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Research Article

On Harmonic Functions Defined by Derivative Operator

K. Al-Shaqsi and M. Darus

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600 Selangor D. Ehsan, Malaysia

Correspondence should be addressed to M. Darus, maslina@pkrisc.cc.ukm.my

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Let $\mathcal{S}_{\mathcal{H}}$ denote the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U} = \{z : |z| < 1\}$, where $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $g(z) = \sum_{k=1}^{\infty} b_k z^k$ ($|b_1| < 1$). In this paper, we introduce the class $M_{\mathcal{H}}(n, \lambda, \alpha)$ of functions $f = h + \overline{g}$ which are harmonic in $\mathbb{U}$. A sufficient coefficient of this class is determined. It is shown that this coefficient bound is also necessary for the class $M_{\overline{\mathcal{H}}}(n, \lambda, \alpha)$ if $f_n(z) = h + \overline{g_n} \in M_{\mathcal{H}}(n, \lambda, \alpha)$, where $h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$, $g_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k$ and $n \in \mathbb{N}_0$. Coefficient conditions, such as distortion bounds, convolution conditions, convex combination, extreme points, and neighborhood for the class $M_{\overline{\mathcal{H}}}(n, \lambda, \alpha)$, are obtained.

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathfrak{D} \subset \mathbb{C}$, we can write $f = h + \overline{g}$, where h and g are analytic in $\mathfrak{D}$. We call h the analytic part and g the coanalytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathfrak{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathfrak{D}$; see [2].

Denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f = h + \overline{g}$ that are harmonic, univalent, and sense-preserving in the unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

Observe that $\mathcal{S}_{\mathcal{H}}$ reduces to $\mathcal{S}$, the class of normalized univalent analytic functions, if the coanalytic part of f is zero. Also, denote by $S^*_{\mathcal{H}}$ the subclasses of $\mathcal{S}_{\mathcal{H}}$ consisting of functions f that map $\mathbb{U}$ onto starlike domain.

For $f = h + \overline{g}$ given by (1.1), we define the derivative operator introduced by authors (see [1]) of f as

$$\mathfrak{D}^n_\lambda f(z) = \mathfrak{D}^n_\lambda h(z) + (-1)^n \overline{\mathfrak{D}^n_\lambda g(z)}, \quad n, \lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ z \in \mathbb{U}, \tag{1.2}$$

where $\mathfrak{D}^n_\lambda h(z) = z + \sum_{k=2}^{\infty} k^n C(\lambda, k) a_k z^k$, $\mathfrak{D}^n_\lambda g(z) = \sum_{k=1}^{\infty} k^n C(\lambda, k) b_k z^k$, and $C(\lambda, k) = \binom{k+\lambda-1}{\lambda}$.

We let $M_{\mathcal{H}}(n, \lambda, \alpha)$ denote the family of harmonic functions f of the form (1.1) such that

$$\operatorname{Re}\left\{ \frac{\mathfrak{D}^{n+1}_\lambda f(z)}{\mathfrak{D}^n_\lambda f(z)} \right\} > \alpha, \quad 0 \le \alpha < 1, \tag{1.3}$$

where $\mathfrak{D}^n_\lambda f$ is defined by (1.2).

If the coanalytic part of $f = h + \overline{g}$ is identically zero, then the class $M_{\mathcal{H}}(n, \lambda, \alpha)$ turns out to be the class $\mathcal{R}^n_\lambda(\alpha)$ introduced by Al-Shaqsi and Darus [1] for the analytic case.

Let $M_{\overline{\mathcal{H}}}(n, \lambda, \alpha)$ denote that the subclass of $M_{\mathcal{H}}(n, \lambda, \alpha)$ consists of harmonic functions $f_n = h + \overline{g_n}$ such that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \qquad g_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k. \tag{1.4}$$

It is clear that the class $M_{\mathcal{H}}(n, \lambda, \alpha)$ includes a variety of well-known subclasses of $\mathcal{S}_{\mathcal{H}}$. For example, $M_{\mathcal{H}}(0, 0, \alpha) \equiv S^*_{\mathcal{H}}(\alpha)$ is the class of sense-preserving, harmonic, univalent functions f which are starlike of order α in $\mathbb{U}$, that is, $(\partial/\partial\theta)\{\arg(f(re^{i\theta}))\} > \alpha$, and $M_{\mathcal{H}}(1, 0, \alpha) \equiv M_{\mathcal{H}}(0, 1, \alpha) \equiv \mathcal{HK}(\alpha)$ is the class of sense-preserving, harmonic, univalent functions f which are convex of order α in $\mathbb{U}$, that is, $(\partial/\partial\theta)\{\arg((\partial/\partial\theta) f(re^{i\theta}))\} > \alpha$. Note that the classes $S^*_{\mathcal{H}}$ and $\mathcal{HK}(\alpha)$ were introduced and studied by Jahangiri [3]. Also we notice that the class $M_{\overline{\mathcal{H}}}(n, 0, \alpha)$ is the class of Salagean-type harmonic univalent functions introduced by Jahangiri et al. [4]; and $M_{\overline{\mathcal{H}}}(0, \lambda, \alpha)$ is the class of Ruscheweyh-type harmonic univalent functions studied by Murugusundaramoorthy and Vijaya [5].

In 1984, Clunie and Sheil-Small [2] investigated the class $\mathcal{S}_{\mathcal{H}}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on $\mathcal{S}_{\mathcal{H}}$ and its subclasses such that Silverman [6], Silverman and Silvia [7], and Jahangiri [3, 8] studied the harmonic univalent functions. Jahangiri and Silverman [9] prove the following theorem.

Theorem 1.1. *Let $f = h + \overline{g}$ given by* (1.1). *If*

$$\sum_{k=2}^{\infty} k(|a_k| + |b_k|) \le 1 - |b_1|, \tag{1.5}$$

then f is sense-preserving, harmonic, and univalent in $\mathbb{U}$ and $f \in S^*_{\mathcal{H}}$ consists of functions in $\mathcal{S}_{\mathcal{H}}$ which are starlike in $\mathbb{U}$.

The condition (1.5) is also necessary if $f \in \mathcal{T}H \equiv M_{\overline{\mathcal{H}}}(0, 0, 0)$.

In this paper, we will give sufficient condition for functions $f = h + \overline{g}$, where h and g are given by (1.1) to be in the class $M_{\mathcal{H}}(n, \lambda, \alpha)$; and it is shown that this coefficient condition is also necessary for functions in the class $M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$. Also, we obtain distortion theorems and characterize the extreme points for functions in $M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$. Closure theorems and application of neighborhood are also obtained.

2. Coefficient bounds

We begin with a sufficient coefficient condition for functions in $M_{\mathcal{H}}(n,\lambda,\alpha)$.

Theorem 2.1. *Let $f=h+\overline{g}$ be given by* (1.1). *If*

$$\sum_{k=1}^{\infty}\left[(k-\alpha)|a_k|+(k+\alpha)|b_k|\right]k^nC(\lambda,k)\le 2(1-\alpha), \tag{2.1}$$

where $a_1=1$, $n,\lambda\in\mathbb{N}_0$, $C(\lambda,k)=\binom{k+\lambda-1}{\lambda}$, and $0\le\alpha<1$, then f is sense-preserving, harmonic, univalent in $\mathbb{U}$, and $f\in M_{\mathcal{H}}(n,\lambda,\alpha)$.

Proof. If $z_1\neq z_2$, then

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|\\ &=1-\left|\frac{\sum_{k=1}^{\infty}b_k\left(z_1^k-z_2^k\right)}{(z_1-z_2)+\sum_{k=2}^{\infty}a_k\left(z_1^k-z_2^k\right)}\right|\\ &>1-\frac{\sum_{k=1}^{\infty}k|b_k|}{1-\sum_{k=2}^{\infty}k|a_k|}\\ &\ge 1-\frac{\sum_{k=1}^{\infty}\left((k+\alpha)k^nC(\lambda,k)/(1-\alpha)\right)|b_k|}{1-\sum_{k=2}^{\infty}\left((k-\alpha)k^nC(\lambda,k)/(1-\alpha)\right)|a_k|}\ge 0,\end{aligned} \tag{2.2}$$

which proves univalence. Note that f is sense-preserving in $\mathbb{U}$. This is because

$$\begin{aligned}|h'(z)| &\ge 1-\sum_{k=2}^{\infty}k|a_k||z|^{k-1}\\ &>1-\sum_{k=2}^{\infty}\frac{(k-\alpha)k^nC(\lambda,k)}{1-\alpha}|a_k|\\ &\ge\sum_{k=1}^{\infty}\frac{(k+\alpha)k^nC(\lambda,k)}{1-\alpha}|b_k|\\ &>\sum_{k=1}^{\infty}\frac{(k+\alpha)k^nC(\lambda,k)}{1-\alpha}|b_k||z|^{k-1}\ge\sum_{k=1}^{\infty}k|b_k||z|^{k-1}\ge|g'(z)|.\end{aligned} \tag{2.3}$$

Using the fact that $\operatorname{Re}w>\alpha$ if and only if $|1-\alpha+w|\ge|1+\alpha-w|$, it suffices to show that

$$\left|(1-\alpha)\mathfrak{D}_{\lambda}^{n}f(z)+\mathfrak{D}_{\lambda}^{n+1}f(z)\right|-\left|(1+\alpha)\mathfrak{D}_{\lambda}^{n}f(z)-\mathfrak{D}_{\lambda}^{n+1}f(z)\right|\ge 0. \tag{2.4}$$

Substituting $\mathfrak{D}_\lambda^n f(z)$ in (2.4) yields, by (2.1), we obtain

$$\begin{aligned}
&\left|(1-\alpha)\mathfrak{D}_\lambda^n f(z)+\mathfrak{D}_\lambda^{n+1} f(z)\right|-\left|(1+\alpha)\mathfrak{D}_\lambda^n f(z)-\mathfrak{D}_\lambda^{n+1} f(z)\right| \\
&\quad=\left|(2-\alpha)z+\sum_{k=2}^{\infty}(k+1-\alpha)k^nC(\lambda,k)a_kz^k-(-1)^n\sum_{k=1}^{\infty}(k-1+\alpha)k^nC(\lambda,k)\overline{b_kz^k}\right| \\
&\qquad-\left|-\alpha z+\sum_{k=2}^{\infty}(k-1-\alpha)k^nC(\lambda,k)a_kz^k-(-1)^n\sum_{k=1}^{\infty}(k+1+\alpha)k^nC(\lambda,k)\overline{b_kz^k}\right| \\
&\quad\geq 2(1-\alpha)|z|\left\{1-\sum_{k=2}^{\infty}\frac{(k-\alpha)k^nC(\lambda,k)}{1-\alpha}|a_k||z|^{k-1}\sum_{k=1}^{\infty}\frac{(k+\alpha)k^nC(\lambda,k)}{1-\alpha}|b_k||z|^{k-1}\right\} \\
&\quad\geq 2(1-\alpha)\left\{1-\sum_{k=2}^{\infty}\frac{(k-\alpha)k^nC(\lambda,k)}{1-\alpha}|a_k|-\sum_{k=1}^{\infty}\frac{(k+\alpha)k^nC(\lambda,k)}{1-\alpha}|b_k|\right\}.
\end{aligned} \tag{2.5}$$

This last expression is nonnegative by (2.1), and so the proof is complete. □

The harmonic function

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\alpha}{(k-\alpha)k^nC(\lambda,k)}x_kz^k+\sum_{k=1}^{\infty}\frac{1-\alpha}{(k+\alpha)k^nC(\lambda,k)}\overline{y_kz^k}, \tag{2.6}$$

where $n,\lambda\in\mathbb{N}_0$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$ show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.6) are in $M_{\mathcal{H}}(n,\lambda,\alpha)$ because

$$\sum_{k=1}^{\infty}\left[\frac{k-\alpha}{1-\alpha}|a_k|+\frac{k+\alpha}{1-\alpha}|b_k|\right]k^nC(\lambda,k)=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=2. \tag{2.7}$$

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_n=h+\overline{g_n}$, where h and g_n are of the form (1.4).

Theorem 2.2. *Let $f_n=h+\overline{g_n}$ be given by* (1.4)*. Then $f_n\in M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\left[(k-\alpha)|a_k|+(k+\alpha)|b_k|\right]k^nC(\lambda,k)\leq 2(1-\alpha), \tag{2.8}$$

where $a_1=1$, $n,\lambda\in\mathbb{N}_0$, $C(\lambda,k)=\binom{k+\lambda-1}{\lambda}$, and $0\leq\alpha<1$.

Proof. Since $M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)\subset M_{\mathcal{H}}(n,\lambda,\alpha)$, we only need to prove the "if and only if" part of the theorem. To this end, for functions f_n of the form (1.4), we notice that the condition (1.3) is equivalent to

$$\operatorname{Re}\left\{\frac{(1-\alpha)z-\sum_{k=2}^{\infty}(k-\alpha)k^nC(\lambda,k)a_kz^k-(-1)^{2n}\sum_{k=1}^{\infty}(k+\alpha)k^nC(\lambda,k)b_k\overline{z^k}}{z-\sum_{k=2}^{\infty}k^nC(\lambda,k)a_kz^k+(-1)^{2n}\sum_{k=1}^{\infty}k^nC(\lambda,k)b_k\overline{z^k}}\right\}\geq 0. \tag{2.9}$$

The above required condition (2.9) must hold for all values of z in $\mathbb{U}$. Upon choosing the values of z on the positive real axis, where $0 \leq z = r < 1$, we must have

$$\frac{1-\alpha-\sum_{k=2}^{\infty}(k-\alpha)k^nC(\lambda,k)a_kr^{k-1}-\sum_{k=1}^{\infty}(k+\alpha)k^nC(\lambda,k)b_kr^{k-1}}{1-\sum_{k=2}^{\infty}k^nC(\lambda,k)a_kr^{k-1}+\sum_{k=1}^{\infty}k^nC(\lambda,k)\overline{b_kr^{k-1}}} \geq 0. \tag{2.10}$$

If the condition (2.8) does not hold, then the numerator in (2.10) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.8) is negative. This contradicts the required condition for $f_n \in M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$ and so the proof is complete. □

3. Distortion bounds

In this section, we will obtain distortion bounds for functions in $M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$.

Theorem 3.1. *Let $f_n \in M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$. Then for $|z| = r < 1$, one has*

$$\begin{aligned}|f_n(z)| &\leq (1+|b_1|)r+\frac{1}{2^n(\lambda+1)}\left(\frac{1-\alpha}{2-\alpha}-\frac{1+\alpha}{2-\alpha}|b_1|\right)r^2,\\ |f_n(z)| &\geq (1-|b_1|)r-\frac{1}{2^n(\lambda+1)}\left(\frac{1-\alpha}{2-\alpha}-\frac{1+\alpha}{2-\alpha}|b_1|\right)r^2.\end{aligned} \tag{3.1}$$

Proof. We only prove the left-hand inequality. The proof for the right-hand inequality is similar and will be omitted. Let $f_n \in M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$. Taking the absolute value of f_n, we obtain

$$\begin{aligned}|f_n(z)| &= \left|z-\sum_{k=2}^{\infty}a_kz^k+(-1)^n\sum_{k=1}^{\infty}b_k\overline{z}^k\right|\\ &\geq (1-|b_1|)r-\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k\\ &\geq (1-|b_1|)r-r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|)\\ &\geq (1-|b_1|)r-\frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)}\left(\sum_{k=2}^{\infty}\frac{(2-\alpha)2^n(\lambda+1)}{1-\alpha}|a_k|+\frac{(2-\alpha)2^n(\lambda+1)}{1-\alpha}|b_k|\right)r^2\\ &\geq (1-|b_1|)r-\frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)}\left(\sum_{k=2}^{\infty}\frac{(k-\alpha)k^nC(\lambda,k)}{1-\alpha}|a_k|+\frac{(k+\alpha)k^nC(\lambda,k)}{1-\alpha}|b_k|\right)r^2\\ &\geq (1-|b_1|)r-\frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)}\left(1-\frac{1+\alpha}{1-\alpha}|b_1|\right)r^2.\end{aligned} \tag{3.2}$$

The functions

$$\begin{aligned}f(z) &= z+|b_1|\overline{z}+\frac{1}{2^n(\lambda+1)}\left(\frac{1-\alpha}{2-\alpha}-\frac{1+\alpha}{2-\alpha}|b_1|\right)\overline{z}^2,\\ f(z) &= (1-|b_1|)z-\frac{1}{2^n(\lambda+1)}\left(\frac{1-\alpha}{2-\alpha}-\frac{1+\alpha}{2-\alpha}|b_1|\right)z^2\end{aligned} \tag{3.3}$$

for $|b_1| \leq (1-\alpha)/(1+\alpha)$ show that the bounds given in Theorem 3.1 are sharp. □

The following covering result follows from the left-hand inequality in Theorem 3.1.

Corollary 3.2. *If the function* $f_n = h + \overline{g_n}$, *where* h *and* g *given by* (1.4) *are in* $M_{\overline{\mathcal{H}}}(n, \lambda, \alpha)$, *then*

$$\left\{ w : |w| < \frac{2^{n+1}(\lambda+1) - 1 - (2^n(\lambda+1) - 1)\alpha}{2^n(\lambda+1)(2-\alpha)} - \frac{2^{n+1}(\lambda+1) - 1 - (2^n(\lambda+1) + 1)\alpha}{2^n(\lambda+1)(2-\alpha)} |b_1| \right\} \subset f_n(\mathbb{U}). \tag{3.4}$$

4. Convolution, convex combination, and extreme points

In this section, we show that the class $M_{\overline{\mathcal{H}}}(n, \lambda, \alpha)$ is invariant under convolution and convex combination of its member.

For harmonic functions $f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k \overline{z}^k$ and $F_n(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^n \sum_{k=1}^{\infty} B_k \overline{z}^k$, the convolution of f_n and F_n is given by

$$(f_n * F_n)(z) = f_n(z) * F_n(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k B_k \overline{z}^k. \tag{4.1}$$

Theorem 4.1. *For* $0 \le \beta \le \alpha < 1$, *let* $f_n \in M_{\overline{\mathcal{H}}}(n, \lambda, \alpha)$ *and* $F_n \in M_{\overline{\mathcal{H}}}(n, \lambda, \beta)$. *Then* $f_n * F_n \in M_{\overline{\mathcal{H}}}(n, \lambda, \alpha) \subset M_{\overline{\mathcal{H}}}(n, \lambda, \beta)$.

Proof. We wish to show that the coefficients of $f_n * F_n$ satisfy the required condition given in Theorem 2.2. For $F_n \in M_{\overline{\mathcal{H}}}(n, \lambda, \beta)$, we note that $|A_k| \le 1$ and $|B_k| \le 1$. Now, for the convolution function $f_n * F_n$, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{(k-\beta)k^n C(\lambda,k)}{1-\beta} |a_k||A_k| + \sum_{k=1}^{\infty} \frac{(k+\beta)k^n C(\lambda,k)}{1-\beta} |b_k||B_k| \\
&\quad \le \sum_{k=2}^{\infty} \frac{(k-\beta)k^n C(\lambda,k)}{1-\beta} |a_k| + \sum_{k=1}^{\infty} \frac{(k+\beta)k^n C(\lambda,k)}{1-\beta} |b_k| \\
&\quad \le \sum_{k=2}^{\infty} \frac{(k-\alpha)k^n C(\lambda,k)}{1-\alpha} |a_k| + \sum_{k=1}^{\infty} \frac{(k+\alpha)k^n C(\lambda,k)}{1-\alpha} |b_k| \le 1,
\end{aligned} \tag{4.2}$$

since $0 \le \beta \le \alpha < 1$ and $f_n \in M_{\overline{\mathcal{H}}}(n, \lambda, \alpha)$. Therefore $f_n * F_n \in M_{\overline{\mathcal{H}}}(n, \lambda, \alpha) \subset M_{\overline{\mathcal{H}}}(n, \lambda, \beta)$.

We now examine the convex combination of $M_{\overline{\mathcal{H}}}(n, \lambda, \alpha)$.

Let the functions $f_{n_j}(z)$ be defined, for $j = 1, 2, \ldots$, by

$$f_{n_j}(z) = z - \sum_{k=2}^{\infty} |a_{k,j}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{k,j}| \overline{z}^k. \tag{4.3}$$

□

Theorem 4.2. *Let the functions* $f_{n_j}(z)$ *defined by* (4.3) *be in the class* $M_{\overline{\mathcal{H}}}(n, \lambda, \alpha)$ *for every* $j = 1, 2, \ldots, m$. *Then the functions* $t_j(z)$ *defined by*

$$t_j(z) = \sum_{j=1}^{m} c_j f_{n_j}(z), \quad 0 \le c_j \le 1 \tag{4.4}$$

are also in the class $M_{\overline{\mathcal{H}}}(n, \lambda, \alpha)$, *where* $\sum_{j=1}^{m} c_j = 1$.

Proof. According to the definition of t_j, we can write

$$t_j(z) = z - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m} c_j a_{k,j}\right) z^k + (-1)^n \sum_{k=1}^{\infty}\left(\sum_{j=1}^{m} c_j b_{n,j}\right)\overline{z}^k. \tag{4.5}$$

Further, since $f_{n_j}(z)$ are in $M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$ for every $j = 1,2,\ldots$, then by (2.8), we have

$$\begin{aligned}
\sum_{k=1}^{\infty}&\left\{\left[(k-\alpha)\left(\sum_{j=1}^{m} c_j\left|a_{k,j}\right|\right) + (k+\alpha)\left(\sum_{j=1}^{m} c_j\left|b_{k,j}\right|\right)\right]k^n C(\lambda,k)\right\}\\
&= \sum_{j=1}^{m} c_j\left(\sum_{k=1}^{\infty}\left[(k-\alpha)\left|a_{n,j}\right| + (k+\alpha)\left|b_{n,j}\right|\right]k^n C(\lambda,k)\right)\\
&\leq \sum_{j=1}^{m} c_j 2(1-\alpha) \leq 2(1-\alpha).
\end{aligned} \tag{4.6}$$

Hence the theorem follows. □

Corollary 4.3. *The class $M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$ is closed under convex linear combination.*

Proof. Let the functions $f_{n_j}(z)$ $(j = 1,2)$ defined by (4.1) be in the class $M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$. Then the function $\Psi(z)$ defined by

$$\Psi(z) = \mu f_{n_1}(z) + (1-\mu) f_{n_2}(z), \quad 0 \leq \mu \leq 1 \tag{4.7}$$

is in the class $M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$. Also, by taking $m = 2$, $t_1 = \mu$, and $t_2 = (1-\mu)$ in Theorem 4.1, we have the corollary.

Next we determine the extreme points of closed convex hulls of $M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$ denoted by $\operatorname{clco} M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$. □

Theorem 4.4. *Let f_n be given by* (1.4). *Then $f_n \in M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$ if and only if*

$$f_n(z) = \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_{n_k}(z)\right), \tag{4.8}$$

where $h_1(z) = z$, $h_k(z) = z - ((1-\alpha)/(k-\alpha)k^n C(\lambda,k))z^k$, $k = 2,3,\ldots$, $g_{n_k}(z) = z + (-1)^n((1-\alpha)/(k+\alpha)k^n C(\lambda,k))\overline{z}^k$, $k = 1,2,3,\ldots$, and $\sum_{k=1}^{\infty}(X_k + Y_k) = 1$, $X_k \geq 0$, $Y_k \geq 0$. In particular, the extreme points of $M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. For the functions f_n of the form (4.8), we have

$$\begin{aligned}
f_n(z) &= \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_{n_k}(z)\right)\\
&= \sum_{k=1}^{\infty}(X_k + Y_k)z - \sum_{k=2}^{\infty}\frac{1-\alpha}{(k-\alpha)k^n C(\lambda,k)}X_k z^k + (-1)^n\sum_{k=1}^{\infty}\frac{1-\alpha}{(k+\alpha)k^n C(\lambda,k)}Y_k\overline{z}^k.
\end{aligned} \tag{4.9}$$

Then

$$\sum_{k=2}^{\infty}\frac{(k-\alpha)k^n C(\lambda,k)}{1-\alpha}\left|a_k\right| + \sum_{k=1}^{\infty}\frac{(k+\alpha)k^n C(\lambda,k)}{1-\alpha}\left|b_k\right| = \sum_{k=2}^{\infty}X_k + \sum_{k=1}^{\infty}Y_k = 1 - X_1 \leq 1, \tag{4.10}$$

and so $f_n \in \text{clco}M_{\overline{\mathscr{H}}}(n,\lambda,\alpha)$.

Conversely, suppose that $f_n \in \text{clco}M_{\overline{\mathscr{H}}}(n,\lambda,\alpha)$. Setting

$$\begin{aligned} X_k &= \frac{(k-\alpha)k^n C(\lambda,k)}{1-\alpha}|a_k|, \quad 0 \le X_k \le 1,\ k=2,3,\ldots, \\ Y_k &= \frac{(k+\alpha)k^n C(\lambda,k)}{1-\alpha}|b_k|, \quad 0 \le Y_k \le 1,\ k=1,2,3,\ldots, \end{aligned} \tag{4.11}$$

and $X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k$. Therefore, f_n can be written as

$$\begin{aligned} f_n(z) &= z - \sum_{k=2}^{\infty}|a_k|z^k + (-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^k \\ &= z - \sum_{k=2}^{\infty}\frac{(1-\alpha)X_k}{(k-\alpha)k^n C(\lambda,k)}z^k + (-1)^n\sum_{k=1}^{\infty}\frac{(1-\alpha)Y_k}{(k+\alpha)k^n C(\lambda,k)}\overline{z}^k \\ &= z + \sum_{k=2}^{\infty}\left(h_k(z)-z\right)X_k + \sum_{k=1}^{\infty}\left(g_{n_k}(z)-z\right)Y_k \\ &= \sum_{k=2}^{\infty}h_k(z)X_k + \sum_{k=1}^{\infty}g_{n_k}(z)Y_k + z\left(1-\sum_{k=2}^{\infty}X_k - \sum_{k=1}^{\infty}Y_k\right) \\ &= \sum_{k=1}^{\infty}\left(h_k(z)X_k + g_{n_k}(z)Y_k\right), \text{ as required.} \end{aligned} \tag{4.12}$$

□

Using Corollary 4.3 we have $\text{clco}M_{\overline{\mathscr{H}}}(n,\lambda,\alpha) = M_{\overline{\mathscr{H}}}(n,\lambda,\alpha)$. Then the statement of Theorem 4.4 is really for $f \in M_{\overline{\mathscr{H}}}(n,\lambda,\alpha)$.

5. An application of neighborhood

In this section, we will prove that the functions in a neighborhood of $M_{\overline{\mathscr{H}}}(n,\lambda,\alpha)$ are starlike harmonic functions.

Following [10], we defined the δ-neighborhood of a function $f \in \mathcal{T}H$ by

$$\mathcal{N}_\delta(f) = \left\{F(z) = z - \sum_{k=2}^{\infty}A_k z^k - \sum_{k=1}^{\infty}B_k\overline{z}^k, \sum_{k=2}^{\infty}k\left[|a_k - A_k| + |b_k - B_k|\right] + |b_1 - B_1| \le \delta\right\}, \tag{5.1}$$

where $\delta > 0$.

Theorem 5.1. *Let*

$$\delta = \frac{(2-\alpha)2^n(\lambda+1) - 1 + \alpha - \left((2-\alpha)2^n(\lambda+1) - 1 - \alpha\right)|b_1|}{(2-\alpha)2^n(\lambda+1)}. \tag{5.2}$$

Then $\mathcal{N}_\delta(M_{\overline{\mathscr{H}}}(n,\lambda,\alpha)) \subset \mathcal{T}H$.

Proof. Suppose $f_n \in M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$. Let $F_n = H + \overline{G_n} \in \mathcal{N}_\delta(f_n)$, where $H = z - \sum_{k=2}^{\infty} A_k z^k$ and $G_n = (-1)^n \sum_{k=1}^{\infty} B_k z^k$. We need to show that $F_n \in \mathcal{T}H$. In other words, it suffices to show that F_n satisfies the condition $\mathcal{T}(F) = \sum_{k=2}^{\infty} k[|A_k| + |B_k|] + |B_1| \leq 1$. We observe that

$$\begin{aligned}
\mathcal{T}(F) &= \sum_{k=2}^{\infty} k\left[\left|A_k\right| + \left|B_k\right|\right] + \left|B_1\right| \\
&= \sum_{k=2}^{\infty} k\left[\left|A_k - a_k + a_k\right| + \left|B_k - b_k + b_k\right|\right] + \left|B_1 - b_1 + b_1\right| \\
&= \sum_{k=2}^{\infty} k\left[\left|A_k - a_k\right| + \left|B_k - b_k\right|\right] + \sum_{k=2}^{\infty} k\left[\left|a_k\right| + \left|b_k\right|\right] + \left|B_1 - b_1\right| + \left|b_1\right| \\
&= \left(\sum_{k=2}^{\infty} k\left[\left|A_k - a_k\right| + \left|B_k - b_k\right|\right] + \left|B_1 - b_1\right|\right) + \sum_{k=2}^{\infty} k\left[\left|a_k\right| + \left|b_k\right|\right] + \left|b_1\right| \\
&= \delta + \left|b_1\right| + \sum_{k=2}^{\infty} k\left[\left|a_k\right| + \left|b_k\right|\right] \\
&= \delta + \left|b_1\right| + \frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)} \sum_{k=2}^{\infty} \left[\frac{2-\alpha}{1-\alpha}\left|a_k\right| + \frac{2+\alpha}{1-\alpha}\left|b_k\right|\right] 2^n(\lambda+1) \\
&\leq \delta + \left|b_1\right| + \frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)} \sum_{k=2}^{\infty} \left[\frac{k-\alpha}{1-\alpha}\left|a_k\right| + \frac{k+\alpha}{1-\alpha}\left|b_k\right|\right] k^n C(\lambda,k) \\
&\leq \delta + \left|b_1\right| + \frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)} \left(1 - \frac{1+\alpha}{1-\alpha}\left|b_1\right|\right).
\end{aligned} \tag{5.3}$$

Now this last expression is never greater than one if

$$\begin{aligned}
\delta &\leq 1 - \left|b_1\right| - \frac{1-\alpha}{(2-\alpha)2^n(\lambda+1)} \left(1 - \frac{1+\alpha}{1-\alpha}\left|b_1\right|\right) \\
&= \frac{(2-\alpha)2^n(\lambda+1) - 1 + \alpha - \left((2-\alpha)2^n(\lambda+1) - 1 - \alpha\right)\left|b_1\right|}{(2-\alpha)2^n(\lambda+1)}.
\end{aligned} \tag{5.4}$$

□

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