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Research Article

A New Subclass of Analytic Functions Involving Al-Oboudi Differential Operator

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The main object of this paper is to introduce and investigate a new subclass of normalized analytic functions in the open unit disc $\mathbb{U}$ which is defined by Al-Oboudi differential operator. Coefficient inequalities, extreme points, and integral means inequalities for fractional derivative for this class are given.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions f normalized by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$.

For $f \in \mathcal{A}$, Al-Oboudi [1] introduced the following operator:

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z), \quad \delta \geq 0 \tag{1.3}$$

$$D^n f(z) = D_\delta\big(D^{n-1} f(z)\big), \quad (n \in \mathbb{N} = 1, 2, 3, \ldots). \tag{1.4}$$

If f is given by (1.1), then from (1.3) and (1.4) we see that

$$D^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\delta]^n a_j z^j, \quad \big(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\big). \tag{1.5}$$

When $\delta = 1$, we get Sălăgean differential operator [2].

Definition 1.1. Let $\mathcal{S}_{m,n,\delta}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of functions f which satisfy the inequality

$$\operatorname{Re}\left(\frac{D^m f(z)}{D^n f(z)}\right) > \alpha \tag{1.6}$$

for some $0 \le \alpha < 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, and all $z \in \mathbb{U}$.

The object of the present paper is to investigate the coefficient bounds, extreme points, and integral mean inequalities for fractional derivatives of functions belonging to the class $\mathcal{S}_{m,n,\delta}(\alpha)$.

2. Coefficient inequalities

Our first theorem gives a sufficient condition for $f \in \mathcal{A}$ to belong to the class $\mathcal{S}_{m,n,\delta}(\alpha)$.

Theorem 2.1. *Let $f(z) \in \mathcal{A}$ satisfy*

$$\sum_{j=2}^{\infty} \Psi(m,n,j,\delta,\alpha)|a_j| \le 2(1-\alpha), \tag{2.1}$$

where

$$\Psi(m,n,j,\delta,\alpha) = \left|[1+(j-1)\delta]^m - (1+\alpha)[1+(j-1)\delta]^n\right| + [1+(j-1)\delta]^m + (1-\alpha)[1+(j-1)\delta]^n \tag{2.2}$$

for some α $(0 \le \alpha < 1)$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, δ $(\delta \ge 0)$. Then $f(z) \in \mathcal{S}_{m,n,\delta}(\alpha)$.

Proof. Suppose that (2.1) is true for $\alpha(0 \le \alpha < 1)$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $\delta(\delta \ge 0)$. For $f(z) \in \mathcal{A}$, define the function $F(z)$ by

$$F(z) = \frac{D^m f(z)}{D^n f(z)} - \alpha. \tag{2.3}$$

It suffices to show that

$$\left|\frac{F(z)-1}{F(z)+1}\right| < 1 \quad (z \in \mathbb{U}). \tag{2.4}$$

We note that

$$\begin{aligned}
\left|\frac{F(z)-1}{F(z)+1}\right| &= \left|\frac{D^m f(z)/D^n f(z) - \alpha - 1}{D^m f(z)/D^n f(z) - \alpha + 1}\right| \\
&= \left|\frac{D^m f(z) - (1+\alpha)D^n f(z)}{D^m f(z) + (1-\alpha)D^n f(z)}\right| \\
&= \left|\frac{\alpha - \sum_{j=2}^{\infty}\left([1+(j-1)\delta]^m - (1+\alpha)[1+(j-1)\delta]^n\right)a_j z^{j-1}}{(2-\alpha) + \sum_{j=2}^{\infty}\left([1+(j-1)\delta]^m + (1-\alpha)[1+(j-1)\delta]^n\right)a_j z^{j-1}}\right| \\
&\le \frac{\alpha + \sum_{j=2}^{\infty}\left|[1+(j-1)\delta]^m - (1+\alpha)[1+(j-1)\delta]^n\right||a_j||z|^{j-1}}{(2-\alpha) - \sum_{j=2}^{\infty}\left([1+(j-1)\delta]^m + (1-\alpha)[1+(j-1)\delta]^n\right)|a_j||z|^{j-1}} \\
&< \frac{\alpha + \sum_{j=2}^{\infty}\left|[1+(j-1)\delta]^m - (1+\alpha)[1+(j-1)\delta]^n\right||a_j|}{(2-\alpha) - \sum_{j=2}^{\infty}\left([1+(j-1)\delta]^m + (1-\alpha)[1+(j-1)\delta]^n\right)|a_j|}.
\end{aligned} \tag{2.5}$$

The last expression is bounded above by 1 if

$$\begin{aligned}\alpha + \sum_{j=2}^{\infty}\left|[1+(j-1)\delta]^m - (1+\alpha)[1+(j-1)\delta]^n\right||a_j| \\ \le (2-\alpha) - \sum_{j=2}^{\infty}\left([1+(j-1)\delta]^m + (1-\alpha)[1+(j-1)\delta]^n\right)|a_j|\end{aligned} \tag{2.6}$$

which is equivalent to condition (2.1). This completes the proof of Theorem 2.1. □

Example 2.2. The function $f(z)$ given by

$$f(z) = z + \sum_{j=2}^{\infty}\frac{2(2+\gamma)(1-\alpha)\epsilon_j}{(j+\gamma)(j+1+\gamma)\Psi(m,n,j,\delta,\alpha)}z^j \tag{2.7}$$

belongs to the class $\mathcal{S}_{m,n,\delta}(\alpha)$ for $\gamma > -2$, $0 \le \alpha < 1$, $\epsilon_j \in \mathbb{C}$, and $|\epsilon_j| = 1$.

We now derive the coefficient inequalities for $f(z)$ belonging to the class $\mathcal{S}_{m,n,\delta}(\alpha)$.

Theorem 2.3. *If $f(z) \in \mathcal{S}_{m,n,\delta}(\alpha)$, then for $k \ge 2$,*

$$\begin{aligned}|a_k| \le \frac{\beta}{|v_k|}\Bigg\{1 &+ \beta\sum_{j=2}^{k-1}\frac{[1+(j-1)\delta]^n}{|v_j|} + \beta^2\sum_{j_2>j_1}^{k-1}\sum_{j_1=2}^{k-2}\frac{([1+(j_1-1)\delta][1+(j_2-1)\delta])^n}{|v_{j_1}v_{j_2}|} \\ &+ \beta^3\sum_{j_3>j_2}^{k-1}\sum_{j_2>j_1}^{k-2}\sum_{j_1=2}^{k-3}\frac{([1+(j_1-1)\delta][1+(j_2-1)\delta][1+(j_3-1)\delta])^n}{|v_{j_1}v_{j_2}v_{j_3}|} + \cdots \\ &+ \beta^{k-2}\prod_{j=2}^{k-1}\frac{[1+(j-1)\delta]^n}{|v_j|}\Bigg\},\end{aligned} \tag{2.8}$$

where $\beta = 2(1-\alpha)$ and $v_k = [1+(k-1)\delta]^m - [1+(k-1)\delta]^n$.

Proof. Define the function $p(z)$ by

$$p(z) = \frac{1}{1-\alpha}\left(\frac{D^m f(z)}{D^n f(z)} - \alpha\right) = 1 + \sum_{j=1}^{\infty}c_j z^j. \tag{2.9}$$

Since $p(z)$ is the Carathéodory function, we have that

$$|c_j| \le 2 \quad (j = 1,2,3,\ldots). \tag{2.10}$$

The definition of $p(z)$ implies that

$$\frac{1}{(1-\alpha)}\left(D^m f(z) - \alpha D^n f(z)\right) = D^n f(z)\left(1 + \sum_{j=1}^{\infty}c_j z^j\right). \tag{2.11}$$

Since

$$D^n f(z) = z + \sum_{j=2}^{\infty} \left[1 + (j-1)\delta\right]^n a_j z^j \quad (n \in \mathbb{N}_0), \tag{2.12}$$

we have

$$\begin{aligned} \frac{D^m f(z) - \alpha D^n f(z)}{1-\alpha} &= z + \frac{(1+\delta)^m - \alpha(1+\delta)^n}{1-\alpha} a_2 z^2 + \frac{(1+2\delta)^m - \alpha(1+2\delta)^n}{1-\alpha} a_3 z^3 + \cdots \\ &\quad + \frac{[1+(k-1)\delta]^m - \alpha[1+(k-1)\delta]^n}{1-\alpha} a_k z^k + \cdots, \\ D^n f(z)\left(1 + \sum_{j=1}^{\infty} c_j z^j\right) &= \left(z + \sum_{j=2}^{\infty} \left[1+(j-1)\delta\right]^n a_j z^j\right)\left(1 + c_1 z + \cdots + c_k z^k + \cdots\right). \end{aligned} \tag{2.13}$$

Therefore, (2.11) shows that

$$\begin{aligned} &z + \frac{(1+\delta)^m - \alpha(1+\delta)^n}{1-\alpha} a_2 z^2 + \frac{(1+2\delta)^m - \alpha(1+2\delta)^n}{1-\alpha} a_3 z^3 + \cdots + \frac{[1+(k-1)\delta]^m - \alpha[1+(k-1)\delta]^n}{1-\alpha} a_k z^k + \cdots \\ &\quad = \left(z + \sum_{j=2}^{\infty} \left[1+(j-1)\delta\right]^n a_j z^j\right)\left(1 + c_1 z + \cdots + c_k z^k + \cdots\right). \end{aligned} \tag{2.14}$$

If we consider the coefficients of z^k of the both sides in the above equality, then we find that

$$\left(\frac{\left[1+(k-1)\delta\right]^m - \alpha\left[1+(k-1)\delta\right]^n}{1-\alpha} - \left[1+(k-1)\delta\right]^n\right) a_k = \sum_{j=1}^{k-1} \left[1+(k-j-1)\delta\right]^n a_{k-j} c_j. \tag{2.15}$$

Therefore,

$$\begin{aligned} |a_k| &= \frac{1-\alpha}{\left|[1+(k-1)\delta]^m - [1+(k-1)\delta]^n\right|} \left|\sum_{j=1}^{k-1} [1+(k-j-1)\delta]^n a_{k-j} c_j\right| \\ &\le \frac{1-\alpha}{\left|[1+(k-1)\delta]^m - [1+(k-1)\delta]^n\right|} \left(\sum_{j=1}^{k-1} [1+(k-j-1)\delta]^n |a_{k-j}| |c_j|\right) \\ &\le \frac{2(1-\alpha)}{\left|[1+(k-1)\delta]^m - [1+(k-1)\delta]^n\right|} \left(\sum_{j=1}^{k-1} [1+(k-j-1)\delta]^n |a_{k-j}|\right), \end{aligned} \tag{2.16}$$

since $|c_j| \le 2\ (j = 1, 2, 3\ldots)$. Thus, for $\beta = 2(1-\alpha)$ and $v_k = [1+(k-1)\delta]^m - [1+(k-1)\delta]^n$, we obtain

$$\begin{aligned}
|a_k| \le \beta\frac{1}{|v_k|}\Bigg\{ & 1 + (1+\delta)^n\frac{\beta}{|v_2|} + (1+2\delta)^n\frac{\beta}{|v_3|} + (1+3\delta)^n\frac{\beta}{|v_4|} + \cdots + (1+(k-2)\delta)^n\frac{\beta}{|v_{k-1}|} \\
& + (1+\delta)^n(1+2\delta)^n\frac{\beta^2}{|v_2v_3|} + (1+\delta)^n(1+3\delta)^n\frac{\beta^2}{|v_2v_4|} \\
& + (1+\delta)^n(1+4\delta)^n\frac{\beta^2}{|v_2v_5|} + \cdots + (1+\delta)^n(1+(k-2)\delta)^n\frac{\beta^2}{|v_2v_{k-1}|} \\
& + (1+2\delta)^n(1+3\delta)^n\frac{\beta^2}{|v_3v_4|} + (1+2\delta)^n(1+4\delta)^n\frac{\beta^2}{|v_3v_5|} + \cdots \\
& + (1+2\delta)^n(1+(k-2)\delta)^n\frac{\beta^2}{|v_3v_{k-1}|} + \cdots \\
& + (1+\delta)^n(1+2\delta)^n(1+3\delta)^n\frac{\beta^3}{|v_2v_3v_4|} + (1+\delta)^n(1+3\delta)^n(1+4\delta)^n\frac{\beta^3}{|v_2v_4v_5|} + \cdots \\
& + (1+\delta)^n(1+(k-3)\delta)^n(1+(k-2)\delta)^n\frac{\beta^3}{|v_2v_{k-2}v_{k-1}|} + \beta^{k-2}\prod_{j=2}^{k-1}\frac{[1+(j-1)\delta]^n}{|v_j|}\Bigg\} \\
= \frac{\beta}{|v_k|}\Bigg\{ & 1 + \beta\sum_{j=2}^{k-1}\frac{[1+(j-1)\delta]^n}{|v_j|} + \beta^2\sum_{j_2>j_1}^{k-1}\sum_{j_1=2}^{k-2}\frac{([1+(j_1-1)\delta][1+(j_2-1)\delta])^n}{|v_{j_1}v_{j_2}|} \\
& + \beta^3\sum_{j_3>j_2}^{k-1}\sum_{j_2>j_1}^{k-2}\sum_{j_1=2}^{k-3}\frac{([1+(j_1-1)\delta, 1+(j_2-1)\delta, 1+(j_3-1)\delta])^n}{|v_{j_1}v_{j_2}v_{j_3}|} + \cdots + \beta^{k-2}\prod_{j=2}^{k-1}\frac{[1+(j-1)\delta]^n}{|v_j|}\Bigg\}.
\end{aligned} \tag{2.17}$$

This completes the proof of Theorem 2.3. □

If we take $\delta = 1$ in Theorems 2.1 and 2.3, we can get the results due to Sümer Eker and Owa [3].

3. Extreme points

In view of Theorem 2.1, we now introduce the subclass $\widetilde{\mathcal{S}}_{m,n,\delta}(\alpha) \subset \mathcal{S}_{m,n,\delta}(\alpha)$, which consists of function

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad (a_j \ge 0) \tag{3.1}$$

whose Taylor-Maclaurin coefficients satisfy inequality (2.1). Now, let us determine extreme points of the class $\widetilde{\mathcal{S}}_{m,n,\delta}(\alpha)$.

Theorem 3.1. *Let* $f_1(z) = z$ *and*

$$f_j(z) = z + \frac{2(1-\alpha)}{\Psi(m,n,j,\delta,\alpha)} z^j \quad (j = 2, 3, \ldots), \tag{3.2}$$

where $\Psi(m,n,j,\delta,\alpha)$ *is given by* (2.2).

Then $f \in \widetilde{\mathcal{S}}_{m,n}(\alpha)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{j=1}^{\infty} \eta_j f_j(z), \tag{3.3}$$

where $\eta_j > 0$ *and* $\sum_{j=1}^{\infty} \eta_j = 1$.

Proof. Suppose that

$$f(z) = \sum_{j=1}^{\infty} \eta_j f_j(z) = z + \sum_{j=2}^{\infty} \eta_j \frac{2(1-\alpha)}{\Psi(m,n,j,\delta,\alpha)} z^j. \tag{3.4}$$

Then

$$\sum_{j=2}^{\infty} \Psi(m,n,j,\delta,\alpha) \frac{2(1-\alpha)}{\Psi(m,n,j,\delta,\alpha)} \eta_j = 2(1-\alpha) \sum_{j=2}^{\infty} \eta_j = 2(1-\alpha)(1-\eta_1) < 2(1-\alpha), \tag{3.5}$$

which shows that f satisfies condition (2.1) and therefore $f \in \widetilde{\mathcal{S}}_{m,n,\delta}(\alpha)$.

Conversely, suppose that $f \in \widetilde{\mathcal{S}}_{m,n,\delta}(\alpha)$. Since

$$a_j \le \frac{2(1-\alpha)}{\Psi(m,n,j,\delta,\alpha)} \quad (j = 2, 3, \ldots), \tag{3.6}$$

we may set

$$\begin{aligned} \eta_j &= \frac{\Psi(m,n,j,\delta,\alpha)}{2(1-\alpha)} a_j, \\ \eta_1 &= 1 - \sum_{j=2}^{\infty} \eta_j. \end{aligned} \tag{3.7}$$

Then we obtain

$$f(z) = \sum_{j=1}^{\infty} \eta_j f_j(z), \tag{3.8}$$

which completes the proof of Theorem 3.1. □

Corollary 3.2. *The extreme points of* $\widetilde{\mathcal{S}}_{m,n,\delta}(\alpha)$ *are the functions* $f_1(z) = z$ *and*

$$f_j(z) = z + \frac{2(1-\alpha)}{\Psi(m,n,j,\delta,\alpha)} z^j \quad (j = 2, 3, \ldots), \tag{3.9}$$

where $\Psi(m,n,j,\delta,\alpha)$ *is given by* (2.2).

4. Integral means inequalities for fractional derivative

We will make use of the following definitions of fractional derivatives by Owa [4], and Srivastava and Owa [5].

Definition 4.1. The fractional derivative of order λ is defined, for a function f, by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\lambda} d\xi \quad (0 \leq \lambda < 1), \tag{4.1}$$

where f is an analytic function in a simply connected region of z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 4.2. Under the hypotheses of Definition 4.1, the fractional derivative of order $p+\lambda$ is defined, for a function f, by

$$D_z^{p+\lambda} f(z) = \frac{d^p}{dz^p} D_z^\lambda f(z) \quad (0 \leq \lambda < 1; p \in \mathbb{N}_0). \tag{4.2}$$

It readily follows from (4.1) that

$$D_z^\lambda z^k = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda} \quad (0 \leq \lambda < 1, k \in \mathbb{N}). \tag{4.3}$$

Further, we need the concept of subordination between analytic functions [6] and a subordination theorem of Littlewood in our investigation.

Definition 4.3. For two functions f and g, analytic in $\mathbb{U}$, say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \tag{4.4}$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \tag{4.5}$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{4.6}$$

In 1925, Littlewood [7] proved the following subordination theorem.

Lemma 4.4. *If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^\mu d\theta \leqq \int_0^{2\pi} |g(z)|^\mu d\theta. \tag{4.7}$$

Theorem 4.5. *Let $f(z) \in \widetilde{\mathcal{S}}_{m,n,\delta}(\alpha)$ and suppose that*

$$\sum_{j=2}^{\infty}(j-p)_{p+1}a_j \leq \frac{2(1-\alpha)\Gamma(k+1)\Gamma(3-\lambda-p)}{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda-p)\Gamma(2-p)} \tag{4.8}$$

for some $j \geq p$, $0 \leq \lambda < 1$, and $(j-p)_{p+1}$ denote the Pochhammer symbol defined by $(j-p)_{p+1} = (j-p)(j-p+1)\cdots j$. Also let the function

$$f_k(z) = z + \frac{2(1-\alpha)}{\Psi(m,n,k,\delta,\alpha)}z^k \quad (k \geq 2). \tag{4.9}$$

If there exists an analytic function $w(z)$ given by

$$(w(z))^{k-1} = \frac{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)}\sum_{j=2}^{\infty}(j-p)_{p+1}\frac{\Gamma(j-p)}{\Gamma(j+1-\lambda-p)}a_j z^{j-1}, \quad (k \geq p), \tag{4.10}$$

then for $z = re^{i\theta}$ and $0 < r < 1$,

$$\int_0^{2\pi}\left|D_z^{p+\lambda}f(z)\right|^{\mu}d\theta \leq \int_0^{2\pi}\left|D_z^{p+\lambda}f_k(z)\right|^{\mu}d\theta \quad (0 \leq \lambda < 1, \mu > 0). \tag{4.11}$$

Proof. By virtue of the fractional derivative formula (4.3) and Definition 4.2, we find from (3.1) that

$$\begin{aligned} D_z^{p+\lambda}f(z) &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left\{1+\sum_{j=2}^{\infty}\frac{\Gamma(2-\lambda-p)\Gamma(j+1)}{\Gamma(j+1-\lambda-p)}a_j z^{j-1}\right\} \\ &= \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left\{1+\sum_{j=2}^{\infty}\Gamma(2-\lambda-p)(j-p)_{p+1}\Phi(j)a_j z^{j-1}\right\}, \end{aligned} \tag{4.12}$$

where

$$\Phi(j) = \frac{\Gamma(j-p)}{\Gamma(j+1-\lambda-p)} \quad (0 \leq \lambda < 1; j \geq p). \tag{4.13}$$

Since $\Phi(j)$ is a decreasing function of j, we have

$$0 < \Phi(j) \leq \Phi(2) = \frac{\Gamma(2-p)}{\Gamma(3-\lambda-p)}. \tag{4.14}$$

Similarly, from (4.3), (4.9), and Definition 4.2, we obtain

$$D_z^{p+\lambda}f_k(z) = \frac{z^{1-\lambda-p}}{\Gamma(2-\lambda-p)}\left\{1+\frac{2(1-\alpha)\Gamma(2-\lambda-p)\Gamma(k+1)}{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda-p)}z^{k-1}\right\}. \tag{4.15}$$

For $z = re^{i\theta}$, $0 < r < 1$, we must show that

$$\begin{aligned}
&\int_0^{2\pi} \left| 1 + \sum_{j=2}^{\infty} \Gamma(2-\lambda-p)(j-p)_{p+1}\Phi(j)a_j z^{j-1} \right|^{\mu} d\theta \\
&\quad \le \int_0^{2\pi} \left| 1 + \frac{2(1-\alpha)\Gamma(2-\lambda-p)\Gamma(k+1)}{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda-p)} z^{k-1} \right|^{\mu} d\theta \quad (\mu > 0).
\end{aligned} \tag{4.16}$$

Thus by applying Littlewood's subordination theorem, it would be suffice to show that

$$1 + \sum_{j=2}^{\infty} \Gamma(2-\lambda-p)(j-p)_{p+1}\Phi(j)a_j z^{j-1} \prec 1 + \frac{2(1-\alpha)\Gamma(2-\lambda-p)\Gamma(k+1)}{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda-p)} z^{k-1}. \tag{4.17}$$

By setting

$$1 + \sum_{j=2}^{\infty} \Gamma(2-\lambda-p)(j-p)_{p+1}\Phi(j)a_j z^{j-1} = 1 + \frac{2(1-\alpha)\Gamma(2-\lambda-p)\Gamma(k+1)}{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda-p)} w(z)^{k-1}, \tag{4.18}$$

we find that

$$(w(z))^{k-1} = \frac{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)} \sum_{j=2}^{\infty} (j-p)_{p+1}\Phi(j)a_j z^{j-1} \tag{4.19}$$

which readily yields $w(0) = 0$. Further, we prove that the analytic function $w(z)$ satisfies $|w(z)| < 1$, $z \in \mathbb{U}$ for (4.10). We know that

$$\begin{aligned}
|w(z)|^{k-1} &\le \left| \frac{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)} \sum_{j=2}^{\infty} (j-p)_{p+1}\Phi(j)a_j z^{j-1} \right| \\
&\le \frac{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)} \sum_{j=2}^{\infty} (j-p)_{p+1}\Phi(j)a_j |z|^{j-1} \\
&\le |z| \frac{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)} \Phi(2) \sum_{j=2}^{\infty} (j-p)_{p+1} a_j \\
&= |z| \frac{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda-p)}{2(1-\alpha)\Gamma(k+1)} \frac{\Gamma(2-p)}{\Gamma(3-\lambda-p)} \sum_{j=2}^{\infty} (j-p)_{p+1} a_j \\
&\le |z| < 1
\end{aligned} \tag{4.20}$$

by means of the hypothesis of Theorem 4.5.

As special case $p = 0$, Theorem 4.5 readily yields. □

Corollary 4.6. *Let $f(z) \in \widetilde{\mathcal{S}}_{m,n,\delta}(\alpha)$ and suppose that*

$$\sum_{j=2}^{\infty} j a_j \le \frac{2(1-\alpha)\Gamma(k+1)\Gamma(3-\lambda)}{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda)} \tag{4.21}$$

for some $0 \leq \lambda < 1$. *Also let the function*

$$f_k(z) = z + \frac{2(1-\alpha)}{\Psi(m,n,k,\delta,\alpha)} z^k \quad (k \geq 2). \tag{4.22}$$

If there exists an analytic function $w(z)$ *given by*

$$(w(z))^{k-1} = \frac{\Psi(m,n,k,\delta,\alpha)\Gamma(k+1-\lambda)}{2(1-\alpha)\Gamma(k+1)} \sum_{j=2}^{\infty} \frac{\Gamma(j+1)}{\Gamma(j+1-\lambda)} a_j z^{j-1}, \tag{4.23}$$

then for $z = re^{i\theta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left| D_z^{\lambda} f(z) \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| D_z^{\lambda} f_k(z) \right|^{\mu} d\theta \quad (0 \leq \lambda < 1, \mu > 0). \tag{4.24}$$

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