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Research Article

Approximation of Fixed Points of Nonexpansive Mappings and Solutions of Variational Inequalities

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Let E be a real q-uniformly smooth Banach space with constant d_q, $q \geq 2$. Let $T : E \to E$ and $G : E \to E$ be a nonexpansive map and an η-strongly accretive map which is also κ-Lipschitzian, respectively. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ that satisfies the following condition: $C1$: $\lim \lambda_n = 0$ and $\sum \lambda_n = \infty$. For $\delta \in (0, (q\eta/d_q k^q)^{1/(q-1)})$ and $\sigma \in (0,1)$, define a sequence $\{x_n\}$ iteratively in E by $x_0 \in E, x_{n+1} = T^{\lambda_{n+1}} x_n = (1-\sigma)x_n + \sigma[Tx_n - \delta\lambda_{n+1}G(Tx_n)]$, $n \geq 0$. Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality problem $\mathrm{VI}(G,K)$ (search for $x^* \in K$ such that $\langle Gx^*, j_q(y-x^*)\rangle \geq 0$ for all $y \in K$), where $K := \mathrm{Fix}(T) = \{x \in E : Tx = x\} \neq \emptyset$. A convergence theorem related to finite family of nonexpansive maps is also proved.

1. Introduction

Let E be a real-normed space and let E^* be its dual space. For some real number q $(1 < q < \infty)$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between elements of E and elements of E^*.

Let K be a nonempty closed convex subset of E, and let $S : E \to E$ be a nonlinear operator. The variational inequality problem is formulated as follows. Find a point $x^* \in K$ such that

$$\mathrm{VI}(S,K) : \langle Sx^*, j_q(y-x^*)\rangle \geq 0 \quad \forall y \in K. \tag{1.2}$$

If $E = H$, a real Hilbert space, the variational inequality problem reduces to the following. Find a point $x^* \in K$ such that

$$\mathrm{VI}(S,K) : \langle Sx^*, y - x^* \rangle \geq 0 \quad \forall y \in K. \tag{1.3}$$

A mapping $G : D(G) \subset E \to E$ is said to be *accretive* if for all $x, y \in D(G)$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Gx - Gy, j_q(x-y) \rangle \geq 0, \tag{1.4}$$

where $D(G)$ denotes the domain of G. For some real number $\eta > 0$, G is called *η-strongly accretive* if for all $x, y \in D(G)$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Gx - Gy, j_q(x-y) \rangle \geq \eta \|x-y\|^q. \tag{1.5}$$

G is *κ-Lipschitzian* if for some $\kappa > 0$, $\|G(x) - G(y)\| \leq \kappa\|x-y\|$ for all $x, y \in D(G)$ and G is called *nonexpansive* if $k = 1$.

In Hilbert spaces, accretive operators are called *monotone* where inequalities (1.4) and (1.5) hold with j_q replaced by the identity map of H.

It is known that if S is Lipschitz and *strongly accretive*, then $\mathrm{VI}(S,K)$ has a unique solution. An important problem is how to find a solution of $\mathrm{VI}(S,K)$ whenever it exists. Considerable efforts have been devoted to this problem (see, e.g., [1, 2] and the references contained therein).

It is known that in a real Hilbert space, the $\mathrm{VI}(S,K)$ is equivalent to the following fixed-point equation:

$$x^* = P_K(x^* - \delta S x^*), \tag{1.6}$$

where $\delta > 0$ is an arbitrary fixed constant and P_K is the *nearest point projection map* from H onto K, that is, $P_K x = y$, where $\|x-y\| = \inf_{u \in K}\|x-u\|$ for $x \in H$. Consequently, under appropriate conditions on S and δ, fixed-point methods can be used to find or approximate a solution of $\mathrm{VI}(S,K)$. For instance, if S is strongly monotone and Lipschitz, then a mapping $G : H \to H$, defined by $Gx = P_K(x - \delta Sx)$, $x \in H$ with $\delta > 0$ sufficiently small, is a strict contraction. Hence, the *Picard iteration*, $x_0 \in H$, $x_{n+1} = Gx_n$, $n \geq 0$ of the classical Banach contraction mapping principle, converges to the unique solution of the $\mathrm{VI}(K,S)$.

It has been observed that the projection operator P_K in the fixed-point formulation (1.6) may make the computation of the iterates difficult due to possible complexity of the convex set K. In order to reduce the possible difficulty with the use of P_K, Yamada [2] recently introduced a hybrid descent method for solving the $\mathrm{VI}(K,S)$. Let $T : H \to H$ be a map and let $K := \{x \in H : Tx = x\} \neq \emptyset$. Let S be η-strongly monotone and κ-Lipschitz on H. Let $\delta \in (0, 2\eta/\kappa^2)$ be arbitrary but fixed real number and let a sequence $\{\lambda_n\}$ in $(0,1)$ satisfy the following conditions:

$$C1\colon \lim \lambda_n = 0; \quad C2\colon \sum \lambda_n = \infty; \quad C3\colon \lim \frac{\lambda_n - \lambda_{n+1}}{\lambda_n^2} = 0. \tag{1.7}$$

Starting with an arbitrary initial guess $x_0 \in H$, let a sequence $\{x_n\}$ be generated by the following algorithm:

$$x_{n+1} = Tx_n - \lambda_{n+1}\delta S(Tx_n), \quad n \geq 0. \tag{1.8}$$

Then, Yamada [2] proved that $\{x_n\}$ converges strongly to the unique solution of $\mathrm{VI}(K,S)$.

In the case that $K = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$, where $\{T_i\}_{i=1}^{r}$ is a finite family of nonexpansive mappings, Yamada [2] studied the following algorithm:

$$x_{n+1} = T_{[n+1]}x_{n+1} - \lambda_{n+1}\delta S(T_{[n+1]}x_n), \quad n \geq 0, \tag{1.9}$$

where $T_{[k]} = T_{k \bmod r}$ for $k \geq 1$, with the mod *function* taking values in the set $\{1, 2, \dots, r\}$, where the sequence $\{\lambda_n\}$ satisfies the conditions $C1$, $C2$, and $C4$: $\sum |\lambda_n - \lambda_{n+N}| < \infty$. Under these conditions, he proved the strong convergence of $\{x_n\}$ to the unique solution of the VI(K, S).

Recently, Xu and Kim [1] studied the convergence of the algorithms (1.8) and (1.9), still in the framework of Hilbert spaces, and proved strong convergence with condition $C3$ replaced by $C5$: $\lim((\lambda_n - \lambda_{n+1})/\lambda_{n+1}) = 0$ and with condition $C4$ replaced by $C6$: $\lim((\lambda_n - \lambda_{n+r})/\lambda_{n+r}) = 0$. These are improvements on the results of Yamada. In particular, the canonical choice $\lambda_n := 1/(n+1)$ is applicable in the results of Xu and Kim but is not in the result of Yamada [2]. For further recent results on the schemes (1.8) and (1.9), still in the framework of Hilbert spaces, the reader my consult Wang [3], Zeng and Yao [4], and the references contained in them.

Recently, the present authors [5] extended the results of Xu and Kim [1] to q-uniformly smooth Banach spaces, $q \geq 2$. In particular, they proved theorems which are applicable in L_p spaces, $2 \leq p < \infty$ under conditions $C1$, $C2$, and $C5$ or $C6$ as in the result of Xu and Kim.

It is our purpose in this paper to modify the schemes (1.8) and (1.9) and prove strong convergence theorems for the unique solution of the variational inequality VI(K, S). Furthermore, in the case $T_i : E \to E$, $i = 1, 2, \dots, r$, is a family of nonexpansive mappings with $K = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$, we prove a convergence theorem where condition $C6$ is replaced by $\lim_{n\to\infty} \|T_{n+1}x_n - T_n x_n\| = 0$. An example satisfying this condition is given see, for example, [6]. All our theorems are proved in q-uniformly smooth spaces, $q \geq 2$. In particular, our theorems are applicable in L_p spaces, $2 \leq p < \infty$.

2. Preliminaries

Let E be a real Banach space and let K be a nonempty, closed, and convex subset of E. Let P be a mapping of E onto K. Then, P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for all $x \in E$ and $t \geq 0$. A mapping P of E into E is said to be a *retraction* if $P^2 = P$. A subset K is said to be *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto K. A retraction P is said to be *orthogonal* if for each x, $x - P(x)$ is normal to K in the sense of James [7].

It is well known (see [8]) that if E is uniformly smooth and there exists a nonexpansive retraction of E onto K, then there exists a nonexpansive projection of E onto K. If E is a real smooth Banach space, then P is an orthogonal retraction of E onto K if and only if $P(x) \in K$ and $\langle P(x) - x, j_q(P(x) - y)\rangle \leq 0$ for all $y \in K$. It is also known (see, e.g., [9]) that if K is a convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and $T : K \to K$ is nonexpansive with $F(T) \neq \emptyset$, then $F(T)$ is a nonexpansive retract of K.

Let K be a nonempty closed convex and bounded subset of a Banach space E and let the diameter of K be defined by $d(K) := \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := \sup\{\|x - y\| : y \in K\}$ and let $r(K) := \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The *normal structure coefficient* $N(E)$ of E (see, e.g., [10]) is defined by $N(E) := \inf\{d(K)/r(K) : K$ is a closed convex and bounded subset of E with $d(K) > 0\}$. A space E such that $N(E) > 1$ is said to have uniform normal structure. It is known that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, e.g., [11, 12]).

We will denote a Banach limit by μ. Recall that μ is an element of $(l^\infty)^*$ such that $\|\mu\| = 1$, $\liminf_{n\to\infty} a_n \le \mu_n a_n \le \limsup_{n\to\infty} a_n$ and $\mu_n a_n = \mu_{n+1} a_n$ for all $\{a_n\}_{n\ge 0} \in l^\infty$ (see, e.g., [11, 13]).

Let E be a normed space with $\dim E \ge 2$. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1;\ \|y\| = \tau\right\}. \tag{2.1}$$

The space E is called *uniformly smooth* if and only if $\lim_{t\to 0^+}(\rho_E(t)/t) = 0$. For some positive constant q, E is called *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho_E(t) \le ct^q$, $t > 0$. It is known that

$$L_p \text{ or } (l_p) \text{ spaces are } \begin{cases} 2\text{-uniformly smooth} & \text{if } 2 \le p < \infty, \\ p\text{-uniformly smooth} & \text{if } 1 < p \le 2 \end{cases} \tag{2.2}$$

(see, e.g., [13]). It is well known that if E is smooth, then the duality mapping is singled-valued, and if E is uniformly smooth, then the duality mapping is norm-to-norm uniformly continuous on bounded subset of E.

We will make use of the following well-known results.

Lemma 2.1. *Let E be a real-normed linear space. Then, the following inequality holds:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle \quad \forall x,y \in E,\ \forall j(x+y) \in J(x+y). \tag{2.3}$$

In the sequel, we will also make use of the following lemmas.

Lemma 2.2 (see [14]). *Let $(a_0, a_1, \ldots) \in l^\infty$ such that $\mu_n(a_n) \le 0$ for all Banach limit μ and $\limsup_{n\to\infty}(a_{n+1} - a_n) \le 0$. Then, $\limsup_{n\to\infty} a_n \le 0$.*

Lemma 2.3 (see [15]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf \beta_n \le \limsup \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integers $n \ge 0$ and $\limsup(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim\|y_n - x_n\| = 0$.*

Lemma 2.4 (see [16]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \ge 0, \tag{2.4}$$

where (i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \le 0$; (iii) $\gamma_n \ge 0$; $(n \ge 0)$, $\sum \gamma_n < \infty$. *Then, $a_n \to 0$ as $n \to \infty$.*

Lemma 2.5 (see [17]). *Let E be a real q-uniformly smooth Banach space for some $q > 1$, then there exists some positive constant d_q such that*

$$\|x+y\|^q \le \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q \quad \forall x,y \in E,\ j_q(x) \in J_q(x). \tag{2.5}$$

Lemma 2.6 (see [12, Theorem 1]). *Suppose E is a Banach space with uniformly normal structure, K is a nonempty bounded subset of E, and $T : K \to K$ is uniformly k-Lipschitzian mapping with $k < N(E)^{1/2}$. Suppose also that there exists a nonempty bounded closed convex subset of C of K with the following property* (P)*:*

$$x \in C \quad \textit{implies } \omega_w(x) \subset C, \tag{P}$$

where $\omega_w(x)$ is the ω-limi set of T at x, that is, the set

$$\Big\{y \in E : y = \textit{weak-}\lim_j T^{n_j}x \textit{ for some } n_j \longrightarrow \infty\Big\}. \tag{2.6}$$

Then, T has a fixed point in C.

3. Main results

We first prove the following lemma which will be central in the sequel.

Lemma 3.1. *Let E be a real q-uniformly smooth Banach space with constant d_q, $q \geq 2$. Let $T : E \to E$ and $G : E \to E$ be a nonexpansive map and an η-strongly accretive map which is also κ-Lipschitzian, respectively. For $\delta \in (0, (q\eta/d_q\kappa^q)^{1/(q-1)})$, $\sigma \in (0,1)$, and $\lambda \in (0, 2/p(p-1))$, define a map $T^\lambda : E \to E$ by $T^\lambda x = (1-\sigma)x + \sigma[Tx - \lambda\delta G(Tx)]$, $x \in E$. Then, T^λ is a strict contraction. Furthermore,*

$$\|T^\lambda x - T^\lambda y\| \leq (1-\lambda\alpha)\|x-y\|, \quad x, y \in E, \tag{3.1}$$

where $\alpha = q/2 - \sqrt{q^2/4 - \sigma\delta(q\eta - \delta^{q-1}d_q\kappa^q)} \in (0,1)$.

Proof. For $x, y \in E$,

$$\begin{aligned}
\|T^\lambda x - T^\lambda y\|^q &= \|(1-\sigma)(x-y) + \sigma[Tx - Ty - \lambda\delta(G(Tx) - G(Ty))]\|^q \\
&\leq (1-\sigma)\|x-y\|^q + \sigma\big[\|Tx-Ty\|^q - q\lambda\delta\langle G(Tx) - G(Ty), j_q(Tx-Ty)\rangle \\
&\qquad + d_q\lambda^q\delta^q\|G(Tx) - G(Ty)\|^q\big] \\
&\leq (1-\sigma)\|x-y\|^q + \sigma\big[\|Tx-Ty\|^q - q\lambda\delta\eta\|Tx-Ty\|^q + d_q\lambda^q\delta^q\kappa^q\|Tx-Ty\|^q\big] \\
&\leq \big[1 - \sigma\lambda\delta(q\eta - d_q\lambda^{q-1}\delta^{q-1}\kappa^q)\big]\|x-y\|^q \\
&\leq \big[1 - \sigma\lambda\delta(q\eta - d_q\delta^{q-1}\kappa^q)\big]\|x-y\|^q.
\end{aligned} \tag{3.2}$$

Define

$$f(\lambda) := 1 - \sigma\lambda\delta(q\eta - d_q\delta^{q-1}\kappa^q) = (1-\lambda\tau)^q \quad \text{for some } \tau \in (0,1) \text{ say}. \tag{3.3}$$

Then, there exists $\xi \in (0,\lambda)$ such that

$$1 - \sigma\lambda\delta(q\eta - d_q\delta^{q-1}\kappa^q) = 1 - q\tau\lambda + \frac{1}{2}q(q-1)(1-\xi\tau)^{q-2}\lambda^2\tau^2. \tag{3.4}$$

This implies that

$$1-\sigma\lambda\delta\left(q\eta-d_q\delta^{q-1}\kappa^q\right)\leq 1-q\tau\lambda+\frac{1}{2}q(q-1)\lambda^2\tau^2. \tag{3.5}$$

Then, we have $\tau \leq q/2-\sqrt{q^2/4-\sigma\delta(q\eta-d_q\delta^{q-1}\kappa^q)}$.

Set

$$\alpha := \frac{q}{2}-\sqrt{\frac{q^2}{4}-\sigma\delta\left(q\eta-d_q\delta^{q-1}\kappa^q\right)}, \tag{3.6}$$

and the proof is complete. □

We note that in L_p spaces, $2 \leq p < \infty$, the following inequality holds (see, e.g., [13]). For each $x, y \in L_p$, $2 \leq p < \infty$,

$$\|x+y\|^2 \leq \|x\|^2+2\langle y, j(x)\rangle+(p-1)\|y\|^2. \tag{3.7}$$

Using this inequality and following the method of proof of Lemma 3.1, the following corollary is easily proved.

Corollary 3.2. *Let $E = L_p$, $2 \leq p < \infty$. Let $T : E \to E$, $G : E \to E$ be a nonexpansive map, an η-strongly monotone, and κ-Lipschitzian map, respectively. For $\lambda, \sigma \in (0,1)$ and $\delta \in (0, 2\eta/(p-1)\kappa^2)$, define a map $T^\lambda : E \to E$ by $T^\lambda x = (1-\sigma)x+\sigma[Tx-\lambda\delta G(Tx)]$, $x \in E$. Then, T^λ is a contraction. In particular,*

$$\left\|T^\lambda x-T^\lambda y\right\| \leq (1-\lambda\alpha)\|x-y\|, \quad x, y \in H, \tag{3.8}$$

where $\alpha = 1-\sqrt{1-\sigma\delta(2\eta-(p-1)\delta\kappa^2)} \in (0,1)$.

Corollary 3.3. *Let H be a real Hilbert space, $T : H \to H$, $G : H \to H$ a nonexpansive map and an η-strongly monotone map which is also κ-Lipschitzian, respectively. For $\lambda, \sigma \in (0,1)$ and $\delta \in (0, 2\eta/\kappa^2)$, define a map $T^\lambda : H \to H$ by $T^\lambda x = (1-\sigma)x+\sigma[Tx-\lambda\delta G(Tx)]$, $x \in H$. Then, T^λ is a contraction. In particular,*

$$\left\|T^\lambda x-T^\lambda y\right\| \leq (1-\lambda\alpha)\|x-y\|, \quad x, y \in H, \tag{3.9}$$

where $\alpha = 1-\sqrt{1-\sigma\delta(2\eta-\delta\kappa^2)} \in (0,1)$.

Proof. Set $p = 2$ in Corollary 3.2 and the result follows. □

Corollary 3.3 is a result of Yamada [2] and is the main tool used in [1–4].

We now prove our main theorems.

Theorem 3.4. *Let E be a real q-uniformly smooth Banach space with constant d_q, $q \geq 2$. Let $T : E \to E$ and $G : E \to E$ be a nonexpansive map and an η-strongly accretive map which is also κ-Lipschitzian, respectively. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ satisfying*

$$\text{C1: } \lim \lambda_n = 0; \qquad \text{C2: } \sum \lambda_n = \infty. \tag{3.10}$$

For $\delta \in (0, (q\eta/d_q\kappa^q)^{1/(q-1)})$ and $\sigma \in (0,1)$, define a sequence $\{x_n\}$ iteratively in E by $x_0 \in E$,

$$x_{n+1} = T^{\lambda_{n+1}}x_n = (1-\sigma)x_n+\sigma\left[Tx_n-\delta\lambda_{n+1}G(Tx_n)\right], \quad n \geq 0. \tag{3.11}$$

Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality VI(G, K).

Proof. Let $x^* \in K := \mathrm{Fix}\,(T)$, then the sequence $\{x_n\}$ satisfies

$$\|x_n - x^*\| \le \max\left\{ \|x_0 - x^*\|, \frac{\delta}{\alpha}\|G(x^*)\| \right\}, \quad n \ge 0. \tag{3.12}$$

It is obvious that this is true for $n = 0$. Assume that it is true for $n = k$ for some $k \in \mathbb{N}$.

From the recursion formula (3.11), we have

$$\begin{aligned}
\|x_{k+1} - x^*\| &= \|T^{\lambda_{k+1}} x_k - x^*\| \\
&\le \|T^{\lambda_{k+1}} x_k - T^{\lambda_{k+1}} x^*\| + \|T^{\lambda_{k+1}} x^* - x^*\| \\
&\le (1 - \lambda_{k+1}\alpha)\|x_k - x^*\| + \lambda_{k+1}\delta\|G(x^*)\| \\
&\le \max\left\{ \|x_0 - x^*\|, \frac{\delta}{\alpha}\|G(x^*)\| \right\},
\end{aligned} \tag{3.13}$$

and the claim follows by induction. Thus, the sequence $\{x_n\}$ is bounded and so are $\{Tx_n\}$ and $\{G(Tx_n)\}$.

Define two sequences $\{\beta_n\}$ and $\{y_n\}$ by $\beta_n := (1-\sigma)\lambda_{n+1}+\sigma$ and $y_n := (x_{n+1}-x_n+\beta_n x_n)/\beta_n$. Then,

$$y_n = \frac{(1-\sigma)\lambda_{n+1}x_n + \sigma[Tx_n - \lambda_{n+1}\delta G(Tx_n)]}{\beta_n}. \tag{3.14}$$

Observe that $\{y_n\}$ is bounded and that

$$\begin{aligned}
&\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\
&\quad \le \left|\frac{\sigma}{\beta_{n+1}} - 1\right| \|x_{n+1} - x_n\| + \left|\frac{\sigma}{\beta_{n+1}} - \frac{\sigma}{\beta_n}\right| \|Tx_n\| + \frac{\lambda_{n+2}(1-\sigma)}{\beta_{n+1}}\|x_{n+1} - x_n\| \\
&\qquad + (1-\sigma)\left|\frac{\lambda_{n+2}}{\beta_{n+1}} - \frac{\lambda_{n+1}}{\beta_n}\right| \|x_n\| + \frac{\lambda_{n+1}\sigma\delta}{\beta_n}\|G(Tx_n) - G(Tx_{n+1})\| + \sigma\delta\left|\frac{\lambda_{n+1}}{\beta_n} - \frac{\lambda_{n+2}}{\beta_{n+1}}\right| \|G(Tx_{n+1})\|.
\end{aligned} \tag{3.15}$$

This implies that $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$, and by Lemma 2.3,

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.16}$$

Hence,

$$\|x_{n+1} - x_n\| = \beta_n\|y_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.17}$$

From the recursion formula (3.11), we have that

$$\sigma\|x_{n+1} - Tx_n\| \le (1-\sigma)\|x_{n+1} - x_n\| + \lambda_{n+1}\sigma\delta\|G(Tx_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.18}$$

which implies that

$$\|x_{n+1} - Tx_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.19}$$

From (3.17) and (3.19), we have

$$\|x_n - Tx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.20}$$

We now prove that $\limsup_{n\to\infty}\langle -G(x^*), j(x_{n+1}-x^*)\rangle \leq 0$.

Define a map $\phi : E \to \mathbb{R}$ by

$$\phi(x) = \mu_n\|x_n - x\|^2 \quad \forall x \in E. \tag{3.21}$$

Then, $\phi(x) \to \infty$ as $\|x\| \to \infty$, ϕ is continuous and convex, so as E is reflexive, there exists $y^* \in E$ such that $\phi(y^*) = \min_{u\in E}\phi(u)$. Hence, the set

$$K^* := \left\{x \in E : \phi(x) = \min_{u\in E}\phi(u)\right\} \neq \varnothing. \tag{3.22}$$

By Lemma 2.6, $K^* \cap K \neq \varnothing$. Without loss of generality, assume that $y^* = x^* \in K^* \cap K$. Let $t \in (0,1)$. Then, it follows that $\phi(x^*) \leq \phi(x^* - tG(x^*))$ and using Lemma 2.1, we obtain that

$$\|x_n - x^* + tG(x^*)\|^2 \leq \|x_n - x^*\|^2 + 2t\langle G(x^*), j(x_n - x^* + tG(x^*))\rangle \tag{3.23}$$

which implies that

$$\mu_n\langle -G(x^*), j(x_n - x^* + tG(x^*))\rangle \leq 0. \tag{3.24}$$

Moreover,

$$\begin{aligned}\mu_n\langle -G(x^*), j(x_n - x^*)\rangle &= \mu_n\langle -G(x^*), j(x_n - x^*) - j(x_n - x^* + tG(x^*))\rangle \\ &\quad + \mu_n\langle -G(x^*), j(x_n - x^* + tG(x^*))\rangle \\ &\leq \mu_n\langle -G(x^*), j(x_n - x^*) - j(x_n - x^* + tG(x^*))\rangle.\end{aligned} \tag{3.25}$$

Since j is norm-to-norm uniformly continuous on bounded subsets of E, we have that

$$\mu_n\langle -G(x^*), j(x_n - x^*)\rangle \leq 0. \tag{3.26}$$

Furthermore, since $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$, we also have

$$\limsup_{n\to\infty}\left(\langle -G(x^*), j(x_n - x^*)\rangle - \langle -G(x^*), j(x_{n+1} - x^*)\rangle\right) \leq 0, \tag{3.27}$$

and so we obtain by Lemma 2.2 that $\limsup_{n\to\infty}\langle -G(x^*), j(x_n - x^*)\rangle \leq 0$.

From the recursion formula (3.11) and Lemma 2.1, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|T^{\lambda_{n+1}}x_n - T^{\lambda_{n+1}}x^* + T^{\lambda_{n+1}}x^* - x^*\|^2 \\ &\leq \|T^{\lambda_{n+1}}x_n - T^{\lambda_{n+1}}x^*\|^2 + 2\lambda_{n+1}\delta\langle -G(x^*), j(x_{n+1} - x^*)\rangle \\ &\leq (1 - \lambda_{n+1}\alpha)\|x_n - x^*\|^2 + 2\lambda_{n+1}\delta\langle -G(x^*), j(x_{n+1} - x^*)\rangle,\end{aligned} \tag{3.28}$$

and by Lemma 2.4, we have that $x_n \to x^*$as $n \to \infty$. This completes the proof. □

The following corollaries follow from Theorem 3.4.

Corollary 3.5. *Let $E = L_p$, $2 \le p < \infty$. Let $T : E \to E$ and $G : E \to E$ be a nonexpansive map and an η-strongly accretive map which is also κ-Lipschitzian, respectively. Let $\{\lambda_n\}$ be a real sequence in* $[0,1]$ *that satisfies conditions* C1 *and* C2 *as in Theorem 3.4. For $\delta \in (0, 2\eta/(p-1)\kappa^2)$ and $\sigma \in (0,1)$, define a sequence $\{x_n\}$ iteratively in E by* (3.11). *Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality* VI(G, K).

Corollary 3.6. *Let $E = H$ be a real Hilbert space. Let $T : H \to H$ and $G : H \to H$ be a nonexpansive map and an η-strongly monotone map which is also κ-Lipschitzian, respectively. Let $\{\lambda_n\}$ be a real sequence in* $[0,1]$ *that satisfies conditions* C1 *and* C2 *as in Theorem 3.4. For $\delta \in (0, 2\eta/\kappa^2)$ and $\sigma \in (0,1)$, define a sequence $\{x_n\}$ iteratively in H by* (3.11). *Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality* VI(G, K).

Finally, we prove the following more general theorem.

Theorem 3.7. *Let E be a real q-uniformly smooth Banach space with constant d_q, $q \ge 2$. Let $T_i : E \to E$, $i = 1, 2, \ldots, r$, be a finite family of nonexpansive mappings with $K := \bigcap_{i=1}^{r} \mathrm{Fix}(T_i) \ne \emptyset$. Let $G : E \to E$ be an η-strongly accretive map which is also κ-Lipschitzian. Let $\{\lambda_n\}$ be a real sequence in* $[0,1]$ *satisfying*

$$\text{C1: } \lim \lambda_n = 0; \qquad \text{C2: } \sum \lambda_n = \infty. \tag{3.29}$$

For a fixed real number $\delta \in (0, (q\eta/d_q\kappa^q)^{1/(q-1)})$, define a sequence $\{x_n\}$ iteratively in E by $x_0 \in E$:

$$x_{n+1} = T_{[n+1]}^{\lambda_{n+1}} x_n = (1-\sigma)x_n + \sigma\left[T_{[n+1]}x_n - \delta\lambda_n G\left(T_{[n+1]}x_n\right)\right], \quad n \ge 0, \tag{3.30}$$

where $T_{[n]} = T_{n \bmod r}$. Assume also that

$$K = \mathrm{Fix}\left(T_r T_{r-1} \cdots T_1\right) = \mathrm{Fix}\left(T_1 T_r \cdots T_2\right) = \cdots = \mathrm{Fix}\left(T_{r-1} T_{r-2} \cdots T_r\right) \tag{3.31}$$

and $\lim_{n\to\infty} \|T_{n+1}x_n - T_n x_n\| = 0$. Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality VI(G, K).

Proof. Let $x^* \in K$, then the sequence $\{x_n\}$ satisfies that

$$\left\|x_n - x^*\right\| \le \max\left\{\left\|x_0 - x^*\right\|, \frac{\delta}{\alpha}\left\|G\left(x^*\right)\right\|\right\}, \quad n \ge 0. \tag{3.32}$$

It is obvious that this is true for $n = 0$. Assume it is true for $n = k$ for some $k \in \mathbb{N}$.

From the recursion formula (3.30), we have

$$\begin{aligned}
\left\|x_{k+1} - x^*\right\| &= \left\|T_{[k+1]}^{\lambda_{k+1}} x_k - x^*\right\| \\
&\le \left\|T_{[k+1]}^{\lambda_{k+1}} x_k - T_{[k+1]}^{\lambda_{k+1}} x^*\right\| + \left\|T_{[k+1]}^{\lambda_{k+1}} x^* - x^*\right\| \\
&\le \left(1 - \lambda_{k+1}\alpha\right)\left\|x_k - x^*\right\| + \lambda_{k+1}\delta\left\|G\left(x^*\right)\right\| \\
&\le \max\left\{\left\|x_0 - x^*\right\|, \frac{\delta}{\alpha}\left\|G\left(x^*\right)\right\|\right\},
\end{aligned} \tag{3.33}$$

and the claim follows by induction. Thus, the sequence $\{x_n\}$ is bounded and so are $\{T_{[n]}x_n\}$ and $\{G(T_{[n]}x_n)\}$.

Define two sequences $\{\beta_n\}$ and $\{y_n\}$ by $\beta_n := (1-\sigma)\lambda_{n+1}+\sigma$ and $y_n := (x_{n+1}-x_n+\beta_n x_n)/\beta_n$. Then,

$$y_n = \frac{(1-\sigma)\lambda_{n+1}x_n + \sigma\left[T_{[n+1]}x_n - \lambda_{n+1}\delta G(T_{[n+1]}x_n)\right]}{\beta_n}. \tag{3.34}$$

Observe that $\{y_n\}$ is bounded and that

$$\begin{aligned}\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\| \le{}& \left|\frac{\sigma}{\beta_{n+1}}-1\right|\|x_{n+1}-x_n\| \\ &+ \frac{\sigma}{\beta_{n+1}}\|T_{[n+2]}x_n - T_{[n+1]}x_n\| + \left|\frac{\sigma}{\beta_{n+1}}-\frac{\sigma}{\beta_n}\right|\|T_{[n+1]}x_n\| \\ &+ \frac{\lambda_{n+2}(1-\sigma)}{\beta_{n+1}}\|x_{n+1}-x_n\| + (1-\sigma)\left|\frac{\lambda_{n+2}}{\beta_{n+1}}-\frac{\lambda_{n+1}}{\beta_n}\right|\|x_n\| \\ &+ \frac{\lambda_{n+1}\sigma\delta}{\beta_n}\|G(T_{[n+1]}x_n) - G(T_{[n+2]}x_{n+1})\| \\ &+ \sigma\delta\left|\frac{\lambda_{n+1}}{\beta_n}-\frac{\lambda_{n+2}}{\beta_{n+1}}\right|\|G(T_{[n+2]}x_{n+1})\|.\end{aligned} \tag{3.35}$$

This implies that $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \le 0$, and by Lemma 2.3,

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.36}$$

Hence,

$$\|x_{n+1}-x_n\| = \beta_n\|y_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.37}$$

From the recursion formula (3.30), we have that

$$\sigma\|x_{n+1} - T_{[n+1]}x_n\| \le (1-\sigma)\|x_{n+1}-x_n\| + \lambda_{n+1}\sigma\delta\|G(T_{[n+1]}x_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{3.38}$$

which implies that

$$\|x_{n+1} - T_{[n+1]}x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.39}$$

From (3.37) and (3.39), we have

$$\|x_n - T_{[n+1]}x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - T_{[n+1]}x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.40}$$

Also,

$$\|x_{n+r}-x_n\| \le \|x_{n+r}-x_{n+r-1}\| + \|x_{n+r-1}-x_{n+r-2}\| + \cdots + \|x_{n+1}-x_n\|, \tag{3.41}$$

and so

$$\|x_{n+r}-x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.42}$$

Using the fact that T_i is nonexpansive for each i, we obtain the following finite table:

$$\begin{gathered}
x_{n+r} - T_{n+r}x_{n+r-1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty; \\
T_{n+r}x_{n+r-1} - T_{n+r}T_{n+r-1}x_{n+r-2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty; \\
\vdots \\
T_{n+r}T_{n+r-1}\cdots T_{n+2}x_{n+1} - T_{n+r}T_{n+r-1}\cdots T_{n+2}T_{n+1}x_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty;
\end{gathered} \tag{3.43}$$

and adding up the table yields

$$x_{n+r} - T_{n+r}T_{n+r-1}\cdots T_{n+1}x_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.44}$$

Using this and (3.42), we get that $\lim_{n\to\infty}\|x_n - T_{n+r}T_{n+r-1}\cdots T_{n+1}x_n\| = 0$.

Carrying out similar arguments as in the proof of Theorem 3.4, we easily get that

$$\limsup_{n\to\infty}\langle -G(x^*), j(x_{n+1} - x^*)\rangle \leq 0. \tag{3.45}$$

From the recursion formula (3.30), and Lemma 2.1, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|T^{\lambda_{n+1}}_{[n+1]}x_n - T^{\lambda_{n+1}}_{[n]}x^* + T^{\lambda_{n+1}}_{[n+1]}x^* - x^*\|^2 \\
&\leq \|T^{\lambda_{n+1}}_{[n+1]}x_n - T^{\lambda_{n+1}}_{[n+1]}x^*\|^2 + 2\lambda_{n+1}\sigma\delta\langle -G(x^*), j(x_{n+1} - x^*)\rangle \\
&\leq (1 - \lambda_{n+1}\alpha)\|x_n - x^*\|^2 + 2\lambda_{n+1}\sigma\delta\langle -G(x^*), j(x_{n+1} - x^*)\rangle,
\end{aligned} \tag{3.46}$$

and by Lemma 2.4, we have that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

The following corollaries follow from Theorem 3.7.

Corollary 3.8. *Let $E = L_p$, $2 \leq p < \infty$. Let $T_i : E \to E$, $i = 1, 2, \ldots, r$, be a finite family of nonexpansive mappings with $K = \bigcap_{i=1}^r \mathrm{Fix}(T_i) \neq \varnothing$. Let $G : E \to E$ be an η-strongly accretive map which is also κ-Lipschitzian. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ that satisfies conditions C1 and C2 as in Theorem 3.7 and also $\lim_{n\to\infty}\|T_{n+1}x_n - T_nx_n\| = 0$. For $\delta \in (0, 2\eta/(p-1)\kappa^2)$, define a sequence $\{x_n\}$ iteratively in E by (3.30). Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality $\mathrm{VI}(G, K)$.*

Corollary 3.9. *Let $E = H$ be a real Hilbert space. Let $T_i : H \to H$, $i = 1, 2, \ldots, r$, be a finite family of nonexpansive mappings with $K = \bigcap_{i=1}^r \mathrm{Fix}(T_i) \neq \varnothing$. Let $G : H \to H$ be an η-strongly monotone map which is also κ-Lipschitzian. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ that satisfies conditions C1 and C2 as in Theorem 3.7 and also $\lim_{n\to\infty}\|T_{n+1}x_n - T_nx_n\| = 0$. For $\delta \in (0, 2\eta/\kappa^2)$, define a sequence $\{x_n\}$ iteratively in H by (3.30). Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality $\mathrm{VI}(G, K)$.*

Remark 3.10. Observe that condition C6 in Theorem 3.2 of [1] is dropped in Corollary 3.9, being replaced by condition $\lim_{n\to\infty}\|T_{n+1}x_n - T_nx_n\| = 0$ on the mappings $\{T_i\}_{i=1}^r$.

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