tu_n^{(2)})\rangle}{\tau^{(1)}(1-\alpha_n^{(1)})} + \frac{2(1-\alpha_n^{(2)})(1-\lambda_{n+1}^{(1)}\tau^{(1)})^2(\sum_{j=2}^m t^{(j)})M^2}{\tau^{(1)}}\right.$$
$$\left.+\frac{(1-\alpha_n^{(2)})^2(1-\lambda_{n+1}^{(1)}\tau^{(1)})^2(\max_{2\le j\le m}\{\lambda_{n+1}^{(j)}\})(\sum_{j=2}^m t^{(j)})^2M^2}{\tau^{(1)}}\right)$$
$$\le \limsup_{n\to\infty}\frac{2t^{(1)}}{\tau^{(1)}(1-\alpha_n^{(1)})}\cdot\langle -F(u^*), Tu_n^{(2)} - u^*\rangle + \limsup_{n\to\infty}\frac{2(t^{(1)})^2\lambda_{n+1}^{(1)}}{\tau^{(1)}(1-\alpha_n^{(1)})}\cdot\langle -F(u^*), -F(Tu_n^{(2)})\rangle$$
$$\le 0+0=0. \tag{3.31}$$

Consequently from Lemma 2.1, we obtain $\|u_n^{(1)} - u^*\| \to 0$ and hence it follows from $\|u_n^{(k)} - u_n^{(1)}\| \to 0$, for $k = 2,3,\ldots,m$, that $\|u_n^{(k)} - u^*\| \to 0$, for $k = 2,3,\ldots,m$.

Acknowledgment

This research was partially supported by Grant no. NSC95-2115-M-039-001- from the National Science Council of Taiwan.

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