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Research Article

A Univalence Preserving Integral Operator

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We define an integral operator denoted by I, and we give sufficient conditions such that $F = I(f_1, f_2, \dots, f_m)$ is univalent.

1. Introduction and preliminaries

Let U denote the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}. \tag{1.1}$$

Let $\mathscr{H}(U)$ denote the space of holomorphic functions in U and let

$$A_n = \{f \in \mathscr{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \cdots,\ z \in U\} \tag{1.2}$$

with $A_1 = A$.

Let

$$S = \{f \in A : f \text{ is univalent in } U\}. \tag{1.3}$$

Definition 1.1 (Ruscheweyh [1]). For $f \in A$, $n \in \mathbb{N} \cup \{0\}$, let R^n be the operator defined by $R^n : A \to A$,

$$\begin{gathered} R^0 f(z) = f(z), \\ (n+1)R^{n+1} f(z) = z\left[R^n f(z)\right]' + nR^n f(z), \quad z \in U. \end{gathered} \tag{1.4}$$

Remark 1.2. If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in U, \tag{1.5}$$

then

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j, \quad z \in U. \tag{1.6}$$

In order to prove our main results, we shall use the following lemmas.

Lemma A (see [2]). *Let α be a complex number,* $\operatorname{Re}\alpha > 0$, *and $f \in A$. If*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \le 1, \quad \forall z \in U, \tag{1.7}$$

then the function

$$F_\alpha(z) = \left[\alpha \int_0^z t^{\alpha-1} f'(t) dt \right]^{1/\alpha} \tag{1.8}$$

is in the class S.

Lemma B (see [3]). *Let α be a complex number,* $\operatorname{Re}\alpha > 0$, *and let $f(z) = z + a_2 z^2 + \cdots$ be a regular function in U. If*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \le 1, \quad \forall z \in U, \tag{1.9}$$

then, for any complex number β with $\operatorname{Re}\beta \ge \operatorname{Re}\alpha$, *the function*

$$F_\beta(z) = \left[\beta \int_0^\beta t^{\beta-1} f'(t) dt \right]^{1/\beta} \tag{1.10}$$

is in the class S.

2. Main results

By using the Ruscheweyh differential operator given by Definition 1.1, we introduce the following integral operator.

Definition 2.1. Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, 3, \ldots, m\}$, $\alpha_i \in \mathbb{C}$, $\alpha \in \mathbb{C}$, with $\operatorname{Re}\alpha > 0$, $A^m = \underbrace{A \times A \times \cdots \times A}_{m \text{ times}}$. We let $I : A^m \to A$ be the integral operator given by

$$I(f_1, f_2, \ldots, f_m)(z) = F(z) = \left[\alpha \int_0^z t^{\alpha-1} \left(\frac{R^n f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{R^n f_m(t)}{t}\right)^{\alpha_m} dt\right]^{1/\alpha}, \tag{2.1}$$

where $f_i \in A$, $i \in \{1, 2, 3, \ldots, n\}$ and R^n is the Ruscheweyh differential operator.

Remark 2.2. (i) For $n = 0$, $m = 1$, $\alpha = 1$, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$, and $f(z) \in A$, we obtain Alexander integral operator introduced in 1915 in [4]:

$$I(z) = \int_0^z \frac{f(t)}{t} dt, \quad z \in U. \tag{2.2}$$

(ii) For $n = 0$, $m = 1$, $\alpha = 1$, $\alpha_1 = \beta \in [0, 1]$, $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$, and $f(z) \in S$, we obtain the integral operator

$$I(z) = \int_0^z \left[\frac{f(t)}{t}\right]^\beta dt, \quad z \in U, \tag{2.3}$$

studied in [5].

For $\beta \in \mathbb{C}$ with $|\beta| \le 1/4$, this integral operator was studied in [6, 7] and for $|\beta| \le 1/3$, in [8].

(iii) For $n = 1$, $m = 1$, $\alpha = 1$, $\alpha_1 = \beta \in \mathbb{C}$, $|\beta| \le 1/4$, $\alpha_2 = \cdots = \alpha_m = 0$, $R^1 f(z) = z f'(z)$, $z \in U$, $f \in S$, we obtain the integral operator

$$I(z) = \int_0^z [f'(t)]^\beta dt, \quad z \in U \tag{2.4}$$

studied in [9].

(iv) For $n = 0$, $m \in \mathbb{N} \cup \{0\}$, $\alpha = 1$, $\alpha_i > 0$, $i \in \{1, 2, \ldots, m\}$, we obtain the integral operator

$$F(z) = \int_0^z \left[\frac{f_1(t)}{t}\right]^{\alpha_1} \cdots \left[\frac{f_m(t)}{t}\right]^{\alpha_m} dt, \tag{2.5}$$

studied in [10].

(v) For $n = 0$, $m \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{R}$, $\operatorname{Re}\alpha > 0$, $\alpha_i \in \mathbb{C}$, $f_i \in S$, $i \in \{1, 2, \ldots, n\}$, we obtain the integral operator introduced in [10] by D. Breaz and N. Breaz:

$$G(z) = \left[\alpha \int_0^z t^{\alpha-1} \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\alpha_m} dt\right]^{1/\alpha}. \tag{2.6}$$

(vi) For $n, m \in \mathbb{N} \cup \{0\}$, $\alpha = 1$, $\alpha_i \in \mathbb{R}$, $i \in \{1, 2, \ldots, m\}$ with $\alpha_i \geq 0$, we obtain the integral operator studied in [11, 12]

$$I(f_1, f_2, \ldots, f_m)(z) = \int_0^z \left[\frac{R^n f_1(t)}{t}\right]^{\alpha_1} \cdots \left[\frac{R^n f_m(t)}{t}\right]^{\alpha_m} dt. \tag{2.7}$$

(vii) For $n = 0$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_m = 0$, $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha \geq 3$, we obtain the integral operator studied in [7, 13]

$$G_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{g(u)}{u}\right) du\right]^{1/\alpha}. \tag{2.8}$$

(viii) For $n = 1$, $m = 1$, $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$, $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_m = 0$, we obtain the integral operator

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du\right]^{1/\alpha} \tag{2.9}$$

studied in [14, 15].

We study the conditions for the integral operator introduced in Definition 2.1 to be univalent.

Theorem 2.3. *Let $n, m \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$, $f_i \in A$, $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \ldots, m\}$ with $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \leq 1$.*

If

$$\left|\frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1\right| \leq 1, \quad z \in U,\ i \in \{1, 2, \ldots, m\}, \tag{2.10}$$

then $F(z)$ given by (2.1) *belongs to class S.*

Proof. Let

$$f(z) = \int_0^z \left[\frac{R^n f_1(t)}{t}\right]^{\alpha_1} \cdots \left[\frac{R^n f_m(t)}{t}\right]^{\alpha_m} dt, \quad z \in U. \tag{2.11}$$

By differentiating (2.11), we obtain

$$f'(z) = \left[\frac{R^n f_1(z)}{z}\right]^{\alpha_1} \cdots \left[\frac{R^n f_m(z)}{z}\right]^{\alpha_m} = 1 + A_2 z + A_3 z^2 + \cdots, \tag{2.12}$$

$z \in U$.

From (2.11), (2.12), and the condition in the theorem, we have that $f'(z) \neq 0$, $z \in U$.

Then using (2.12), we obtain

$$\log f'(z) = \alpha_1\left[\log R^n f_1(z) - \log z\right] + \cdots + \alpha_m\left[\log R^n f_m(z) - \log z\right], \quad z \in U. \tag{2.13}$$

By differentiating (2.13), after a short calculation, we have

$$\frac{zf''(z)}{f'(z)} = \alpha_1\left[\frac{z\left(R^n f_1(z)\right)'}{R^n f_1(z)} - 1\right] + \cdots + \alpha_m\left[\frac{z\left(R^n f_m(z)\right)'}{R^n f_m(z)} - 1\right], \quad z \in U. \tag{2.14}$$

Using the conditions given by the hypothesis of Theorem 2.3, we obtain

$$\begin{aligned}\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| &= \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left[|\alpha_1|\left|\frac{z\left(R^n f_1(z)\right)'}{R^n f_1(z)} - 1\right| + \cdots + |\alpha_m|\left|\frac{z\left(R^n f_m(z)\right)'}{R^n f_m(z)} - 1\right|\right]\\ &\le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left[|\alpha_1| + |\alpha_2| + \cdots + |\alpha_m|\right]\\ &\le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \le \frac{1}{\operatorname{Re}\alpha} \le 1.\end{aligned} \tag{2.15}$$

Using (2.12), the conditions in Lemma A are satisfied, hence $F(z)$ belongs to the class S. □

Remark 2.4. For $\alpha_i \in \mathbb{R}$, $i \in \{1, 2, \ldots, m\}$, then Theorem 2.3 can be rewritten as following.

Corollary 2.5. *Let $n, m \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{C}$, with $\operatorname{Re}\alpha > 0$, let $\alpha_i \in \mathbb{R}$, $\alpha_i \ge 0$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_m \le 1$ and let $f_i \in A$, $i \in \{1, 2, \ldots, m\}$.*

If

$$\left|\frac{z\left(R^n f_i(z)\right)'}{R^n f_i(z)} - 1\right| \le 1, \quad z \in U,\ i \in \{1, 2, \ldots, m\}, \tag{2.16}$$

then $F(z)$ given by (2.1) belongs to the class S, where R^n is the Ruscheweyh differential operator.

Example 2.6. Let $n \in \mathbb{N} \cup \{0\}$, $m = 2$, $\alpha = 2 + 3i$, $\alpha_1 = 1/4 + i(\sqrt{3}/4)$, $\alpha_2 = 1/5 - i(2/5)$, $|\alpha_1| + |\alpha_2| = (5 + 2\sqrt{5})/10 < 1$, $f_1(z) = z + az^2$, $R^n f_1(z) = z + (n+1)az^2$, $f_2(z) = z + bz^2$, $R^n f_2(z) = z + (n+1)bz^2$, $a, b \in \mathbb{C}$, with $|a| \le 1/2(n+1)$, $|b| \le 1/2(n+1)$, $z \in U$. Then

$$\left|\frac{z\left[z + (n+1)bz^2\right]'}{z\left[1 + (n+1)bz\right]} - 1\right| = \left|\frac{1 + 2(n+1)bz}{1 + (n+1)bz} - 1\right| = \left|\frac{(n+1)bz}{1 + (n+1)bz}\right|, \quad z \in U. \tag{2.17}$$

But

$$\left|\frac{(n+1)bz}{1+(n+1)bz}\right|^2 = \frac{|(n+1)bz|^2}{|1+(n+1)bz|^2}$$
$$= \frac{(n+1)bz(n+1)\overline{b}\overline{z}}{(1+(n+1)bz)(1+(n+1)\overline{b}\overline{z})}$$
$$= \frac{(n+1)^2|b|^2|z|^2}{1+(n+1)bz+(n+1)\overline{b}\overline{z}+(n+1)^2|b|^2|z|^2}$$
$$= \frac{(n+1)^2|b|^2|z|^2}{1+2(n+1)\operatorname{Re} bz+(n+1)^2|b|^2|z|^2} \tag{2.18}$$
$$\leq \frac{(n+1)^2|b|^2|z|^2}{1-2(n+1)|b||z|+(n+1)^2|b|^2|z|^2} = \frac{(n+1)^2|b|^2|z|^2}{(1-(n+1)|b||z|)^2}$$
$$\leq \frac{(n+1)|b|^2}{[1-(n+1)|b|]^2} \leq \frac{(n+1)\cdot 1/4(n+1)^2}{[1-(n+1)\cdot 1/2(n+1)]^2}$$
$$= \frac{1/4(n+1)^2}{1/4} = \frac{1}{(n+1)^2} \leq 1.$$

It implies that

$$\left|\frac{(n+1)bz}{1+(n+1)bz}\right| \leq 1, \quad z \in U. \tag{2.19}$$

Similarly, we obtain

$$\left|\frac{z[z+(n+1)az^2]'}{z[1+(n+1)az]} - 1\right| = \left|\frac{(n+1)az}{1+(n+1)az}\right| \leq 1, \quad z \in U. \tag{2.20}$$

Using Theorem 2.3, we have

$$I(f_1, f_2) = F(z) = \left[(2+3i)\int_0^z t^{1+3i}(1+at)^{1/4+i(\sqrt{3}/4)}(1+bt)^{1/5-i(2/5)}dt\right]^{1/(2+3i)} \in S, \tag{2.21}$$

for all $z \in U$.

Theorem 2.7. *Let* $n, m \in \mathbb{N} \cup \{0\}$, $\alpha, \beta \in \mathbb{C}$, *with* $\operatorname{Re}\beta \geq \operatorname{Re}\alpha > 0$, *let* $f_i \in A$, *and let* $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \ldots, m\}$, *with* $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \leq 1$.

If

$$\left|\frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1\right| \leq 1, \quad z \in U, \ i \in \{1, 2, \ldots, m\}, \tag{2.22}$$

where R^n is the Ruscheweyh differential operator, then the function given by

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{R^n f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{R^n f_m(t)}{t}\right)^{\alpha_m} dt\right]^{1/\beta} \tag{2.23}$$

belongs to the class S.

Proof. Using (2.14) and (2.22), from the proof of Theorem 2.3, we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left|\frac{zf''(z)}{f'(z)}\right| \le 1, \quad z \in U. \tag{2.24}$$

Using (2.12), the conditions from Lemma B are satisfied and by applying it we have that the function $F_\beta(z)$ given by (2.23) belongs to the class S. □

Example 2.8. Let $n \in \mathbb{N} \cup \{0\}$, $m = 2$, $\beta = 4 - i$, $\alpha = 2 + 3i$, $\operatorname{Re}\beta > \operatorname{Re}\alpha > 0$, $\alpha_1 = 1/4 + i(\sqrt{3}/4)$, $\alpha_2 = 1/5 - i(2/5)$, $|\alpha_1| + |\alpha_2| = (5 + 2\sqrt{5})/10 < 1$, $f_1(z) = z + az^2$, $f_2(z) = z + bz^2$, $R^n f_1(z) = z + (n+1)az^2$, $R^n f_2(z) = z + (n+1)bz^2$, $a, b \in \mathbb{C}$, $|a| \le 1/2(n+1)$, $|b| \le 1/2(n+1)$, $z \in U$. Then

$$\begin{aligned} &\left|\frac{z(z+(n+1)az^2)'}{z+(n+1)az^2} - 1\right| = \left|\frac{(n+1)az}{1+(n+1)az}\right| \le 1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z \in U. \\ &\left|\frac{z(z+(n+1)bz^2)'}{z(1+(n+1)bz)} - 1\right| = \left|\frac{(n+1)bz}{1+(n+1)bz}\right| \le 1, \end{aligned} \tag{2.25}$$

Using Theorem 2.7, we have

$$F_\beta(z) = \left[(4-i)\int_0^z t^{3-i}(1+at)^{1/4+i(\sqrt{3}/4)}(1+bt)^{1/5-i(2/5)} dt\right]^{1/(4-i)} \in S, \tag{2.26}$$

for all $z \in U$.

Theorem 2.9. *Let $n, m \in \mathbb{N} \cup \{0\}$, $\mu > 0$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, let $f_i \in A$ and let $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \ldots, m\}$ with $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \le 1/(2\mu + 1)$.*

If

$$\begin{aligned} &\text{(i)}\ \left|R^n f_i(z)\right| \le \mu, \\ &\text{(ii)}\ \left|\frac{z^2(R^n f_i(z))'}{[R^n f_i(z)]^2} - 1\right| \le 1, \quad z \in U,\ i \in \{1, 2, \ldots, m\}, \end{aligned} \tag{2.27}$$

where R^n is the Ruscheweyh differential operator, then the function $F(z)$ given by (2.1) *belongs to the class S.*

Proof. Using (2.14), we have

$$\left|\frac{zf''(z)}{f'(z)}\right| = |\alpha_1|\left|\frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1\right| + \cdots + |\alpha_m|\left|\frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1\right|, \quad z \in U. \tag{2.28}$$

Using the conditions from Theorem 2.9 and (2.28) we calculate

$$\begin{aligned}
&\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \\
&= \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha_1|\left|\frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1\right| + \cdots + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha_m|\left|\frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1\right| \\
&\leq \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha_1|\left[\left|\frac{z(R^n f_1(z))}{R^n f_1(z)}\right| + 1\right] + \cdots + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha_m|\left[\left|\frac{z(R^n f_m(z))}{R^n f_m(z)}\right| + 1\right] \\
&= \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha_1|\left|\frac{z(R^n f_1(z))'}{R^n f_1(z)}\right| + \cdots + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha_m|\left|\frac{z(R^n f_m(z))'}{R^n f_m(z)}\right| \\
&\quad + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(|\alpha_1| + \cdots + |\alpha_m|\right) \\
&= \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha_1|\left|\frac{z^2(R^n f_1(z))'}{(R^n f_1(z))^2}\right|\frac{|R^n f_1(z)|}{|z|} + \cdots + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha_m|\left|\frac{z^2(R^n f_m(z))'}{(R^n f_m(z))^2}\right|\frac{|R^n f_m(z)|}{|z|} \\
&\quad + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(|\alpha_1| + \cdots + |\alpha_m|\right) \\
&\leq \mu\left[\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{z^2(R^n f_1(z))'}{(R^n f_1(z))^2}\right| + \cdots + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{z^2(R^n f_m(z))'}{(R^n f_m(z))^2}\right|\right] \\
&\quad + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(|\alpha_1| + \cdots + |\alpha_m|\right) \\
&= \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left[\left|\frac{z^2(R^n f_1(z))'}{(R^n f_1(z))^2}\right| - 1 + 1\right]\cdot\mu + \cdots + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left[\left|\frac{z^2(R^n f_m(z))'}{(R^n f_m(z))^2}\right| - 1 + 1\right]\cdot\mu \\
&\quad + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(|\alpha_1| + \cdots + |\alpha_m|\right) \\
&\leq \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha_1|\left|\frac{z^2(R^n f_1(z))'}{(R^n f_1(z))^2} - 1\right|\cdot\mu + \cdots + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\alpha_m|\left|\frac{z^2(R^n f_m(z))'}{(R^n f_m(z))^2} - 1\right|\cdot\mu \\
&\quad + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\cdot\left(|\alpha_1| + \cdots + |\alpha_m|\right) + \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\cdot\mu\left(|\alpha_1| + \cdots + |\alpha_m|\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(\left|\alpha_1\right|+\cdots+\left|\alpha_m\right|\right)2\mu+\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\cdot\left(\left|\alpha_1\right|+\cdots+\left|\alpha_m\right|\right)\\
&\leq \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left(\left|\alpha_1\right|+\cdots+\left|\alpha_m\right|\right)(2\mu+1)\\
&\leq \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\leq\frac{1}{\operatorname{Re}\alpha}\leq 1,\quad z\in U.
\end{aligned}
\tag{2.29}
$$

By using (2.12) and (2.29) and by applying Lemma A, we obtain that the function $F(z)$ given by (2.1) belongs to the class S. □

Theorem 2.10. *Let $n,m\in\mathbb{N}\cup\{0\}$, $\mu>0$, $\alpha,\beta\in\mathbb{C}$ with $\operatorname{Re}\beta\geq\operatorname{Re}\alpha>0$, and let $\alpha_i\in\mathbb{C}$, $i\in\{1,2,3,\ldots,m\}$, with $|\alpha_1|+|\alpha_2|+\cdots+|\alpha_m|\leq 1/(2\mu+1)$.*

If

$$
\begin{aligned}
&\text{(i)}\ \left|R^n f_i(z)\right|\leq\mu,\\
&\text{(ii)}\ \left|\frac{z^2R^nf_i(z)}{\left(R^nf_i(z)\right)^2}-1\right|\leq 1,\quad z\in U,\ i\in\{1,2,\ldots,m\},
\end{aligned}
\tag{2.30}
$$

where R^n is the Ruscheweyh differential operator, then function $F_\beta(z)$ given by (2.23) *belongs to the class S.*

Proof. By using (2.30) and (2.12) and from the proof of Theorem 2.9, we have

$$
\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|<1,\quad z\in U,
\tag{2.31}
$$

and applying Lemma B we obtain that the $F_\beta(z)$ given by (2.23) belongs to the class S. □

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