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Research Article

Two Inequalities for ${}_r\phi_r$ and Applications

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We use the q-binomial formula to establish two inequalities for the basic hypergeometric series ${}_r\phi_r$. As applications of the inequalities, we discuss the convergence of q-series.

1. Introduction and main results

q-series, which is also called basic hypergeometric series, plays a very important role in many fields, such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials, physics, and so on. Inequality technique is one of the useful tools in the study of special functions. There are many papers about it [1–6]. In [1], the authors gave some inequalities for hypergeometric functions. In this paper, we derive two inequalities for the basic hypergeometric series ${}_r\phi_r$, which can be used to study the convergence of q-series.

The main results of this paper are the following two inequalities.

Theorem 1.1. *Suppose a_i, b_i, and z are any real numbers such that $|b_i|<1$ with $i=1,2,\ldots,r$. Then*

$$\left|{}_r\phi_r\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_r\end{matrix};q,z\right)\right| \le (-|z|;q)_\infty \prod_{i=1}^{r}\frac{(-|a_i|;q)_\infty}{(|b_i|;q)_\infty}. \tag{1.1}$$

Theorem 1.2. *Suppose a_i, b_i, and z are any real numbers such that $z<0$ and $|a_i|<1, |b_i|<1$ with $i=1,2,\ldots,r$. Then*

$${}_r\phi_r\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_r\end{matrix};q,z\right) \ge (z;q)_\infty \prod_{i=1}^{r}\frac{(|a_i|;q)_\infty}{(-|b_i|;q)_\infty}. \tag{1.2}$$

Before the proof of the theorems, we recall some definitions, notations, and known results which will be used in this paper. Throughout the whole paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.3}$$

We also adopt the following compact notation for multiple q-shifted factorial:

$$(a_1,a_2,\ldots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \tag{1.4}$$

where n is an integer or ∞.

The q-binomial theorem [7]

$$\sum_{k=0}^{\infty}\frac{(a;q)_k z^k}{(q;q)_k} = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1. \tag{1.5}$$

Replacing a with $1/a$, and z with az and then setting $a=0$, we get

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} z^k}{(q;q)_k} = (z;q)_\infty. \tag{1.6}$$

Heine introduced the ${}_r\phi_s$ basic hypergeometric series, which is defined by [7]

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix};q,z\right) = \sum_{k=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_k}{(q,b_1,b_2,\ldots,b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k. \tag{1.7}$$

2. The proof of Theorem 1.1

In this section, we use the q-binomial formula (1.6) to prove Theorem 1.1.

Proof. Since

$$\begin{aligned}\left|(a;q)_n\right| &= \prod_{i=0}^{n-1}\left|1-aq^i\right| \le \prod_{i=0}^{n-1}(1+|a|q^i) = (-|a|;q)_n \le (-|a|;q)_\infty,\\ \left|(b;q)_n\right| &= \prod_{i=0}^{n-1}\left|1-bq^i\right| \ge \prod_{i=0}^{n-1}(1-|b|q^i) = (|b|;q)_n \ge (|b|;q)_\infty > 0,\end{aligned} \tag{2.1}$$

we have

$$\left|\frac{(a;q)_n}{(b;q)_n}\right| \le \frac{(-|a|;q)_\infty}{(|b|;q)_\infty}. \tag{2.2}$$

Hence,

$$\left|\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_r;q)_n}\right| \le \prod_{i=1}^{r}\frac{(-|a_i|;q)_\infty}{(|b_i|;q)_\infty}. \tag{2.3}$$

Multiplying both sides of (2.3) by

$$\frac{q^{\binom{n}{2}}|-z|^n}{(q;q)_n} \tag{2.4}$$

gives

$$\left|\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_r;q)_n}(-1)^n q^{\binom{n}{2}} z^n\right| \le \frac{q^{\binom{n}{2}}|z|^n}{(q;q)_n}\cdot\prod_{i=1}^{r}\frac{(-|a_i|;q)_\infty}{(|b_i|;q)_\infty}. \tag{2.5}$$

Consequently,

$$\begin{aligned}
&\left|{}_r\phi_r\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ q,b_1,b_2,\ldots,b_r\end{matrix};q,z\right)\right| \\
&\quad=\left|\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_r;q)_n}(-1)^n q^{\binom{n}{2}} z^n\right| \le \sum_{n=0}^{\infty}\left|\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_r;q)_n}(-1)^n q^{\binom{n}{2}} z^n\right| \\
&\quad=\sum_{n=0}^{\infty}\left\{\frac{q^{\binom{n}{2}}|z|^n}{(q;q)_n}\cdot\left|\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_r;q)_n}\right|\right\} \le \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}|z|^n}{(q;q)_n}\cdot\prod_{i=1}^{r}\frac{(-|a_i|;q)_\infty}{(|b_i|;q)_\infty}.
\end{aligned} \tag{2.6}$$

Using the q-binomial theorem (1.6) obtains

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}|z|^n}{(q;q)_n} = (-|z|;q)_\infty. \tag{2.7}$$

Substituting (2.7) into (2.6) gets (1.1). Thus, we complete the proof. □

3. The proof of Theorem 1.2

In this section, we use again the q-binomial formula (1.6) in order to prove Theorem 1.2.

Proof. Since

$$\begin{aligned}
(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) \ge \prod_{i=0}^{n-1}(1-|a|q^i) = (|a|;q)_n \ge (|a|;q)_\infty > 0,\\
0<(b;q)_n = \prod_{i=0}^{n-1}(1-bq^i) \le \prod_{i=0}^{n-1}(1+|b|q^i) = (-|b|;q)_n \le (-|b|;q)_\infty,
\end{aligned} \tag{3.1}$$

we have

$$\frac{(a;q)_n}{(b;q)_n} \ge \frac{(|a|;q)_\infty}{(-|b|;q)_\infty}. \tag{3.2}$$

Hence,

$$\frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} \geq \prod_{i=1}^{r} \frac{(|a_i|; q)_\infty}{(-|b_i|; q)_\infty}. \tag{3.3}$$

Multiplying both sides of (3.3) by

$$\frac{(-1)^n q^{\binom{n}{2}} z^n}{(q;q)_n} > 0 \tag{3.4}$$

gives

$$\frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_r; q)_n} (-1)^n q^{\binom{n}{2}} z^n \geq \frac{(-1)^n q^{\binom{n}{2}} z^n}{(q;q)_n} \cdot \prod_{i=1}^{r} \frac{(|a_i|; q)_\infty}{(-|b_i|; q)_\infty}. \tag{3.5}$$

Consequently, we have

$$\begin{aligned} {}_r\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ q, b_1, b_2, \ldots, b_r \end{matrix}; q, z\right) &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_r; q)_n} (-1)^n q^{\binom{n}{2}} z^n \\ &\geq \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{(q;q)_n} \cdot \prod_{i=1}^{r} \frac{(|a_i|; q)_\infty}{(-|b_i|; q)_\infty}. \end{aligned} \tag{3.6}$$

Using the q-binomial theorem (1.6) obtains

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{(q;q)_n} = (z;q)_\infty. \tag{3.7}$$

Substituting (3.7) into (3.6) gets (1.2). Thus, we complete the proof. □

4. Some applications of the inequalities

Convergence is an important problem in the study of q-series. There are some results about it. For example, Ito used inequality technique to give a sufficient condition for convergence of a special q-series called Jackson integral [8]. In this section, we use the inequalities obtained in this paper to give some sufficient conditions for convergence of a q-series and sufficient conditions for divergence of a q-series.

Theorem 4.1. *Suppose a_i, b_i, and z are any real numbers such that $|b_i| < 1$ with $i = 1, 2, \ldots, r$. Let $\{c_n\}$ and $\{d_n\}$ be any number series. If*

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = p < 1, \quad |d_{n+1}| \leq |d_n|, \quad n = 1, 2, \ldots, \tag{4.1}$$

then the q-series

$$\sum_{n=0}^{\infty} c_n \, {}_r\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_r \end{matrix}; q, d_n\right) \tag{4.2}$$

converges absolutely.

Proof. Letting $z = d_n$ in (1.1) and then multiplying both sides of (1.1) by $|c_n|$ give

$$\left| c_n\, {}_r\phi_r\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; q, d_n\right)\right| \le |c_n|(-|d_n|;q)_\infty \prod_{i=1}^{r} \frac{(-|a_i|;q)_\infty}{(|b_i|;q)_\infty}. \tag{4.3}$$

From $|d_{n+1}| \le |d_n|$, we know

$$\frac{(-|d_{n+1}|;q)_\infty}{(-|d_n|;q)_\infty} \le 1. \tag{4.4}$$

The ratio test shows that the series

$$\sum_{n=0}^{\infty} |c_n|(-|d_n|;q)_\infty \prod_{i=1}^{r} \frac{(-|a_i|;q)_\infty}{(|b_i|;q)_\infty} \tag{4.5}$$

is convergent. From (4.3), it is sufficient to establish that (4.2) is absolutely convergent. □

Theorem 4.2. *Suppose a_i, b_i, and z are any real numbers such that $|a_i| < 1, |b_i| < 1$ with $i = 1, 2, \dots, r$. Let $\{c_n\}$ and $\{d_n\}$ be any number series. If*

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = p > 1, \quad d_{n+1} \le d_n < 0, \quad n = 0, 1, 2, \dots, \tag{4.6}$$

then the q-series

$$\sum_{n=0}^{\infty} c_n\, {}_r\phi_r\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; q, d_n\right) \tag{4.7}$$

diverges.

Proof. Letting $z = d_n$ in (1.2) and then multiplying both sides of (1.2) by $|c_n|$ give

$$|c_n|\, {}_r\phi_r\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; q, d_n\right) \ge |c_n|(d_n;q)_\infty \prod_{i=1}^{r} \frac{(|a_i|;q)_\infty}{(-|b_i|;q)_\infty}. \tag{4.8}$$

From $d_{n+1} \le d_n$, we know

$$\frac{(d_{n+1};q)_\infty}{(d_n;q)_\infty} \ge 1. \tag{4.9}$$

Since

$$\lim_{n\to\infty} \inf \frac{|c_{n+1}|(d_{n+1};q)_\infty}{|c_n|(d_n;q)_\infty} \ge \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} > 1, \tag{4.10}$$

there exists an integer N_0 such that, when $n > N_0$,

$$\left|c_n\right|{}_r\phi_r\left(\begin{matrix}a_1, a_2, \ldots, a_r\\ b_1, b_2, \ldots, b_r\end{matrix}; q, d_n\right) \geq \left|c_n\right|\left(d_n; q\right)_\infty \prod_{i=1}^{r} \frac{\left(\left|a_i\right|; q\right)_\infty}{\left(-\left|b_i\right|; q\right)_\infty} > \left|c_{N_0}\right|\left(d_{N_0}; q\right)_\infty \prod_{i=1}^{r} \frac{\left(\left|a_i\right|; q\right)_\infty}{\left(-\left|b_i\right|; q\right)_\infty} > 0. \tag{4.11}$$

So, (4.7) diverges. □

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