

Research Article

On Certain Subclasses of Meromorphic Close-to-Convex Functions

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By using the operator $D_{\lambda}^n f(z)$, $z \in U$, Definition 2.1, we introduce a class of meromorphic functions denoted by $\Sigma(\alpha, \lambda, n)$ and we obtain certain differential subordinations.

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1. Introduction and preliminaries

Denote by U the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}, \quad \dot{U} = U - \{0\}. \quad (1.1)$$

Let $\mathcal{H}(U)$ be the space of holomorphic function in U .

Let

$$A_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\} \quad (1.2)$$

with $A_1 = A$.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U), f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, z \in U\}. \quad (1.3)$$

Let

$$K = \left\{ f \in A, \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \right\} \quad (1.4)$$

denote the class of normalized convex functions in U .

If f and g are analytic functions in U , then we say that f is subordinate to g , written $f < g$, if there is a function w analytic in U , with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f < g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

A function f , analytic in U , is said to be convex if it is univalent and $f(U)$ is convex.

Let $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and let h be univalent in U . If p is analytic in U and satisfies the (second-order) differential subordination,

$$(i) \quad \psi(p(z), zp'(z), z^2p''(z); z) < h(z), \quad z \in U,$$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p < q$ for all p satisfying (i).

A dominant \tilde{q} that satisfies $\tilde{q} < q$ for all dominants q of (i) is said to be the best dominant of (i). (Note that the best dominant is unique up to a rotation of U .) In order to prove the original results, we use the following lemmas.

Lemma 1.1 (see [1, Theorem 3.1.6, page 71, and the references therein]). *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $\operatorname{Re} \gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) < h(z), \quad z \in U, \quad (1.5)$$

then

$$p(z) < q(z) < h(z), \quad z \in U, \quad (1.6)$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1} dt, \quad z \in U. \quad (1.7)$$

The function q is convex and the best dominant.

Lemma 1.2 (see [2, Lemma 13.5.1, page 375, and the references therein]). *Let g be a convex function in U , and let*

$$h(z) = g(z) + n\alpha zg'(z), \quad z \in U, \quad (1.8)$$

where $\alpha > 0$, and n is a positive integer.

If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots, \quad z \in U \quad (1.9)$$

is holomorphic in U , and

$$p(z) + \alpha zp'(z) < h(z), \quad z \in U, \quad (1.10)$$

then

$$p(z) < g(z), \quad (1.11)$$

and this result is sharp.

Lemma 1.3 (see [1, Corollary 2.6.g.2, page 66]). *Let $f \in A$ and*

$$F(z) = \frac{2}{z} \int_0^z f(t) dt, \quad z \in U. \quad (1.12)$$

If

$$\operatorname{Re} \left(\frac{zf''(z)}{f'(z)} + 1 \right) > -\frac{1}{2}, \quad (1.13)$$

then

$$F \in K. \quad (1.14)$$

Lemma 1.4 (see [3, Lemma 1.5]). *Let $\operatorname{Re} c > 0$, and let*

$$w = \frac{k^2 + |c|^2 - |k^2 - c^2|}{4k \operatorname{Re} c}. \quad (1.15)$$

Let h be an analytic function in U with $h(0) = 1$, and suppose that

$$\operatorname{Re} \left(\frac{zh''(z)}{h'(z)} + 1 \right) > -w, \quad z \in U. \quad (1.16)$$

If $p(z) = 1 + p_k z^k + \dots$ ($k \geq 1$ integer) is analytic in U and

$$p(z) + \frac{1}{c} z p'(z) < h(z), \quad z \in U, \quad (1.17)$$

then

$$p(z) < q(z), \quad z \in U, \quad (1.18)$$

where q is the solution of the differential equation:

$$q(z) + \frac{k}{c} z q'(z) = h(z), \quad q(z) = 1, \quad (1.19)$$

given by

$$q(z) = \frac{c}{kz^{c/k}} \int_0^z t^{c/k-1} h(t) dt. \quad (1.20)$$

Moreover, q is the best dominant.

Definition 1.5 (see [4]). For $f \in A$, $n \in \mathbb{N}^* \cup \{0\}$, the operator $S^n f$ is defined by $S^n : A \rightarrow A$

$$\begin{aligned} S^0 f(z) &= f(z), \\ S^1 f(z) &= z f'(z), \\ &\dots \\ S^{n+1} f(z) &= z [S^n f(z)]', \quad z \in U. \end{aligned} \quad (1.21)$$

Remark 1.6. If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (1.22)$$

then

$$S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j, \quad z \in U. \quad (1.23)$$

Definition 1.7 (see [5]). For $f \in A$, $n \in \mathbb{N}^* \cup \{0\}$, the operator $R^n f$ is defined by $R^n : A \rightarrow A$

$$\begin{aligned} R^0 f(z) &= f(z), \\ R^1 f(z) &= z f'(z), \\ &\dots \\ (n+1)R^{n+1} f(z) &= z [R^n f(z)]' + n R^n f(z), \quad z \in U. \end{aligned} \quad (1.24)$$

Remark 1.8. If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (1.25)$$

then

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^n a_j z^j, \quad z \in U. \quad (1.26)$$

2. Main results

Definition 2.1. Let $n \in \mathbb{N}^* \cup \{0\}$ and $\lambda \geq 0$. Let $D_\lambda^n f$ denote the operator defined by $D_\lambda^n : A \rightarrow A$

$$D_\lambda^n f(z) = (1 - \lambda) S^n f(z) + \lambda R^n f(z), \quad z \in U, \quad (2.1)$$

where the operators $S^n f$ and $R^n f$ are given by Definitions 1.5 and 1.7, respectively.

Remark 2.2. We observe that D_λ^n is a linear operator and for

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (2.2)$$

we have

$$D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} [(1 - \lambda) j^n + \lambda C_{n+j-1}^n] a_j z^j. \quad (2.3)$$

Also, it is easy to observe that if we consider $\lambda = 1$ in Definition 2.1, we obtain the Ruscheweyh differential operator, and if we consider $\lambda = 0$ in Definition 2.1, we obtain the Sălăgean differential operator.

Remark 2.3. For $n = 0$,

$$D_{\lambda}^0 f(z) = (1 - \lambda)S^0 f(z) + \lambda R^0 f(z) = f(z) = S^0 f(z) = R^0 f(z), \quad (2.4)$$

and for $n = 1$,

$$D_{\lambda}^1 f(z) = (1 - \lambda)S^1 f(z) + \lambda R^1 f(z) = z f'(z) = S^1 f(z) = R^1 f(z). \quad (2.5)$$

Remark 2.4. If $f \in \Sigma$,

$$f(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \dots, \quad (2.6)$$

and we let

$$g(z) = z^2 f(z) = z + a_0 z^2 + a_1 z^3 + \dots, \quad z \in U. \quad (2.7)$$

Definition 2.5. If $0 \leq \alpha < 1$, $\lambda \geq 0$, and $n \in \mathbb{N}$, let $\Sigma(\alpha, \lambda, n + 1)$ denote the class of functions $f \in \Sigma$ which satisfy the inequality,

$$\operatorname{Re} \left\{ [D_{\lambda}^{n+1} g(z)]' + \frac{\lambda z n [R^n g(z)]''}{n+1} \right\} > \alpha, \quad (2.8)$$

where $D_{\lambda}^{n+1} g$ is given by Definition 2.1, g is given by (2.7), and $R^n g$ is given by Definition 1.7.

Theorem 2.6. *If $0 \leq \alpha < 1$, $\lambda \geq 0$, and $n \in \mathbb{N}$, then*

$$\Sigma(\alpha, \lambda, n + 1) \subset \Sigma(\delta, \lambda, n + 1), \quad (2.9)$$

where

$$\delta = \delta(\alpha) = 2\alpha - 1 + 2(1 - \alpha) \ln 2. \quad (2.10)$$

Proof. Let $f \in \Sigma(\alpha, \lambda, n + 1)$,

$$g(z) = z^2 f(z) = z + a_0 z^2 + a_1 z^3 + \dots, \quad g \in A. \quad (2.11)$$

Since $f \in \Sigma(\alpha, \lambda, n + 1)$ by using Definition 2.5, we deduce

$$\operatorname{Re} \left\{ [D_{\lambda}^{n+1} g(z)]' + \frac{\lambda n z [R^n g(z)]''}{n+1} \right\} > \alpha, \quad z \in U, \quad (2.12)$$

which is equivalent to

$$[D_{\lambda}^{n+1} g(z)]' + \frac{\lambda n z [R^n g(z)]''}{n+1} < \frac{1 + (2\alpha - 1)z}{1 + z} = h(z), \quad z \in U. \quad (2.13)$$

By using the properties of the operators $D_\lambda^n g$, $S^n g$, and $R^n g$, we have

$$\begin{aligned}
 & [(1-\lambda)S^{n+1}g(z) + \lambda R^{n+1}g(z)]' + \frac{\lambda n z [R^n g(z)]''}{n+1} \\
 &= (1-\lambda)[z(S^n g(z))']' + \lambda \frac{[z(R^n g(z))' + nR^n g(z)]'}{n+1} + \frac{\lambda n z (R^n g(z))''}{n+1} \\
 &= (1-\lambda)[(S^n g(z))' + z(S^n g(z))''] + \lambda \frac{(R^n g(z))' + z(R^n g(z))'' + n[R^n g(z)]'}{n+1} + \frac{\lambda n z (R^n g(z))''}{n+1} \\
 &= (1-\lambda)(S^n g(z))' + \lambda(R^n g(z))' + z[(1-\lambda)(S^n g(z))'' + \lambda(R^n g(z))''], \quad z \in U.
 \end{aligned} \tag{2.14}$$

Using (2.14) in (2.13), we obtain

$$(1-\lambda)(S^n g(z))' + \lambda(R^n g(z))' + z[(1-\lambda)(S^n g(z))'' + \lambda(R^n g(z))''] < \frac{1+(2\alpha-1)z}{1+z}, \quad z \in U. \tag{2.15}$$

Let

$$\begin{aligned}
 p(z) &= [D_\lambda^n g(z)]' \\
 &= (1-\lambda)(S^n g(z))' + \lambda(R^n g(z))' \\
 &= (1-\lambda)\left(z + \sum_{j=2}^{\infty} j^n a_j z^j\right)' + \lambda\left(z + \sum_{j=2}^{\infty} C_{n+j-1}^n a_j z^j\right)' \\
 &= (1-\lambda)\left(1 + \sum_{j=2}^{\infty} j^{n+1} a_j z^{j-1}\right) + \lambda\left(1 + \sum_{j=2}^{\infty} j C_{n+j-1}^n a_j z^{j-1}\right) \\
 &= 1 + \sum_{j=2}^{\infty} [(1-\lambda)j^{n+1} + \lambda j C_{n+j-1}^n] a_j z^{j-1} \\
 &= 1 + b_1 z + b_2 z^2 + \dots, \quad z \in U.
 \end{aligned} \tag{2.16}$$

We have that $p \in \mathcal{H}[1, 1]$. From (2.16), we have

$$p'(z) = (1-\lambda)(S^n g(z))'' + \lambda(R^n g(z))''. \tag{2.17}$$

Using (2.16) and (2.17) in (2.15), we obtain

$$p(z) + z p'(z) < \frac{1+(2\alpha-1)z}{1+z} = h(z), \quad z \in U. \tag{2.18}$$

By using Lemma 1.1, we have

$$p(z) < q(z) < h(z), \quad z \in U, \tag{2.19}$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t) dt = \frac{1}{z} \int_0^z \frac{1 + (2\alpha - 1)t}{1+t} dt = 2\alpha - 1 + 2(1 - \alpha) \frac{\ln(1+z)}{z}, \quad z \in U. \quad (2.20)$$

The function q is convex and best dominant.

Since q is convex and $q(U)$ is symmetric with respect to the real axis, we deduce

$$\operatorname{Re} p(z) > \operatorname{Re} q(1) = \delta = \delta(\alpha) = 2\alpha - 1 + 2(1 - \alpha) \ln 2, \quad (2.21)$$

from which we deduce that $\Sigma(\alpha, \lambda, n+1) \subset \Sigma(\delta, \lambda, n+1)$. \square

Example 2.7. If $n = 0$, $\alpha = 1/2$, $\lambda \geq 0$, then $\delta(1/2) = \ln 2$, and we deduce for $f \in \Sigma$ that

$$\operatorname{Re} [4zf(z) + 5z^2f'(z) + z^3f''(z)] > \frac{1}{2}, \quad z \in U \quad (2.22)$$

implies

$$\operatorname{Re} [2zf(z) + z^2f'(z)] > \ln 2, \quad z \in U. \quad (2.23)$$

Theorem 2.8. Let r be a convex function, $r(0) = 1$, and let h be a function such that

$$h(z) = r(z) + zr'(z), \quad z \in U. \quad (2.24)$$

If $f \in \Sigma$, g is given by (2.7), and the following differential subordination holds

$$[D_\lambda^{n+1}g(z)]' + \frac{\lambda n z [R^n g(z)]''}{n+1} < h(z) = r(z) + zr'(z), \quad z \in U, \quad (2.25)$$

then

$$[D_\lambda^n g(z)]' < r(z), \quad z \in U, \quad (2.26)$$

and this result is sharp.

Proof. By using the properties of the operator $D_\lambda^n g$, we have

$$D_\lambda^{n+1}g(z) = (1 - \lambda)S^{n+1}g(z) + \lambda R^{n+1}g(z). \quad (2.27)$$

By using the properties of operators $S^n g(z)$, $R^n g(z)$, and by differentiating (2.27), we obtain

$$\begin{aligned} [D_\lambda^{n+1}g(z)]' &= [(1 - \lambda)S^{n+1}g(z) + \lambda R^{n+1}g(z)]' \\ &= (1 - \lambda)[(S^n g(z))' + z(S^n g(z))''] + \lambda \frac{(n+1)(R^n g(z))' + z(R^n g(z))''}{n+1}. \end{aligned} \quad (2.28)$$

Using (2.28) in (2.25) and relations (2.16) and (2.17), after a simple calculation, Subordination (2.25) becomes

$$p(z) + zp'(z) < r(z) + zr'(z), \quad z \in U. \quad (2.29)$$

By using Lemma 1.2, we have

$$p(z) < r(z), \quad z \in U, \quad (2.30)$$

that is,

$$[D_{\lambda}^n g(z)]' < r(z), \quad z \in U. \quad (2.31)$$

□

Example 2.9. If $n = 0$, $\lambda \geq 0$, $r(z) = (1+z)/(1-z)$, from Theorem 2.8, we deduce that if $f \in \Sigma$ and

$$4zf(z) + 5z^2f'(z) + z^3f''(z) < \frac{1+2z-z^2}{(1-z)^2}, \quad z \in U, \quad (2.32)$$

then

$$2zf(z) + z^2f'(z) < \frac{1+z}{1-z}, \quad z \in U. \quad (2.33)$$

Theorem 2.10. Let r be a convex function, $r(0) = 1$, and

$$h(z) = r(z) + zr'(z), \quad z \in U. \quad (2.34)$$

If $f \in \Sigma$, g is given by (2.7), and the following differential subordination holds

$$[D_{\lambda}^n g(z)]' < h(z) = r(z) + zr'(z), \quad z \in U, \quad (2.35)$$

then

$$\frac{D_{\lambda}^n g(z)}{z} < r(z), \quad z \in U, \quad (2.36)$$

and this result is sharp.

Proof. We let

$$p(z) = \frac{D_{\lambda}^n g(z)}{z}, \quad z \in U. \quad (2.37)$$

By differentiating (2.37), we obtain

$$[D_{\lambda}^n g(z)]' = p(z) + zp'(z), \quad z \in U. \quad (2.38)$$

Using (2.38), Subordination (2.35) becomes

$$p(z) + zp'(z) < r(z) + zr'(z) = h(z), \quad z \in U. \quad (2.39)$$

By using Lemma 1.2, we have

$$p(z) < r(z), \quad (2.40)$$

that is,

$$\frac{D_{\lambda}^n g(z)}{z} < r(z), \quad z \in U, \quad (2.41)$$

and this result is sharp. □

Example 2.11. If we let $r(z) = 1/(1 - z)$, $n = 1$, $\lambda \geq 0$, then

$$h(z) = \frac{1}{(1 - z)^2}, \quad (2.42)$$

and from Theorem 2.10, we deduce that if $f \in \Sigma$, and

$$4zf(z) + 5z^2f'(z) + z^3f''(z) < \frac{1}{(1 - z)^2}, \quad z \in U, \quad (2.43)$$

then

$$2f(z) + zf'(z) < \frac{1}{1 - z}, \quad z \in U, \quad (2.44)$$

and this result is sharp.

Theorem 2.12. Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$ which verifies the inequality:

$$\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}, \quad z \in U. \quad (2.45)$$

If $f \in \Sigma$, g is given by (2.7) and the following differential subordination holds

$$[D_\lambda^n g(z)]' < h(z), \quad z \in U, \quad (2.46)$$

then

$$\frac{D_\lambda^n g(z)}{z} < q(z), \quad z \in U, \quad (2.47)$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t) dt, \quad z \in U. \quad (2.48)$$

Function q is convex and the best dominant.

Proof. In order to prove Theorem 2.12, we will use Lemmas 1.3 and 1.4. We deduce the value of w from Lemma 1.4 by using the conditions of Theorem 2.12. From (2.37), Definition 2.1 and Remark 2.2, we have

$$\begin{aligned} p(z) &= \frac{D_\lambda^n g(z)}{z} \\ &= \frac{z + \sum_{j=2}^{\infty} [(1 - \lambda)j^n + \lambda C_{n+j-1}^n] a_j z^j}{z} \\ &= 1 + \sum_{j=2}^{\infty} [(1 - \lambda)j^n + \lambda C_{n+j-1}^n] a_j z^{j-1} \\ &= 1 + b_1 z + b_2 z + \dots, \quad z \in U. \end{aligned} \quad (2.49)$$

Using Lemma 1.4, we deduce from (2.49) that $k = 1$. Using (2.38) in Subordination (2.46), we have

$$p(z) + zp'(z) < h(z), \quad z \in U. \quad (2.50)$$

From Subordination (2.50), by using Lemma 1.4, we deduce that $c = 1$. Then,

$$w = \frac{k^2 + c^2 - |k^2 - c^2|}{4k\operatorname{Re} c} = \frac{1 + 1 - |1 - 1|}{4} = \frac{1}{2}. \quad (2.51)$$

Applying Lemma 1.4, from Subordination (2.50), we obtain

$$p(z) < q(z) = \frac{1}{z} \int_0^z h(t) dt, \quad z \in U, \quad (2.52)$$

that is,

$$\frac{D_\lambda^n g(z)}{z} < q(z) = \frac{1}{z} \int_0^z h(t) dt, \quad z \in U, \quad (2.53)$$

where q is the best dominant.

Since the function h verifies the relation (2.45), from Lemma 1.3, we deduce that q is a convex function. \square

Example 2.13. If $n = 0$, $\lambda \geq 0$, $h(z) = e^{(3/2)z} - 1$, from Theorem 2.12, we deduce for $f \in \Sigma$ that if

$$4zf(z) + 5z^2f'(z) + z^3f''(z) < e^{(3/2)z} - 1, \quad z \in U, \quad (2.54)$$

then

$$2f(z) + zf'(z) < \frac{2}{3z}e^{(3/2)z} - \frac{2}{3z} - 1, \quad z \in U. \quad (2.55)$$

Theorem 2.14. Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$ which verifies the inequality:

$$\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}, \quad z \in U. \quad (2.56)$$

If $f \in \Sigma$, g is given by (2.7), and the following differential subordination holds

$$[D_\lambda^{n+1}g(z)]' + \frac{\lambda n z [R^n g(z)]''}{n+1} < h(z), \quad z \in U, \quad (2.57)$$

then

$$[D_\lambda^n g(z)]' < q(z), \quad z \in U, \quad (2.58)$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t) dt \quad (2.59)$$

is convex and the best dominant.

Proof. In order to prove Theorem 2.14, we will use Lemmas 1.3 and 1.4. The value of w is obtained using the conditions of Theorem 2.14.

Using (2.16) and (2.17), Subordination (2.49) becomes

$$p(z) + zp'(z) < h(z), \quad z \in U. \quad (2.60)$$

From Subordination (2.60), by using Lemma 1.4, we deduce that $c = 1$; and from the relation (2.16), Definition 2.1, and Remark 2.2, we obtain

$$\begin{aligned} p(z) &= [D_{\lambda}^n g(z)]' \\ &= \left[z + \sum_{j=2}^{\infty} [(1-\lambda)j^n + \lambda C_{n+j-1}^n] a_j z^j \right]' \\ &= 1 + \sum_{j=2}^{\infty} [(1-\lambda)j^n + \lambda C_{n+j-1}^n] \cdot j \cdot a_j \cdot z^{j-1} \\ &= 1 + c_1 z + c_2 z^2 + \dots, \quad z \in U. \end{aligned} \quad (2.61)$$

From (2.61), by using Lemma 1.4, we deduce that $k = 1$, then

$$w = \frac{k^2 + |c|^2 - |k - c|^2}{4k \operatorname{Re} c} = \frac{1 + 1 - |1 - 1|^2}{4} = \frac{1}{2}. \quad (2.62)$$

Applying Lemma 1.4, from Subordination (2.60), we obtain

$$p(z) < q(z) = \frac{1}{z} \int_0^z h(t) dt, \quad z \in U, \quad (2.63)$$

that is,

$$[D_{\lambda}^n g(z)]' < q(z) = \frac{1}{z} \int_0^z h(t) dt, \quad z \in U, \quad (2.64)$$

where q is the best dominant.

Since the function h verifies the inequality (2.45), from Lemma 1.3, we deduce that q is a convex function. \square

Example 2.15. If $n = 0$, $\lambda \geq 0$, $f \in \Sigma$, $h(z) = (2z + z^2)/(1 + z)^2$, from Theorem 2.14, we deduce that if

$$4zf(z) + 5z^2 f'(z) + z^3 f''(z) < \frac{2z + z^2}{2(1 + z)^2}, \quad z \in U, \quad (2.65)$$

then

$$2f(z) + zf'(z) < \frac{1}{2}z + \frac{1}{2} \frac{1}{1+z} + 1, \quad z \in U. \quad (2.66)$$

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