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Research Article

Strict Stability Criteria for Impulsive Functional Differential Systems

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By using Lyapunov functions and Razumikhin techniques, the strict stability of impulsive functional differential systems is investigated. Some comparison theorems are given by virtue of differential inequalities. The corresponding theorems in the literature can be deduced from our results.

1. Introduction

Since time-delay systems are frequently encountered in engineering, biology, economy, and other disciplines, it is significant to study these systems [1]. On the other hand, because many evolution processes in nature are characterized by the fact that at certain moments of time they experience an abrupt change of state, the study of dynamic systems with impulse effects has been assuming greater importance [2–4]. It is natural to expect that the hybrid systems which are called impulsive functional differential systems can represent a truer framework for mathematical modeling of many real world phenomena. Recently, several papers dealing with stability problem for impulsive functional differential systems have been published [5–10].

The usual stability concepts do not give any information about the rate of decay of the solutions, and hence are not strict concepts. Consequently, strict-stability concepts have been defined and criteria for such notions to hold are discussed in [11]. Till now, to the best of our knowledge, only the following very little work has been done in this direction [12–15].

In this paper, we investigate strict stability for impulsive functional differential systems. The paper is organized as follows. In Section 2, we introduce some basic definitions and notations. In Section 3, we first give two comparison lemmas on differential inequalities. Then, by these lemmas, a comparison theorem is obtained and several direct results are deduced from it. An example is also given to illustrate the advantages of our results.

2. Preliminaries

We consider the following impulsive functional differential system:

$$\begin{aligned} x'(t) &= f(t, x_t), \quad t \neq \tau_k, \\ \triangle x(\tau_k) &\triangleq x(\tau_k) - x(\tau_k^-) = I_k(x(\tau_k^-)), \quad k \in \mathbb{Z}^+, \end{aligned} \tag{2.1}$$

where $\mathbb{Z}^+$ is the set of all positive integers, $f : \mathbb{R}^+ \times D \to \mathbb{R}^n$, D is an open set in $PC([-\tau, 0], \mathbb{R}^n)$, here $\mathbb{R}^+ = [0, \infty)$, $\tau > 0$, and $PC([-\tau, 0], \mathbb{R}^n) = \{\phi : [-\tau, 0] \to \mathbb{R}^n, \phi(t)$ is continuous everywhere except for a finite number of points $\hat{t}$ at which $\phi(\hat{t}^+)$ and $\phi(\hat{t}^-)$ exist and $\phi(\hat{t}^+) = \phi(\hat{t})\}$. $I_k : S(\rho_0) \to \mathbb{R}^n$ for each $k \in \mathbb{Z}^+$, where $S(\rho_0) = \{x \in \mathbb{R}^n : \|x\| < \rho_0, \|\cdot\|$ denotes the norm of vector in $\mathbb{R}^n\}$, $0 = \tau_0 \leq \tau_1 < \tau_2 < \cdots < \tau_k < \cdots$ with $\tau_k \to \infty$ as $k \to \infty$ and $x'(t)$ denotes the right-hand derivative of $x(t)$. For each $t \in \mathbb{R}^+$, $x_t \in PC$ is defined by $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$. For $\phi \in PC$, $|\phi|_1 = \sup_{-\tau \leq s \leq 0} \|\phi(s)\|$, $|\phi|_2 = \inf_{-\tau \leq s \leq 0} \|\phi(s)\|$. We assume that $f(t, 0) \equiv 0$ and $I_k(0) \equiv 0$, so that $x(t) \equiv 0$ is a solution of (2.1), which we call the zero solution.

Let $t_0 \in [\tau_{m-1}, \tau_m)$ for some $m \in \mathbb{Z}^+$ and $\varphi \in D$, a function $x(t) : [t_0 - \tau, \beta) \to \mathbb{R}^n$ ($\beta \leq \infty$) is said to be a solution of (2.1) with the initial condition

$$x_{t_0} = \varphi, \tag{2.2}$$

if it is continuous and satisfies the differential equation $x'(t) = f(t, x_t)$ in each $[t_0, \tau_m)$, $[\tau_i, \tau_{i+1})$, $i = m, m+1, \ldots$, and at $t = \tau_i$ it satisfies $\Delta x(\tau_i) = I_i(x(\tau_i^-))$.

Throughout this paper, we always assume the following conditions hold to ensure the global existence and uniqueness of solution of (2.1) through (t_0, φ).

- (H_1) f is continuous on $[\tau_{k-1}, \tau_k) \times D$ for each $k \in \mathbb{Z}^+$ and for all $k \in \mathbb{Z}^+$ and $\varphi \in D$, the limits $\lim_{(t,\phi) \to (\tau_k^-, \varphi)} f(t, \phi) = f(\tau_k^-, \varphi)$ exist.
- (H_2) $f(t, \phi)$ is Lipschitzian in ϕ in each compact set in D.
- (H_3) $I_k(x) \in C[S(\rho_0), \mathbb{R}^n]$ for all $k \in \mathbb{Z}^+$ and there exists $\rho_0 \leq \rho$ such that $x \in S(\rho_0)$ implies that $x + I_k(x) \in S(\rho)$ for all $k \in \mathbb{Z}^+$.

The function $V(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ belongs to class V_0 if the following hold.

- (A_1) V is continuous on each of the sets $[\tau_{k-1}, \tau_k) \times \mathbb{R}^n$ and for each $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}^+$, $\lim_{(t,y) \to (\tau_k^-, x)} V(t, y) = V(\tau_k^-, x)$ exists.
- (A_2) $V(t, x)$ is locally Lipschitzian in $x \in \mathbb{R}^n$ and for $t \in \mathbb{R}^+$, $V(t, 0) \equiv 0$. Let $V \in V_0$, D^+V along the solution $x(t)$ of (2.1) is defined as

$$D^+V(t, x(t)) = \lim_{\delta \to 0^+} \sup \frac{1}{\delta} \left[V(t + \delta,\, x(t + \delta)) - V(t, x(t)) \right]. \tag{2.3}$$

Let us introduce the following notations for further use:

- (i) $K_0 = \{a(u) \in C[\mathbb{R}^+, \mathbb{R}^+] :$ increasing and $a(0) = 0\}$;
- (ii) $K = \{a(u) \in K_0 :$ strictly increasing$\}$;
- (iii) $K_1 = \{a(u) \in K_0 : a(u) \leq u$ and $a(u) > 0$ for $u > 0\}$;

(iv) $K_2 = \{a(u) \in K : a(u) \geq u\}$;

(v) $PC_1(\rho) = \{\phi \in PC([-\tau, 0], \mathbb{R}^n) : |\phi|_1 < \rho\}$;

(vi) $PC_2(\theta) = \{\phi \in PC([-\tau, 0], \mathbb{R}^n) : |\phi|_2 > \theta > 0\}$.

Definition 2.1. The zero solution of (2.1) is said to be strictly stable (SS), if for any $t_0 \in \mathbb{R}^+$ and $\varepsilon_1 > 0$, there exists a $\delta_1 = \delta_1(t_0, \varepsilon_1) > 0$ such that $\varphi \in PC_1(\delta_1)$ implies $\|x(t; t_0, \varphi)\| < \varepsilon_1$ for $t \geq t_0$, and for every $0 < \delta_2 \leq \delta_1$, there exists an $0 < \varepsilon_2 < \delta_2$ such that

$$\varphi \in PC_2(\delta_2) \text{ implies } \varepsilon_2 < \|x(t; t_0, \varphi)\|, \quad t \geq t_0. \tag{2.4}$$

Definition 2.2. The zero solution of (2.1) is said to be strictly uniformly stable (SUS), if δ_1, δ_2, and ε_2 in (SS) are independent of t_0.

Remark 2.3. If in (SS) or (SUS), $\varepsilon_2 = 0$, we obtain nonstrict stabilities, that is, the usual stability or uniform stability, respectively. Moreover, strict stability immediately implies that the zero solution is not asymptotically stable.

The preceding notions imply that the motion remains in the tube like domains. To obtain sufficient conditions for such stability concepts to hold, it is necessary to simultaneously obtain both lower and upper bounds of the derivative of Lyapunov function. Thus, we need to consider the following two auxiliary systems:

$$\begin{aligned} v' &= g_1(t, v), \quad t \neq \tau_k, \\ v(\tau_k) &= \phi_k(v(\tau_k^-)), \\ v(t_0) &= v_0 \geq 0, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} u' &= g_2(t, u), \quad t \neq \tau_k, \\ u(\tau_k) &= \psi_k(u(\tau_k^-)), \\ u(t_0) &= u_0 \geq 0, \end{aligned} \tag{2.6}$$

where $g_1, g_2 \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}]$, $g_1(t, u) \leq g_2(t, u)$, $g_1(t, 0) \equiv g_2(t, 0) \equiv 0$, $\phi_k, \psi_k : \mathbb{R}^+ \to \mathbb{R}^+$, $\phi_k(u) \leq \psi_k(u)$ for each $k \in \mathbb{Z}^+$.

From the theory of impulsive differential systems [2], we obtain that

$$\rho(t; t_0, v_0) \leq \gamma(t; t_0, u_0), \quad t \geq t_0 \text{ whenever } v_0 \leq u_0, \tag{2.7}$$

where $\rho(t; t_0, v_0)$ and $\gamma(t; t_0, u_0)$ are the minimal and maximal solutions of (2.5), (2.6), respectively.

The corresponding definitions of strict stability of the auxiliary systems (2.5), (2.6) are as follows.

Definition 2.4. The zero solutions of comparison systems (2.5), (2.6), as a system, are said to be strictly stable (SS*), if for any $t_0 \in \mathbb{R}^+$ and $\varepsilon_1 > 0$, there exist a $\delta_1 = \delta_1(t_0, \varepsilon_1), \delta_2 = \delta_2(t_0, \varepsilon_1)$, and $\varepsilon_2 = \varepsilon_2(t_0, \varepsilon_1)$ satisfying $0 < \varepsilon_2 < \delta_2 < \delta_1 < \varepsilon_1$ such that

$$\varepsilon_2 < \rho(t; t_0, v_0) \leq \gamma(t; t_0, u_0) < \varepsilon_1, \quad t \geq t_0, \text{ provided } \delta_2 < v_0 \leq u_0 < \delta_1. \tag{2.8}$$

Definition 2.5. The zero solutions of comparison systems (2.5),(2.6), as a system , are said to be strictly uniformly stable (SUS*), if δ_1, δ_2, and ε_2 in (SS*) are independent of t_0.

3. Main results

We first give two Razumikhin-type comparison lemmas on differential inequalities.

Lemma 3.1. *Assume that*

(i) $g_1, g_2 \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}]$, $-g_1(t,\cdot)$, $g_2(t,\cdot) \in K_0$ *for each* t;

(ii) *there exists* $m_i : \mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1,2)$, *where* $m_i(t)$ $(i = 1,2)$ *are continuous on* $[\tau_{k-1}, \tau_k)$ *and* $\lim_{t\to\tau_k^-} m_i(t) = m_i(\tau_k^-)$ $(i = 1,2)$ *exist*, $k \in \mathbb{Z}^+$, *satisfying*

$$\begin{aligned} g_1(t, m_1(t)) &\le D^+ m_1(t), \\ D^+ m_2(t) &\le g_2(t, m_2(t)). \end{aligned} \tag{3.1}$$

Then

$$\rho(t) \le m_1(t) \quad \text{if } \inf_{-\tau\le s\le 0} m_1(t_0 + s) \ge v_0, \tag{3.2}$$

$$m_2(t) \le \gamma(t) \quad \text{if } \sup_{-\tau\le s\le 0} m_2(t_0 + s) \le u_0, \tag{3.3}$$

where $\rho(t) = \rho(t; t_0, v_0)$ *and* $\gamma(t) = \gamma(t; t_0, u_0)$ *are the minimal and maximal solutions of systems* (3.4) *and* (3.5), *respectively*,

$$\begin{aligned} v' &= g_1(t, v), \\ v(t_0) &= v_0 \ge 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} u' &= g_2(t, u), \\ u(t_0) &= u_0 \ge 0. \end{aligned} \tag{3.5}$$

Proof. First, we prove that (3.2) holds. Otherwise, there exist $t_0 \le t_1 < t_2$ such that

(a) $\rho(t_1) = m_1(t_1)$,

(b) $m_1(t + s) \ge m_1(t)$, $s \in [-\tau, 0]$, $t \in [t_1, t_2]$, and

(c) $\rho(t_2) < m_1(t_2)$.

By (a), (b), and (ii), applying the classical comparison theorem, we have

$$\rho(t) \le m_1(t), \quad t \in [t_1, t_2], \tag{3.6}$$

which contradicts (c). So (3.2) is correct. Equation (3.3) can be proved in the same way as above. Then Lemma 3.1 holds. □

Lemma 3.2. *Assume that (i) in Lemma 3.1 holds. Suppose further that*

(ii) *there exists* $V_1 \in V_0$ *satisfying*

$$\phi_k(V_1(\tau_k^-, x)) \le V_1(\tau_k, x + I_k(x)), \quad k \in \mathbb{Z}^+, \tag{3.7}$$

where $\phi_k \in K_1$, *and for any solution* $x(t)$ *of* (2.1), $V_1(t+s, x(t+s)) \ge V_1(t, x(t))$, $s \in [-\tau, 0]$, *implies that*

$$g_1(t, V_1(t, x(t))) \le D^+ V_1(t, x(t)); \tag{3.8}$$

(iii) *there exists* $V_2 \in V_0$ *satisfying*

$$V_2\big(\tau_k, x + I_k(x)\big) \le \psi_k\big(V_2(\tau_k^-, x)\big), \quad k \in \mathbb{Z}^+, \tag{3.9}$$

where $\psi_k \in K_2$, *and for any solution* $x(t)$ *of* (2.1), $V_2(t+s, x(t+s)) \le V_2(t, x(t))$, $s \in [-\tau, 0]$, *implies that*

$$D^+V_2(t, x(t)) \le g_2(t, V_2(t, x(t))). \tag{3.10}$$

Then

$$\rho(t) \le V_1\big(t, x(t)\big) \quad \textit{if} \inf_{-\tau \le s \le 0} V_1\big(t_0 + s,\, x(t_0 + s)\big) \ge v_0, \tag{3.11}$$

$$V_2\big(t, x(t)\big) \le \gamma(t) \quad \textit{if} \sup_{-\tau \le s \le 0} V_2\big(t_0 + s, x(t_0 + s)\big) \le u_0, \tag{3.12}$$

where $\rho(t) = \rho(t; t_0, v_0)$ *and* $\gamma(t) = \gamma(t; t_0, u_0)$ *are the minimal and maximal solutions of* (2.5), (2.6), *respectively.*

Proof. Assume $t_0 \in [\tau_{m-1}, \tau_m)$, $m \in \mathbb{Z}^+$. First, we prove that (3.11) holds for $t \in [t_0, \tau_m)$, that is

$$\rho(t) \le V_1\big(t, x(t)\big), \quad t \in \big[t_0, \tau_m\big). \tag{3.13}$$

Let $m_1(t) = V_1(t, x(t))$, $t \ge t_0$. Equation (3.13) holds obviously by Lemma 3.1 for $t \in [t_0, \tau_m)$. By (ii), $V_1(\tau_m, x(\tau_m)) \ge \phi_m(V_1(\tau_m^-, x(\tau_m^-))) \ge \phi_m(\rho(\tau_m^-)) = \rho(\tau_m)$. The same proof as for $t \in [t_0, \tau_m)$ leads to

$$\rho(t) \le V_1\big(t, x(t)\big), \quad t \in \big[\tau_m, \tau_{m+1}\big). \tag{3.14}$$

By induction, (3.11) is correct. Similarly, (3.12) can be proved by using Lemma 3.1 and assumption (iii). □

Using Lemma 3.2, we can easily get the following theorem about strict stability properties of (2.1).

Theorem 3.3. *Assume that all the conditions of Lemma 3.2 hold. Suppose further that there exist functions* $a_i, b_i \in K, i = 1, 2$, *such that*

(iv) $b_i(\|x\|) \le V_i(t, x) \le a_i(\|x\|)$ *for* $x \in S(\rho)$.

Then the strict stability properties of comparison systems (2.5), (2.6) *imply the corresponding strict stability properties of zero solution of* (2.1).

Proof. First, let us prove strict stability of the zero solution of (2.1). Suppose that $0 < \varepsilon_1 < \rho_0$ and $t_0 \in \mathbb{R}^+$ are given. Assume that (SS*) holds. Then, given $b_2(\varepsilon_1) > 0$, there exists $\widehat{\delta}_1 = \widehat{\delta}_1(t_0, \varepsilon_1), \widehat{\delta}_2 = \widehat{\delta}_2(t_0, \varepsilon_1)$, and $\widehat{\varepsilon}_2 = \widehat{\varepsilon}_2(t_0, \varepsilon_1)$ satisfying $0 < \widehat{\varepsilon}_2 < \widehat{\delta}_2 < \widehat{\delta}_1 < b_2(\varepsilon_1)$ such that

$$\widehat{\varepsilon}_2 < \rho(t) \le \gamma(t) < b_2(\varepsilon_1) \text{ provided } \widehat{\delta}_2 < v_0 \le u_0 < \widehat{\delta}_1, \quad t \ge t_0. \tag{3.15}$$

By (iv), there exist $0 < \delta_2 < \delta_1 < \varepsilon_1$ such that for $s \in [-\tau, 0]$,

$$V_i\big(t_0 + s, x\big) \in PC_2\big(\widehat{\delta}_2\big) \cap PC_1\big(\widehat{\delta}_1\big) \text{ provided } \delta_2 < \|x\| < \delta_1, \quad i = 1, 2. \tag{3.16}$$

Next, choose $\varepsilon_2 = \varepsilon_2(t_0, \varepsilon_1) > 0$ such that $a_1(\varepsilon_2) \le \widehat{\varepsilon}_2$ and $\varepsilon_2 < \delta_2$. We claim that with the choices of ε_2, δ_2, and δ_1, the zero solution of (2.1) is strictly stable. That means that if $x(t) = x(t; t_0, \varphi)$ is any solution of (2.1), $\varphi \in PC_2(\delta_2) \cap PC_1(\delta_1)$ implies that $\varepsilon_2 < \|x(t)\| < \varepsilon_1$, $t \ge t_0$. If not, we have either of the following alternatives.

Case 1. There exists a $t_1 \in [\tau_r, \tau_{r+1})$ such that

$$\varepsilon_2 \geq \|x(t_1)\|. \tag{3.17}$$

Then clearly $\|x(t)\| < \rho_0$, $t_0 \leq t \leq t_1$. Thus, by Lemma 3.2, (i) and (ii) imply that

$$\rho(t) \leq V_1(t, x(t)) \text{ provided } v_0 \leq \inf_{s\in[-\tau,0]} V_1(t_0 + s,\, x(t_0 + s)), \quad t \in [t_0, t_1]. \tag{3.18}$$

Using (3.15)–(3.18) and (iv), we get

$$a_1(\varepsilon_2) \geq a_1(\|x(t_1)\|) \geq V_1(t_1, x(t_1)) \geq \rho(t_1) > \widehat{\varepsilon}_2 \geq a_1(\varepsilon_2), \tag{3.19}$$

which is a contradiction.

Case 2. There exists a $\widehat{t}_2 \in [\tau_s, \tau_{s+1})$ such that

$$\varepsilon_1 \leq \|x(\widehat{t}_2)\|, \tag{3.20}$$

$$\|x(t)\| < \varepsilon_1, \quad t_0 \leq t < \tau_s. \tag{3.21}$$

By (H_3), (3.21) yields

$$\|x(\tau_s)\| = \|x(\tau_s^-) + I_s(x(\tau_s^-))\| < \rho. \tag{3.22}$$

Because of (3.20) and (3.22), there exists a $t_2 \in [\tau_s, \widehat{t}_2]$ such that

$$\varepsilon_1 \leq \|x(t_2)\| < \rho. \tag{3.23}$$

By Lemma 3.2, (i) and (iii) imply that

$$V_2(t, x(t)) \leq \gamma(t) \text{ provided } \sup_{s\in[-\tau,0]} V_2(t_0 + s,\, x(t_0 + s))) \leq u_0, \quad t \in [t_0, t_2]. \tag{3.24}$$

From (3.15), (3.23), (3.24), and (iv), we have the following contradiction:

$$b_2(\varepsilon_1) \leq b_2(\|x(t_2)\|) \leq V_2(t_2, x(t_2)) \leq \gamma(t_2) < b_2(\varepsilon_1). \tag{3.25}$$

We, therefore, obtain the strict stability of the zero solution of (2.1). If we assume that the zero solutions of comparison systems (2.5), (2.6) are (SUS*), since $\widehat{\delta}_1, \widehat{\delta}_2$ are independent of t_0, we obtain, because of (iv), δ_1 and δ_2 in (3.16) are independent of t_0, and hence, (SUS) of (2.1) holds.

□

Using Theorem 3.3, we can get two direct results on strictly uniform stability of zero solution of (2.1) and the first one is Theorem 3.3 in [15].

Corollary 3.4. *In Theorem 3.3, suppose that* $g_1 \equiv g_2 \equiv 0$, $\phi_k(u) = (1 - c_k)u$, $\psi_k(u) = (1 + d_k)u$, $k \in \mathbb{Z}^+$, *where* $0 \leq c_k < 1$, $\sum_{k=1}^{\infty} c_k < \infty$, *and* $d_k \geq 0$, $\sum_{k=1}^{\infty} d_k < \infty$.

Then the zero solution of (2.1) *is strictly uniformly stable.*

Corollary 3.5. *In Theorem 3.3, suppose that* $g_1(t,u) = -M_1'(t)u$, $g_2(t,u) = M_2'(t)u$, *where* $M_i'(t) \in C[\mathbb{R}^+, \mathbb{R}^+]$, $i = 1,2$, *and* $M_i(t)$, $i = 1,2$ *are bounded,* $\phi_k(u)$ *and* $\psi_k(u)$, $k \in \mathbb{Z}^+$ *are just the same as in Corollary 3.4.*

Then the zero solution of (2.1) *is strictly uniformly stable.*

Proof. Under the given hypotheses, it is easy to obtain the solutions of (2.5) and (2.6):

$$\begin{aligned} v(t) &= v_0 \prod_{t_0 \le \tau_k \le t} (1 - c_k) \exp\left[- \left(M_1(t) - M_1(t_0) \right) \right], \\ u(t) &= u_0 \prod_{t_0 \le \tau_k \le t} (1 + d_k) \exp\left[M_2(t) - M_2(t_0) \right]. \end{aligned} \tag{3.26}$$

Since $M_i(t)$, $i = 1,2$, are bounded, there exist two positive constants B_1, B_2 such that $|M_1(t)| \le B_1$, $|M_2(t)| \le B_2$. Also, since $\sum_{k=1}^{\infty} c_k < \infty$, $\sum_{k=1}^{\infty} d_k < \infty$, it follows that $\prod_{k=1}^{\infty}(1 - c_k) = N$ and $\prod_{k=1}^{\infty}(1+d_k) = M$, obviously $0 < N \le 1$, $1 \le M < \infty$. Given $\varepsilon_1 > 0$, choose $\delta_1 = M^{-1}\exp(-2B_2)\varepsilon_1$ and for $0 < \delta_2 < \delta_1$, choose $\varepsilon_2 = \delta_2 N \exp(-2B_1)$. Then, if $\delta_2 < v_0 \le u_0 < \delta_1$, we have

$$\varepsilon_2 < v(t) \le u(t) < \varepsilon_1. \tag{3.27}$$

That is, the zero solutions of (2.5), (2.6) are strictly uniformly stable. Hence, by Theorem 3.3, the zero solution of (2.1) is strictly uniformly stable. □

Example 3.6. Consider the system

$$\begin{aligned} x'(t) &= -a(t)x(t) + b(t)x(t - \tau), \quad t \ne \tau_k,\ t \ge 0, \\ \Delta x(\tau_k) &= I_k\left(x\left(\tau_k^-\right)\right), \quad k \in \mathbb{Z}^+, \end{aligned} \tag{3.28}$$

where $a(t)$, $b(t)$ are continuous on $\mathbb{R}^+$, $b(t) \ge 0$, $I_k(x) \in C[\mathbb{R}, \mathbb{R}]$. Assume that $-1/(1 + t^2) \le -a(t) + b(t) \le 1/(1 + t^2)$, $(1 - c_k)x^2 \le (x + I_k(x))^2 \le (1 + d_k)x^2$ with $0 \le c_k < 1$, $\sum_{k=1}^{\infty} c_k < \infty$, and $d_k \ge 0$, $\sum_{k=1}^{\infty} d_k < \infty$.

Let $V_1(t,x) = V_2(t,x) = V(x) = (1/2)x^2$, then

$$\begin{aligned} (1 - c_k)V(x) &= \frac{1}{2}(1 - c_k)x^2 \le V\left(x + I_k(x)\right) \\ &= \frac{1}{2}\left(x + I_k(x)\right)^2 \le \frac{1}{2}(1 + d_k)x^2 = (1 + d_k)V(x). \end{aligned} \tag{3.29}$$

For any solution $x(t)$ of (3.28) such that $V(x(t + s)) \ge V(x(t))$, $s \in [-\tau, 0]$, we have

$$D^+V\left(x(t)\right) = -a(t)x^2(t) + b(t)x(t)x(t - \tau) \ge \left[- a(t) + b(t) \right] x^2(t) \ge -\frac{2}{1 + t^2} V\left(x(t)\right), \tag{3.30}$$

and if $V(x(t + s)) \le V(x(t))$, $s \in [-\tau, 0]$, we have

$$D^+V\left(x(t)\right) = -a(t)x^2(t) + b(t)x(t)x(t - \tau) \le \left[- a(t) + b(t) \right] x^2(t) \le \frac{2}{1 + t^2} V\left(x(t)\right). \tag{3.31}$$

By Corollary 3.5, the zero solution of (2.1) is strictly uniformly stable.

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References

[1] J. Hale, *Theory of Functional Differential Equations*, vol. 3 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2nd edition, 1977.
[2] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific, Teaneck, NJ, USA, 1989.
[3] X. L. Fu, B. Q. Yan, and Y. S. Liu, *Introduction to Impulsive Differential Systems*, Science Press, Beijing, China, 2005.
[4] X. L. Fu, K. N. Wang, and H. X. Lao, "Boundedness of perturbed systems with impulsive effects," *Acta Mathematica Scientia. Series A*, vol. 24, no. 2, pp. 135–143, 2004.
[5] J. Shen and J. Yan, "Razumikhin type stability theorems for impulsive functional-differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 33, no. 5, pp. 519–531, 1998.
[6] J. H. Shen, "Razumikhin techniques in impulsive functional-differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 36, no. 1, pp. 119–130, 1999.
[7] J. Yan and J. Shen, "Impulsive stabilization of functional-differential equations by Lyapunov-Razumikhin functions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 37, no. 2, pp. 245–255, 1999.
[8] Z. Luo and J. Shen, "New Razumikhin type theorems for impulsive functional differential equations," *Applied Mathematics and Computation*, vol. 125, no. 2–3, pp. 375–386, 2002.
[9] Y. Xing and M. Han, "A new approach to stability of impulsive functional differential equations," *Applied Mathematics and Computation*, vol. 151, no. 3, pp. 835–847, 2004.
[10] K. Liu and X. Fu, "Stability of functional differential equations with impulses," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 2, pp. 830–841, 2007.
[11] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities: Theory and Applications. Vol. I: Ordinary Differential Equations*, vol. 55 of *Mathematics in Science and Engineering*, Academic Press, New York, 1969.
[12] V. Lakshmikantham and J. Vasundhara Devi, "Strict stability criteria for impulsive differential systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 21, no. 10, pp. 785–794, 1993.
[13] V. Lakshmikantham and Y. Zhang, "Strict practical stability of delay differential equation," *Applied Mathematics and Computation*, vol. 122, no. 3, pp. 341–351, 2001.
[14] V. Lakshmikantham and R. N. Mohapatra, "Strict stability of differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 46, no. 7, pp. 915–921, 2001.
[15] Y. Zhang and J. Sun, "Strict stability of impulsive functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 301, no. 1, pp. 237–248, 2005.