

Research Article

Fixed Points in Functional Inequalities

Choonkil Park

Department of Mathematics, Hanyang University, Seoul 133791, South Korea

Correspondence should be addressed to Choonkil Park, baak@hanyang.ac.kr

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Using fixed point methods, we prove the generalized Hyers-Ulam stability of the following functional inequalities $\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|$ and $\|f(x) + f(y) + 2f(z)\| \leq \|2f((x + y)/2 + z)\|$ in the spirit of Th. M. Rassias stability approach.

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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. In 1982, J. M. Rassias [6] followed the innovative approach of the Th. M. Rassias' theorem [4] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p+q \neq 1$.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (1.1)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [7] for mappings $f : X \rightarrow Y$, where X is a normed space and Y is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [9] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems

of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [10–20]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a *generalized metric* on X if d satisfies

- (1) $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1 (see [21, 22]). *Let (X, d) be a complete generalized metric space and let $J : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{1.2}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

- (1) $d(J^n x, J^{n+1} x) < \infty$, for all $n \geq n_0$;
- (2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;
- (3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
- (4) $d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.

In 1996, Isac and Th. M. Rassias [23] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [24–28]).

In [29], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\|, \tag{1.3}$$

then, f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y). \tag{1.4}$$

See also [30]. Gilányi [31] and Fechner [32] proved the generalized Hyers-Ulam stability of the functional inequality (1.3).

In this paper, using fixed point methods, we prove the generalized Hyers-Ulam stability of the functional inequalities

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|, \tag{1.5}$$

$$\|f(x) + f(y) + 2f(z)\| \leq \left\| 2f\left(\frac{x+y}{2} + z\right) \right\|, \tag{1.6}$$

in Banach spaces.

2. Fixed points and generalized Hyers-Ulam stability of the functional inequalities (1.5) and (1.6)

Throughout this paper, assume that X is a normed vector space with norm $\|\cdot\|$ and that Y is a Banach space with norm $\|\cdot\|$.

Using fixed point methods, we prove the generalized Hyers-Ulam stability of the functional inequality (1.5) in Banach spaces.

Theorem 2.1. *Let $f : X \rightarrow Y$ be an odd mapping for which there exists a function $\varphi : X^3 \rightarrow [0, \infty)$ such that there exists an $L < 1$ such that $\varphi(x, y, z) \leq (1/2)L\varphi(2x, 2y, 2z)$ for all $x, y, z \in X$, and*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| + \varphi(x, y, z) \quad (2.1)$$

for all $x, y, z \in X$. Then, there exists a unique Cauchy additive mapping $A : X \rightarrow Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{L}{2 - 2L} \varphi(x, x, -2x) \quad (2.2)$$

for all $x \in X$.

Proof. Consider the set

$$S := \{g : X \rightarrow Y\}, \quad (2.3)$$

and introduce the *generalized metric* on S :

$$d(g, h) = \inf \{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq K\varphi(x, x, -2x), \forall x \in X\}. \quad (2.4)$$

It is easy to show that (S, d) is complete. (See [33, proof of Theorem 2.5].)

Now we consider the linear mapping $J : S \rightarrow S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right) \quad (2.5)$$

for all $x \in X$.

It follows from [21, proof of Theorem 3.1] that

$$d(Jg, Jh) \leq Ld(g, h) \quad (2.6)$$

for all $g, h \in S$.

Since $f : X \rightarrow Y$ is odd, $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$. Letting $y = x$ and $z = -2x$ in (2.1), we get

$$\|2f(x) - f(2x)\| = \|2f(x) + f(-2x)\| \leq \varphi(x, x, -2x) \quad (2.7)$$

for all $x \in X$.

It follows from (2.7) that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, -x\right) \leq \frac{L}{2}\varphi(x, x, -2x) \quad (2.8)$$

for all $x \in X$. Hence $d(f, Jf) \leq L/2$.

By Theorem 1.1, there exists a mapping $A : X \rightarrow Y$ satisfying the following:

(1) A is a fixed point of J , that is,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \quad (2.9)$$

for all $x \in X$. Then $A : X \rightarrow Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}. \quad (2.10)$$

This implies that A is a unique mapping satisfying (2.9) such that there exists a $K \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \leq K\varphi(x, x, -2x) \quad (2.11)$$

for all $x \in X$;

(2) $d(J^n f, A) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$\lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right) = A(x) \quad (2.12)$$

for all $x \in X$;

(3) $d(f, A) \leq (1/(1-L))d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{2-2L}. \quad (2.13)$$

This implies that the inequality (2.2) holds.

It follows from (2.1) and (2.12) that

$$\|A(x) + A(y) + A(z)\| \leq \|A(x+y+z)\| \quad (2.14)$$

for all $x, y, z \in X$. By [34, Proposition 2.2], the mapping $A : X \rightarrow Y$ is a Cauchy additive mapping.

Therefore, there exists a unique Cauchy additive mapping $A : X \rightarrow Y$ satisfying (2.2), as desired. \square

Corollary 2.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an odd mapping such that*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \quad (2.15)$$

for all $x, y, z \in X$. Then, there exists a unique Cauchy additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^r + 2}{2^r - 2} \theta \|x\|^r \quad (2.16)$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y, z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r) \quad (2.17)$$

for all $x, y, z \in X$. Then, we can choose $L = 2^{1-r}$ and we get the desired result. \square

Corollary 2.3. *Let $r > 1/3$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an odd mapping such that*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| + \theta \cdot \|x\|^r \cdot \|y\|^r \cdot \|z\|^r \quad (2.18)$$

for all $x, y, z \in X$. Then, there exists a unique Cauchy additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^r}{2^{3r} - 2} \theta \|x\|^{3r} \quad (2.19)$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y, z) := \theta \cdot \|x\|^r \cdot \|y\|^r \cdot \|z\|^r \quad (2.20)$$

for all $x, y, z \in X$. Then we can choose $L = 2^{1-3r}$ and we get the desired result. \square

Remark 2.4. Let $f : X \rightarrow Y$ be an odd mapping for which there exists a function $\varphi : X^3 \rightarrow [0, \infty)$ satisfying (2.1). By a similar method to the proof of Theorem 2.1, one can show that if there exists an $L < 1$ such that $\varphi(x, y, z) \leq 2L\varphi(x/2, y/2, z/2)$ for all $x, y, z \in X$, then there exists a unique Cauchy additive mapping $A : X \rightarrow Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{1}{2 - 2L} \varphi(x, x, -2x) \quad (2.21)$$

for all $x \in X$.

For the case $0 < r < 1$, one can obtain a similar result to Corollary 2.2. Let $0 < r < 1$ and $\theta \geq 0$ be real numbers, and let $f : X \rightarrow Y$ be an odd mapping satisfying (2.15). Then, there exists a unique Cauchy additive mapping $A : X \rightarrow Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{2 + 2^r}{2 - 2^r} \theta \|x\|^r \quad (2.22)$$

for all $x \in X$.

For the case $0 < r < 1/3$, one can obtain a similar result to Corollary 2.3. Let $0 < r < 1/3$ and $\theta \geq 0$ be real numbers, and let $f : X \rightarrow Y$ be an odd mapping satisfying (2.18). Then, there exists a unique Cauchy additive mapping $A : X \rightarrow Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{2^r}{2 - 2^{3r}} \theta \|x\|^{3r} \quad (2.23)$$

for all $x \in X$.

Using fixed point methods, we prove the generalized Hyers-Ulam stability of the functional inequality (1.6) in Banach spaces.

Theorem 2.5. *Let $f : X \rightarrow Y$ be an odd mapping for which there exists a function $\varphi : X^3 \rightarrow [0, \infty)$ such that there exists an $L < 1$ such that $\varphi(x, y, z) \leq (1/2)L\varphi(2x, 2y, 2z)$ for all $x, y, z \in X$, and*

$$\|f(x) + f(y) + 2f(z)\| \leq \left\| 2f\left(\frac{x+y}{2} + z\right) \right\| + \varphi(x, y, z) \quad (2.24)$$

for all $x, y, z \in X$. Then, there exists a unique Cauchy additive mapping $A : X \rightarrow Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{L}{2 - 2L} \varphi(0, 2x, -x) \quad (2.25)$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 2.1. □

Corollary 2.6. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an odd mapping such that*

$$\|f(x) + f(y) + 2f(z)\| \leq \left\| 2f\left(\frac{x+y}{2} + z\right) \right\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \quad (2.26)$$

for all $x, y, z \in X$. Then, there exists a unique Cauchy additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^r + 1}{2^r - 2} \theta \|x\|^r \quad (2.27)$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x, y, z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r) \quad (2.28)$$

for all $x, y, z \in X$. Then, we can choose $L = 2^{1-r}$ and we get the desired result. \square

Remark 2.7. Let $f : X \rightarrow Y$ be an odd mapping for which there exists a function $\varphi : X^3 \rightarrow [0, \infty)$ satisfying (2.24). By a similar method to the proof of Theorem 2.5, one can show that if there exists an $L < 1$ such that $\varphi(x, y, z) \leq 2L\varphi(x/2, y/2, z/2)$ for all $x, y, z \in X$, then there exists a unique Cauchy additive mapping $A : X \rightarrow Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{1}{2 - 2L} \varphi(0, 2x, -x) \quad (2.29)$$

for all $x \in X$.

For the case $0 < r < 1$, one can obtain a similar result to Corollary 2.6. Let $0 < r < 1$ and $\theta \geq 0$ be real numbers, and let $f : X \rightarrow Y$ be an odd mapping satisfying (2.26). Then, there exists a unique Cauchy additive mapping $A : X \rightarrow Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{1 + 2^r}{2 - 2^r} \theta \|x\|^r \quad (2.30)$$

for all $x \in X$.

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