Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 346279, 9 pages
doi:10.1155/2008/346279

Research Article

Neighborhoods of Starlike and Convex Functions Associated with Parabola

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Received 16 June 2008; Accepted 11 August 2008

Recommended by Ramm Mohapatra

Let f be a normalized analytic function defined on the unit disk and $f_\lambda(z) := (1-\lambda)z + \lambda f(z)$ for $0 < \lambda \leq 1$. For $\alpha > 0$, a function $f \in \mathcal{SP}(\alpha, \lambda)$ if $zf'(z)/f_\lambda(z)$ lies in the parabolic region $\Omega := \{w : |w - \alpha| < \operatorname{Re} w + \alpha\}$. Let $\mathcal{CP}(\alpha, \lambda)$ be the corresponding class consisting of functions f such that $(zf'(z))'/f'_\lambda(z)$ lies in the region Ω. For an appropriate $\delta > 0$, the δ-neighbourhood of a function $f \in \mathcal{CP}(\alpha, \lambda)$ is shown to consist of functions in the class $\mathcal{SP}(\alpha, \lambda)$.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions $f(z)$ defined on the open unit disk $\Delta := \{z : |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\mathcal{ST}$ and $\mathcal{CV}$ be the well-known subclasses of $\mathcal{S}$, respectively, consisting of starlike and convex functions. Given $\delta \geq 0$, Ruscheweyh [1] defined the δ-*neighbourhood* $N_\delta(f)$ of a function:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A} \tag{1.1}$$

to be the set

$$N_\delta(f) := \left\{ g(z) : g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \text{ and } \sum_{k=2}^{\infty} k\left|a_k - b_k\right| \leq \delta \right\}. \tag{1.2}$$

Ruscheweyh [1] proved among other results that $N_{1/4}(f) \subset \mathcal{ST}$ for $f \in \mathcal{CV}$. Sheil-Small and Silvia [2] introduced more general notions of neighbourhood of an analytic function. These included noncoefficient neighbourhoods as well. Problems related to the neighbourhoods of analytic functions were considered by many others, for example, see [3–12].

An analytic function $f(z) \in \mathcal{S}$ is *uniformly convex* [13] if for every circular arc γ contained in Δ with center $\zeta \in \Delta$, the image arc $f(\gamma)$ is convex. Denote the class of all uniformly convex functions by $\mathcal{UCV}$. In [14, 15], it was shown that a function $f(z)$ is uniformly convex if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \Delta). \tag{1.3}$$

The class $\mathcal{S}_p$ of functions $zf'(z)$ with $f(z)$ in $\mathcal{UCV}$ was introduced in [15] and clearly $f(z)$ is in $\mathcal{S}_p$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)}-1\right| \quad (z \in \Delta). \tag{1.4}$$

The class $\mathcal{UCV}$ of uniformly convex functions and the class $\mathcal{S}_p$ of parabolic starlike functions were investigated in [16–20]. A survey of these functions can be found in [21].

Let $\alpha > 0$ and $0 < \lambda \le 1$. The class $\mathcal{SP}(\alpha,\lambda)$ consists of functions $f \in \mathcal{S}$ satisfying

$$\operatorname{Re}\left\{\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}\right\} + \alpha > \left|\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-\alpha\right| \quad (z \in \Delta). \tag{1.5}$$

By writing $f_\lambda(z) := (1-\lambda)z + \lambda f(z)$, the inequality in (1.5) can be written as

$$\operatorname{Re}\left\{\frac{zf'(z)}{f_\lambda(z)}\right\} + \alpha > \left|\frac{zf'(z)}{f_\lambda(z)}-\alpha\right|. \tag{1.6}$$

Observe that (1.5) defines a parabolic region. More explicitly, $f \in \mathcal{SP}(\alpha,\lambda)$ if and only if the values of the functional $zf'(z)/f_\lambda(z)$ lie in the parabolic region Ω, where

$$\Omega := \{w : |w-\alpha| < \operatorname{Re} w + \alpha\} = \{w = u+iv : v^2 < 4\alpha u\}. \tag{1.7}$$

The geometric properties of the function f_λ when f belongs to certain classes of starlike and convex functions were investigated by several authors [22–27]; in particular, we recall the following result.

Theorem 1.1 (see [25]). *Let $f \in \mathcal{CV}$. Then,*

(1) $f_\lambda(z) := (1-\lambda)z + \lambda f(z) \in \mathcal{ST}$ *if and only if* $\lambda \in \mathbb{C}$ *and* $|\lambda - 1| \le 1/3$*;*

(2) *if* $f''(0) = 0$, *then* $f_\lambda \in \mathcal{ST}$ *for* $\lambda \in [0,1]$.

For $\alpha > 0$ and $0 < \lambda \le 1$, the class $\mathcal{CP}(\alpha,\lambda)$ consists of functions $f \in S$ satisfying

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{f_\lambda'(z)}\right\} + \alpha > \left|\frac{(zf'(z))'}{f_\lambda'(z)}-\alpha\right| \quad (z \in \Delta). \tag{1.8}$$

When $\lambda = 1$, the classes $\mathcal{SP}(\alpha,\lambda)$ and $\mathcal{CP}(\alpha,\lambda)$ reduce, respectively, to the classes introduced in [28, 29]. Besides several other properties, the authors in [28, 29] also gave geometric interpretations, respectively, of the classes $\mathcal{SP}(\alpha) := \mathcal{SP}(\alpha,1)$ and $\mathcal{CP}(\alpha) := \mathcal{CP}(\alpha,1)$.

In this paper, the neighbourhood $N_\delta(f)$ for functions $f \in \mathcal{CP}(\alpha,\lambda)$ is investigated. It is shown that all functions $g \in N_\delta(f)$ are in the class $\mathcal{SP}(\alpha,\lambda)$ for a certain $\delta > 0$. It is of interest to note that the conditions on δ obtained here coincide with those in [30] for corresponding results in the classes $\mathcal{CP}(\alpha)$ and $\mathcal{SP}(\alpha)$.

2. Main results

In order to obtain the main results, a characterization of the class $\mathcal{SP}(\alpha,\lambda)$ in terms of the functions in another class $\mathcal{SP}'(\alpha,\lambda)$ is needed. For a fixed $\alpha > 0$, $0 < \lambda \le 1$, and $t \ge 0$, a function $H_{t,\lambda}$ is said to be in the class $\mathcal{SP}'(\alpha,\lambda)$ if the function $H_{t,\lambda}$ is of the form

$$H_{t,\lambda}(z) := \frac{1}{1-\left(t\pm 2\sqrt{\alpha t}i\right)}\left[\frac{z}{(1-z)^2}-\frac{\left[z-(1-\lambda)z^2\right]}{1-z}\left(t\pm 2\sqrt{\alpha t}i\right)\right] \quad (z\in\Delta). \tag{2.1}$$

Recall that for any two functions $f(z)$ and $g(z)$ given by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \qquad g(z)=z+\sum_{n=2}^{\infty}b_nz^n, \tag{2.2}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z) := z+\sum_{n=2}^{\infty}a_nb_nz^n =: (g*f)(z). \tag{2.3}$$

Lemma 2.1. *Let $\alpha > 0$ and $0 < \lambda \le 1$. A function f is in the class $\mathcal{SP}(\alpha,\lambda)$ if and only if*

$$\frac{1}{z}\left(f*H_{t,\lambda}\right)(z)\neq 0 \quad (z\in\Delta), \tag{2.4}$$

for all $H_{t,\lambda}\in\mathcal{SP}'(\alpha,\lambda)$.

Proof. Let $f\in\mathcal{SP}(\alpha,\lambda)$. Then, the image of Δ under $w = zf'(z)/f_\lambda(z)$ lies in the parabolic region $\Omega(\alpha,\lambda)=\{w : |w-\alpha| < \operatorname{Re} w+\alpha\}$ so that

$$\frac{zf'(z)}{f_\lambda(z)}\neq t\pm 2\sqrt{\alpha t}i \quad (z\in\Delta,\ t\ge 0). \tag{2.5}$$

Thus $f\in\mathcal{SP}(\alpha,\lambda)$ if and only if

$$\frac{zf'(z)-\left[t\pm 2\sqrt{\alpha t}i\right]f_\lambda(z)}{z\left(1-\left[t\pm 2\sqrt{\alpha t}i\right]\right)}\neq 0 \quad (z\in\Delta,\ t\ge 0), \tag{2.6}$$

or equivalently

$$\frac{1}{z}\left(f*H_{t,\lambda}\right)(z)\neq 0 \quad (z\in\Delta,\ t\ge 0), \tag{2.7}$$

for all $H_{t,\lambda}\in\mathcal{SP}'(\alpha,\lambda)$. □

Lemma 2.2. *Let $\alpha > 0$ and $0 < \lambda \le 1$. If*

$$H_{t,\lambda}(z) := z+\sum_{k=2}^{\infty}h_{k,\lambda}(t)z^k\in\mathcal{SP}'(\alpha,\lambda), \tag{2.8}$$

then

$$\left|h_{k,\lambda}(t)\right|\le\begin{cases}\dfrac{k}{2\sqrt{\alpha(1-\alpha)}}, & 0<\alpha<\dfrac{1}{2},\\ k, & \alpha\ge\dfrac{1}{2},\end{cases} \tag{2.9}$$

for all $t\ge 0$.

Proof. Writing $H_{t,\lambda}(z) = z + \sum_{k=2}^{\infty} h_{k,\lambda}(t) z^k$, and comparing coefficients of z^k in (2.1), one obtains

$$h_{k,\lambda}(t) = \frac{k - \lambda\left(t \pm 2\sqrt{\alpha t} i\right)}{1 - \left(t \pm 2\sqrt{\alpha t} i\right)}. \tag{2.10}$$

Thus, for $t \geq 0$ and $0 < \lambda \leq 1$,

$$\begin{aligned} \left|h_{k,\lambda}(t)\right|^2 &= \left|\frac{k - \lambda\left(t \pm 2\sqrt{\alpha t} i\right)}{1 - \left(t \pm 2\sqrt{\alpha t} i\right)}\right|^2 \\ &= \frac{(k - \lambda t)^2 + 4\lambda^2 \alpha t}{(1 - t)^2 + 4\alpha t} \\ &= \lambda^2 + \frac{(k - \lambda)(k + \lambda - 2\lambda t)}{(1 - t)^2 + 4\alpha t} \\ &\leq \lambda^2 + \frac{(k^2 - \lambda^2)}{(1 - t)^2 + 4\alpha t}. \end{aligned} \tag{2.11}$$

It is easy to see that

$$(1 - t)^2 + 4\alpha t \geq \begin{cases} 4\alpha(1 - \alpha), & 0 < \alpha < \dfrac{1}{2}, \\ 1, & \alpha \geq \dfrac{1}{2}. \end{cases} \tag{2.12}$$

Hence, for $0 < \alpha < 1/2$, and $0 < \lambda \leq 1$,

$$\left|h_{k,\lambda}(t)\right|^2 \leq \lambda^2 + \frac{(k^2 - \lambda^2)}{4\alpha(1 - \alpha)} \leq \frac{k^2}{4\alpha(1 - \alpha)}, \tag{2.13}$$

and, for $\alpha \geq 1/2$,

$$\left|h_{k,\lambda}(t)\right|^2 \leq \lambda^2 + k^2 - \lambda^2 = k^2. \tag{2.14}$$

□

Lemma 2.3. *For each complex number ϵ and $f \in \mathcal{A}$, define the function F_ϵ by*

$$F_\epsilon(z) := \frac{f(z) + \epsilon z}{1 + \epsilon}. \tag{2.15}$$

Let $\alpha > 0$, $0 < \lambda \leq 1$, and $F_\epsilon \in \mathcal{SP}(\alpha, \lambda)$ for $|\epsilon| < \delta$ for some $\delta > 0$. Then

$$\left|\frac{1}{z}(f * H_{t,\lambda})(z)\right| \geq \delta \quad (z \in \Delta), \tag{2.16}$$

for every $H_{t,\lambda} \in \mathcal{SP}'(\alpha, \lambda)$.

Proof. If $F_\epsilon \in \mathcal{SP}(\alpha,\lambda)$ for $|\epsilon| < \delta$, where $\delta > 0$ is fixed, then by Lemma 2.1, for all $H_{t,\lambda} \in \mathcal{SP}'(\alpha,\lambda)$, it follows that

$$\frac{1}{z}(F_\epsilon * H_{t,\lambda})(z) \neq 0, \quad (z \in \Delta), \tag{2.17}$$

or equivalently

$$\frac{(f * H_{t,\lambda})(z) + \epsilon z}{(1+\epsilon)z} \neq 0. \tag{2.18}$$

Since $|\epsilon| < \delta$, it easily follows that

$$\left|\frac{1}{z}(f * H_{t,\lambda})(z)\right| \geq \delta. \tag{2.19}$$

□

Theorem 2.4. *Let $\alpha > 0$ and $0 < \lambda \leq 1$. Let $f \in \mathcal{A}$ and $\delta > 0$. For a complex number ϵ with $|\epsilon| < \delta$, let the function F_ϵ, defined by* (2.15)*, be in $\mathcal{SP}(\alpha,\lambda)$. Then, $N_{\delta'}(f) \subset \mathcal{SP}(\alpha,\lambda)$ for*

$$\delta' := \begin{cases} 2\delta\sqrt{\alpha(1-\alpha)}, & 0 < \alpha < \dfrac{1}{2}, \\ \delta, & \alpha \geq \dfrac{1}{2}. \end{cases} \tag{2.20}$$

Proof. Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in N_{\delta'}(f)$. For any $H_{t,\lambda} \in \mathcal{SP}'(\alpha,\lambda)$,

$$\begin{aligned} \left|\frac{1}{z}(g * H_{t,\lambda})(z)\right| &= \left|\frac{1}{z}(f * H_{t,\lambda})(z) + \frac{1}{z}((g-f) * H_{t,\lambda})(z)\right| \\ &\geq \left|\frac{1}{z}(f * H_{t,\lambda})(z)\right| - \left|\frac{1}{z}((g-f) * H_{t,\lambda})(z)\right|. \end{aligned} \tag{2.21}$$

Using Lemma 2.3, it follows that

$$\begin{aligned} \left|\frac{1}{z}(g * H_{t,\lambda})(z)\right| &\geq \delta - \left|\sum_{k=2}^{\infty} \frac{(b_k - a_k) h_{k,\lambda}(t) z^k}{z}\right| \\ &\geq \delta - \sum_{k=2}^{\infty} |b_k - a_k| \left|h_{k,\lambda}(t)\right|. \end{aligned} \tag{2.22}$$

Using Lemma 2.2 and noting that $g \in N_{\delta'}(f)$, and whence $\sum_{k=2}^{\infty} k|b_k - a_k| < \delta'$, thus

$$\left|\frac{1}{z}(g * H_{t,\lambda})(z)\right| \geq \begin{cases} \delta - \dfrac{\delta'}{2\sqrt{\alpha(1-\alpha)}}, & 0 < \alpha < \dfrac{1}{2}, \\ \delta - \delta', & \alpha \geq \dfrac{1}{2}. \end{cases} \tag{2.23}$$

Therefore, $|(1/z)(g * H_{t,\lambda})(z)| \neq 0$ in Δ for all $H_{t,\lambda} \in \mathcal{SP}(\alpha,\lambda)$ if

$$\delta' = \begin{cases} 2\delta\sqrt{\alpha(1-\alpha)}, & 0 < \alpha < \dfrac{1}{2}, \\ \delta, & \alpha \geq \dfrac{1}{2}. \end{cases} \tag{2.24}$$

By Lemma 2.1, this means that $g \in \mathcal{SP}(\alpha,\lambda)$. This proves that $N_{\delta'}(f) \subset \mathcal{SP}(\alpha,\lambda)$. □

We need the following well-known result in [31] concerning convolution of functions.

Lemma 2.5 (see [31]). *Let $f \in \mathcal{CV}$, $g \in \mathcal{ST}$, and suppose F is any analytic function defined on Δ. Then*

$$\frac{f(z)*g(z)F(z)}{f(z)*g(z)} \subset \overline{\mathrm{co}}F(\Delta), \quad (z \in \Delta), \tag{2.25}$$

where $\overline{\mathrm{co}}$ stands for the closed convex hull.

Lemma 2.6. *If $f \in \mathcal{CV}$, $g \in \mathcal{SP}(\alpha,\lambda)$, and $g_\lambda \in \mathcal{ST}$, then $f*g \in \mathcal{SP}(\alpha,\lambda)$.*

Proof. The conclusion $f*g \in \mathcal{SP}(\alpha,\lambda)$ is a consequence of Lemma 2.5 on noting that

$$\frac{z\left(f(z)*g(z)\right)'}{\left(f(z)*g(z)\right)_\lambda} = \frac{f(z)*zg'(z)}{f(z)*g_\lambda(z)} = \frac{f(z)*g_\lambda(z)\left(zg'(z)/g_\lambda(z)\right)}{f(z)*g_\lambda(z)} \subset \overline{\mathrm{co}}\left\{\frac{zg'(z)}{g_\lambda(z)} : z \in \Delta\right\}. \tag{2.26}$$

□

Theorem 2.7. *Let $\alpha > 0$ and $0 \le \lambda \le 1$. If $f \in \mathcal{CP}(\alpha,\lambda)$ and $f_\lambda \in \mathcal{CV}$, then the function F_ϵ defined by* (2.15) *belongs to $\mathcal{SP}(\alpha,\lambda)$ for $|\epsilon| < 1/4$.*

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{CP}(\alpha,\lambda)$. Then,

$$F_\epsilon(z) = \frac{f(z) + \epsilon z}{1+\epsilon} = \left(f*h\right)(z), \tag{2.27}$$

where

$$h(z) := \frac{z - \left(\epsilon/(1+\epsilon)\right)z^2}{1-z} = \frac{z - \rho z^2}{1-z} \quad (z \in \Delta), \tag{2.28}$$

and $\rho := \epsilon/(1+\epsilon)$. Note that

$$\operatorname{Re}\frac{zh'(z)}{h(z)} \ge \frac{1}{2} - \frac{|\rho|}{1-|\rho|} > 0 \quad (z \in \Delta), \tag{2.29}$$

if $|\rho| \le 1/3$. This clearly holds for $|\epsilon| < 1/4$. Thus, the function $h(z)$ is starlike for $|\epsilon| < 1/4$ and whence the function

$$\int_0^z \frac{h(t)}{t}dt = h(z)*\log\frac{1}{1-z} \quad (z \in \Delta) \tag{2.30}$$

is in $\mathcal{CV}$. Since $f(z) \in \mathcal{CP}(\alpha,\lambda)$, the function $zf'(z) \in \mathcal{SP}(\alpha,\lambda)$. Also $f_\lambda(z) \in \mathcal{CV}$ implies that $(zf'(z))_\lambda \in \mathcal{ST}$. By Lemma 2.6,

$$F_\epsilon(z) = (f*h)(z) = zf'(z)*\left(h(z)*\log\frac{1}{1-z}\right) \in \mathcal{SP}(\alpha,\lambda), \tag{2.31}$$

for $|\epsilon| < 1/4$. □

Theorem 2.8. *Let $\alpha > 0$ and $0 \le \lambda \le 1$. If $f \in \mathcal{CP}(\alpha, \lambda)$ and $f_\lambda \in \mathcal{CV}$, then $N_{\delta'}(f) \subset \mathcal{SP}(\alpha, \lambda,)$, where*

$$\delta' := \begin{cases} \frac{1}{2}\sqrt{\alpha(1-\alpha)}, & 0 < \alpha < \frac{1}{2}, \\ \frac{1}{4}, & \alpha \ge \frac{1}{2}. \end{cases} \tag{2.32}$$

Proof. The result follows from Theorems 2.4 and 2.7 by taking $\delta = 1/4$ in Theorem 2.4. □

Remark 2.9. It is interesting to note that the values of δ' in Theorems 2.4 and 2.8 are independent of λ. In fact, the conclusion of Theorems 2.4, 2.7, and 2.8 is the same as found in [29] for the subclasses $\mathcal{SP}(\alpha)$ and $\mathcal{CP}(\alpha)$.

To prove our next result, we need the following results.

Lemma 2.10 (see [32]). *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that the mapping $\Phi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$ satisfies $\Phi(i\rho, \sigma; z) \notin \Omega$ for $z \in \Delta$, and for all real ρ, σ such that $\sigma \le -n(1+\rho^2)/2$. If the function $p(z) = 1 + c_n z^n + \cdots$ is analytic in Δ and $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \Delta$, then $\operatorname{Re} p(z) > 0$.*

Lemma 2.11. *Let $0 \le \lambda \le 1/3$. If $p(z) = 1 + cz + \cdots$ is analytic in Δ and*

$$\operatorname{Re}\left\{ \frac{p(z) + zp'(z)}{(1-\lambda) + \lambda p(z)} \right\} > 0, \tag{2.33}$$

then $\operatorname{Re} p(z) > 0$.

Proof. Let $\Omega := \{w : \operatorname{Re} w > 0\}$ and

$$\psi(r, s) := \frac{r + s}{(1-\lambda) + \lambda r}. \tag{2.34}$$

Then, the given inequality (2.33) can be written as $\psi(p(z), zp'(z); z) \in \Omega$. Since

$$\operatorname{Re} \psi(i\rho, \sigma; z) = \frac{\lambda\rho^2 + \sigma(1-\lambda)}{(1-\lambda)^2 + \lambda^2\rho^2} \le \frac{(3\lambda - 1)\rho^2 - (1-\lambda)}{2\left[(1-\lambda)^2 + \lambda^2\rho^2\right]} \le 0 \tag{2.35}$$

when $\rho \in \Re$ and $\sigma \le -(1+\rho^2)/2$, the condition of Lemma 2.10 is satisfied. Thus, $\operatorname{Re} p(z) > 0$. □

Theorem 2.12. *Let $0 \le \lambda \le 1/3$. If $f \in \mathcal{SP}(\alpha, \lambda)$, then $f_\lambda \in \mathcal{ST}$.*

Proof. If $f \in \mathcal{SP}(\alpha, \lambda)$, then

$$\operatorname{Re}\left\{ \frac{zf'(z)}{f_\lambda(z)} \right\} + \alpha > \left| \frac{zf'(z)}{f_\lambda(z)} - \alpha \right|, \tag{2.36}$$

and hence

$$\operatorname{Re} \frac{zf'(z)}{f_\lambda(z)} > 0. \tag{2.37}$$

Let the analytic function $p(z)$ be defined by

$$p(z) = \frac{f(z)}{z} \quad (z \in \mathcal{U}). \tag{2.38}$$

Computations show that

$$\operatorname{Re} \frac{p(z) + zp'(z)}{(1-\lambda) + \lambda p(z)} = \operatorname{Re} \frac{zf'(z)}{f_\lambda(z)} > 0. \tag{2.39}$$

By Lemma 2.11, we see that $\operatorname{Re} p(z) > 0$ or $\operatorname{Re}(f(z)/z) > 0$ in $\mathcal{U}$.

In view of (2.37), it follows from $\operatorname{Re}(f(z)/z) > 0$ and

$$\frac{zf'_\lambda(z)}{f_\lambda(z)} = \frac{1-\lambda}{1-\lambda+\lambda(f(z)/z)} + \lambda \frac{zf'(z)}{f_\lambda(z)} \tag{2.40}$$

that

$$\operatorname{Re} \frac{zf'_\lambda(z)}{f_\lambda(z)} > 0, \tag{2.41}$$

or equivalently $f_\lambda \in \mathcal{ST}$. □

As an immediate consequence, we have the following corollary.

Corollary 2.13. *Let* $0 \le \lambda \le 1/3$. *If* $f \in \mathcal{CP}(\alpha, \lambda)$, *then* $f_\lambda \in \mathcal{CV}$.

In view of this corollary, the statement that $f_\lambda \in \mathcal{CV}$ can be omitted from Theorems 2.7 and 2.8 if $0 \le \lambda \le 1/3$. Also clearly that $f \in \mathcal{CP}(\alpha, 1)$ implies $f_1 = f \in \mathcal{CV}$. Thus, Theorem 2.8 reduces to the corresponding result in [30] for $\lambda = 1$.

Acknowledgments

The first two authors acknowledge the support from the USM's RU grant, while the fourth author acknowledges the support from the University Research Council of Kent State University and Universiti Sains Malaysia. This work was completed while the third and fourth authors were visiting USM.

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