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Research Article

Bounds for Trivariate Copulas with Given Bivariate Marginals

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We determine two constructions that, starting with two bivariate copulas, give rise to new bivariate and trivariate copulas, respectively. These constructions are used to determine pointwise upper and lower bounds for the class of all trivariate copulas with given bivariate marginals.

1. Introduction

In recent literature, several researchers have focused the attention on constructions and stochastic orders among probability distribution functions with given marginals. These problems are interesting especially for their relevance in finance and quantitative risk management, like models of multivariate portfolios and bounding functions of dependent risks (see, e.g., [1]).

If a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is characterized by a distribution function (= d.f.) F with known univariate marginals, then upper and lower bounds for F were given in early works by Fréchet. When, instead, we have some information about the multivariate marginals of F, then the problem has not been considered extensively in the literature, although it seems natural that for some applications one needs to estimate the joint distribution F of $\mathbf{X}$, when the dependence among some components of F is known. For this discussion, we refer to Rüschendorf [2, 3] and Joe [4, 5].

In this paper, we aim at contributing to this problem by providing lower and upper bounds in the class of continuous trivariate d.f.'s whose bivariate marginals are given, that is, when we have full information about the pairwise dependence among the components of the corresponding random vector. These new bounds improve some estimations given by Joe [5].

We will formulate our results in the class of *copulas*, which are multivariate d.f.'s whose one-dimensional marginals are uniformly distributed on $[0,1]$: see Joe [5]; Nelsen [6]. It is well known that this restriction does not cause any loss of generality in the problem because, thanks to *Sklar's Theorem* [7], any continuous multivariate d.f. can be represented by means of a copula and its one-dimensional marginals. Moreover, in order to obtain our results, we use two constructions that, starting with two bivariate copulas, give rise to new bivariate and trivariate copulas, respectively. These constructions can be seen as generalizations of the product-like operations on copulas considered by Darsow et al. [8] and Kolesárová et al. [9].

2. Preliminaries

Let n be in $\mathbb{N}$, $n \geq 2$, and denote by $\mathbf{x} = (x_1, \ldots, x_n)$ any point in $\mathbb{R}^n$. An *n-dimensional copula* (shortly, *n-copula*) is a mapping $C_n : [0,1]^n \to [0,1]$ satisfying the following conditions:

(C1) $C_n(\mathbf{u}) = 0$ whenever $\mathbf{u} \in [0,1]^n$ has at least one component equal to 0;

(C2) $C_n(\mathbf{u}) = u_i$ whenever all components of $\mathbf{u} \in [0,1]^n$ are equal to 1 except for the ith one, which is equal to u_i;

(C3) C_n is *n-increasing*, viz., for each n-box $B = \times_{i=1}^{n}[u_i, v_i]$ in $[0,1]^n$ with $u_i \leq v_i$ for each $i \in \{1, \ldots, n\}$,

$$V_{C_n}(B) := \sum_{\mathbf{z} \in \times_{i=1}^{n}\{u_i, v_i\}} (-1)^{N(\mathbf{z})} C_n(\mathbf{z}) \geq 0, \tag{2.1}$$

where $N(\mathbf{z}) = \operatorname{card}\{k \mid z_k = u_k\}$.

We denote by $\mathcal{C}_n$ the set of all n-dimensional copulas ($n \geq 2$). For every $C_n \in \mathcal{C}_n$ and for every $\mathbf{u} \in [0,1]^n$, we have that

$$W_n(\mathbf{u}) \leq C_n(\mathbf{u}) \leq M_n(\mathbf{u}), \tag{2.2}$$

where

$$W_n(\mathbf{u}) := \max\left\{\sum_{i=1}^{n} u_i - n + 1, 0\right\}, \qquad M_n(\mathbf{u}) := \min\{u_1, u_2, \ldots, u_n\}. \tag{2.3}$$

Notice that M_n is in $\mathcal{C}_n$, but W_n is in $\mathcal{C}_n$ only for $n = 2$. Another important n-copula is the product $\Pi_n(\mathbf{u}) := \prod_{i=1}^{n} u_i$.

We recall that, for C and C' in $\mathcal{C}_2$, C' is said to be greater than C in the *concordance order*, and we write $C \preceq C'$, if $C(u_1, u_2) \leq C'(u_1, u_2)$ for all $(u_1, u_2) \in [0,1]^2$. Moreover, for D and D' in $\mathcal{C}_3$, D' is said to be greater than D in the *concordance order*, and we write $D \preceq D'$, if $D(\mathbf{u}) \leq D'(\mathbf{u})$ and $\overline{D}(\mathbf{u}) \leq \overline{D'}(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^3$, where $\overline{D}$ is the survival copula of D defined on $[0,1]^3$ by

$$\overline{D}(u_1, u_2, u_3) = 1 - u_1 - u_2 - u_3 + D(u_1, u_2, 1) + D(u_1, 1, u_3) + D(1, u_2, u_3) - D(u_1, u_2, u_3). \tag{2.4}$$

For more details about copulas, see [5, 6].

For each $C_n \in \mathcal{C}_n$ and for each permutation $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ of $(1, 2, \ldots, n)$, the mapping $C_n^{\sigma} : [0,1]^n \to [0,1]$ given by

$$C_n^{\sigma}(u_1, \ldots, u_n) = C_n(u_{\sigma_1}, \ldots, u_{\sigma_n}) \tag{2.5}$$

is also in $\mathcal{C}_n$. For example, if $C_3 \in \mathcal{C}_3$, then we denote by $C_3^{(1,3,2)}$ the 3-copula given by $C_3^{(1,3,2)}(u_1, u_2, u_3) = C_3(u_1, u_3, u_2)$.

For the sequel, we need the following definition.

Definition 2.1. Three 2-copulas C_{12}, C_{13} and C_{23} are *compatible* if, and only if, there exists $\widetilde{C} \in \mathcal{C}_3$ such that, for all u_1, u_2, u_3 in $[0,1]$,

$$\begin{aligned} C_{12}(u_1, u_2) &= \widetilde{C}(u_1, u_2, 1), \\ C_{13}(u_1, u_3) &= \widetilde{C}(u_1, 1, u_3), \\ C_{23}(u_2, u_3) &= \widetilde{C}(1, u_2, u_3). \end{aligned} \tag{2.6}$$

In such a case, C_{12}, C_{13} and C_{23} are called the *bivariate marginals* (briefly, 2-marginals) of $\widetilde{C}$.

In general, it is a difficult problem to determine whether three bivariate copulas are compatible (for some preliminary studies, see [5] and the references therein). Notice that Π_2, Π_2, Π_2 are compatible, because they are the 2-marginals of Π_3. Analogously, M_2, M_2, M_2 are compatible, because they are the 2-marginals of M_3. The copulas W_2, W_2, W_2, however, are not compatible.

If C_{12}, C_{13} and C_{23} in $\mathcal{C}_2$ are compatible, the *Fréchet class* of (C_{12}, C_{13}, C_{23}), denoted by $\mathcal{F}(C_{12}, C_{13}, C_{23})$, is the class of all $\widetilde{C} \in \mathcal{C}_3$ such that (2.6) hold.

In the following result, we present a way for obtaining a 3-copula starting with some suitable 2-copulas. This method can be considered as a direct extension of some results by Darsow et al. [8] and Kolesárová et al. [9].

Proposition 2.2. *Let A and B be in $\mathcal{C}_2$ and let $\mathbf{C} = (C_t)_{t \in [0,1]}$ be a family in $\mathcal{C}_2$. Then the mapping $A \star_{\mathbf{C}} B : [0,1]^3 \to [0,1]$ defined by*

$$(A \star_{\mathbf{C}} B)(u_1, u_2, u_3) = \int_0^{u_2} C_t\left(\frac{\partial}{\partial t} A(u_1, t), \frac{\partial}{\partial t} B(t, u_3)\right) \mathrm{d}t \tag{2.7}$$

is in $\mathcal{C}_3$, provided that the above integral exists and is finite.

Proof. It is immediate that $A \star_{\mathbf{C}} B$ satisfies (C1) and (C2). In order to prove (C3) for $n = 3$, let u_i, v_i be in $[0,1]$ such that $u_i \leq v_i$ for every $i \in \{1, 2, 3\}$. Since A is 2-increasing, we have that

$A(v_1,t) - A(u_1,t)$ is increasing in $t \in [0,1]$, and, therefore, $(\partial/\partial t)A(v_1,t) \geq (\partial/\partial t)A(u_1,t)$ for all $t \in [0,1]$. Analogously, $(\partial/\partial t)B(t,v_3) \geq (\partial/\partial t)B(t,u_3)$ for all $t \in [0,1]$. Then, we have that

$$\begin{aligned} &V_{A\star_{\mathbf{C}}B}([u_1,v_1]\times[u_2,v_2]\times[u_3,v_3]) \\ &\quad = \int_{u_2}^{v_2} V_{C_t}\left(\left[\frac{\partial}{\partial t}A(u_1,t),\frac{\partial}{\partial t}A(v_1,t)\right]\times\left[\frac{\partial}{\partial t}B(t,u_3),\frac{\partial}{\partial t}B(t,v_3)\right]\right)\mathrm{d}t \geq 0, \end{aligned} \tag{2.8}$$

which concludes the proof. □

The copula $A\star_{\mathbf{C}}B$ is called the **C**-*lifting* of the copulas A and B with respect to the family $\mathbf{C} = (C_t)_{t\in[0,1]}$ in $\mathcal{C}_2$. Given $C \in \mathcal{C}_2$, if $C_t = C$ for every t in $[0,1]$, we will write $A\star_{\mathbf{C}}B = A\star_C B$. Notice that, if $C_t = \Pi_2$ for every $t \in [0,1]$, then the operation $\star_{\Pi_2}$ was considered by Darsow et al. [8] and Kolesárová et al. [9]. We easily derive that the 2-marginals of $A\star_{\mathbf{C}}B$ are A, $A*_{\mathbf{C}}B$ and B, where

$$(A*_{\mathbf{C}}B)(u_1,u_2) = \int_0^1 C_t\left(\frac{\partial}{\partial t}A(u_1,t),\frac{\partial}{\partial t}B(t,u_2)\right)\mathrm{d}t \tag{2.9}$$

is called the **C**-*product* of the copulas A and B (see [10] for details).

As we will see in the sequel, every 3-copula can be represented in the form (2.7). In fact, a **C**-lifting $\widetilde{C}$ can be interpreted as mixture of conditional distributions (see [5, Section 4.5] and [11]). Specifically, $\widetilde{C}$ is the d.f. of the random vector (U_1,U_2,U_3), U_i uniformly distributed on $[0,1]$ for $i \in \{1,2,3\}$, characterized by the following property: for every $t \in [0,1]$, the conditional d.f.'s of $[U_1 \mid U_2 = t]$ and $[U_3 \mid U_2 = t]$ are coupled by means of the copula C_t. For instance, if they were (conditionally) independent for every t, then C_t would be equal to Π_2 for every t.

Finally, we show a result that will be useful in next section, concerning the concordance order between two 3-copulas generated by means of the **C**-lifting operation.

Proposition 2.3. *Let $\mathbf{C} = (C_t)_{t\in[0,1]}$ and $\mathbf{C}' = (C'_t)_{t\in[0,1]}$ be two families in $\mathcal{C}_2$. For all $A,B \in \mathcal{C}_2$, suppose that the copulas $A\star_{\mathbf{C}}B$ and $A\star_{\mathbf{C}'}B$ are well defined. If $C_t \preceq C'_t$ for every $t \in [0,1]$, then $A\star_{\mathbf{C}}B \preceq A\star_{\mathbf{C}'}B$.*

Proof. It is immediate that $C_t \preceq C'_t$, for every $t \in [0,1]$, implies $A\star_{\mathbf{C}}B \leq A\star_{\mathbf{C}'}B$ in the pointwise order. Thus, we have only to prove that $\overline{A\star_{\mathbf{C}}B} \leq \overline{A\star_{\mathbf{C}'}B}$. To this end, notice that

$$\begin{aligned} (A\star_{\mathbf{C}}B)(u_1,u_2,1) &= (A\star_{\mathbf{C}'}B)(u_1,u_2,1) = A(u_1,u_2), \\ (A\star_{\mathbf{C}}B)(1,u_2,u_3) &= (A\star_{\mathbf{C}'}B)(1,u_2,u_3) = B(u_2,u_3). \end{aligned} \tag{2.10}$$

Therefore $\overline{A\star_{\mathbf{C}}B}(u_1,u_2,u_3) \leq \overline{A\star_{\mathbf{C}'}B}(u_1,u_2,u_3)$ if, and only if,

$$(A*_{\mathbf{C}}B)(u_1,u_3) - (A\star_{\mathbf{C}}B)(u_1,u_2,u_3) \leq (A*_{\mathbf{C}'}B)(u_1,u_3) - (A\star_{\mathbf{C}'}B)(u_1,u_2,u_3), \tag{2.11}$$

which, in turn, is equivalent to

$$\int_{u_2}^{1} C_t\left(\frac{\partial}{\partial t}A(u_1,t),\frac{\partial}{\partial t}B(t,u_3)\right)\mathrm{d}t \le \int_{u_2}^{1} C'_t\left(\frac{\partial}{\partial t}A(u_1,t),\frac{\partial}{\partial t}B(t,u_3)\right)\mathrm{d}t, \tag{2.12}$$

and this is obviously true since $C_t \le C'_t$ for every $t \in [0,1]$. □

3. Bounds for trivariate copulas

Given three compatible 2-copulas C_{12}, C_{13} and C_{23}, we are now interested in the bounds for the *Fréchet class* $\mathcal{F}(C_{12},C_{13},C_{23})$ of all 3-copulas whose 2-marginals are, respectively, C_{12}, C_{13} and C_{23}.

Theorem 3.1. *For every* $\widetilde{C} \in \mathcal{F}(C_{12},C_{13},C_{23})$ *and for all* u_1,u_2,u_3 *in* $[0,1]$, *one has*

$$C_L(u_1,u_2,u_3) \le \widetilde{C}(u_1,u_2,u_3) \le C_U(u_1,u_2,u_3), \tag{3.1}$$

where

$$\begin{aligned} C_L(u_1,u_2,u_3) &= \max_{(i,j,k)\in\mathcal{P}}\big\{(C_{ij}\star_{W_2}C_{jk})(u_i,u_j,u_k),(C_{ij}\star_{M_2}C_{jk})(u_i,u_j,u_k)\\ &\qquad + C_{ik}(u_i,u_k)-(C_{ij}*_{M_2}C_{jk})(u_i,u_k)\big\},\\ C_U(u_1,u_2,u_3) &= \min_{(i,j,k)\in\mathcal{P}}\big\{(C_{ij}\star_{M_2}C_{jk})(u_i,u_j,u_k),(C_{ij}\star_{W_2}C_{jk})(u_i,u_j,u_k)\\ &\qquad + C_{ik}(u_i,u_k)-(C_{ij}*_{W_2}C_{jk})(u_i,u_k)\big\}, \end{aligned} \tag{3.2}$$

and $\mathcal{P} = \{(1,2,3),(1,3,2),(2,1,3)\}$.

Proof. If $\widetilde{C} \in \mathcal{F}(C_{12},C_{13},C_{23})$, then there exist a probability space $(\Omega,\mathcal{F},P)$ and a random vector $\mathbf{U} = (U_1,U_2,U_3)$, U_i uniformly distributed on $[0,1]$ for each $i \in \{1,2,3\}$, such that, for all u_1,u_2,u_3 in $[0,1]$,

$$\widetilde{C}(u_1,u_2,u_3) = P(U_1 \le u_1, U_2 \le u_2, U_3 \le u_3). \tag{3.3}$$

Moreover, C_{12} is the copula of (U_1,U_2), C_{13} is the copula of (U_1,U_3) and C_{23} is the copula of (U_2,U_3). Then we have that

$$\widetilde{C}(u_1,u_2,u_3) = \int_0^{u_2} C_t^{(2)}\big(P(U_1 \le u_1 \mid U_2 = t), P(U_3 \le u_3 \mid U_2 = t)\big)\mathrm{d}t, \tag{3.4}$$

where, for each $t \in [0,1]$, $C_t^{(2)}$ is the 2-copula associated with the (conditional) distribution function of (U_1,U_3) given $U_2 = t$. But, by simple calculations, we also obtain that, almost surely on $[0,1]$,

$$P(U_1 \le u_1 \mid U_2 = t) = \frac{\partial C_{12}(u_1,t)}{\partial t}, \qquad P(U_3 \le u_3 \mid U_2 = t) = \frac{\partial C_{23}(t,u_3)}{\partial t}. \tag{3.5}$$

Therefore we can rewrite (3.4) in the form

$$\begin{aligned}\widetilde{C}(u_1,u_2,u_3) &= \int_0^{u_2} C_t^{(2)}\left(\frac{\partial}{\partial t}C_{12}(u_1,t), \frac{\partial}{\partial t}C_{23}(t,u_3)\right)\mathrm{d}t \\ &= (C_{12}\star_{\mathbf{C}_2}C_{23})(u_1,u_2,u_3),\end{aligned} \tag{3.6}$$

where $\mathbf{C_2} = (C_t^{(2)})_{t\in[0,1]}$. If we repeat the above procedure by conditioning in (3.4) with respect to $U_1 = t$ and with respect to $U_3 = t$, we obtain that there exist other two families of 2-copulas, $\mathbf{C}_1 = (C_t^{(1)})_{t\in[0,1]}$ and $\mathbf{C}_3 = (C_t^{(3)})_{t\in[0,1]}$, such that

$$\widetilde{C} = (C_{13}\star_{\mathbf{C}_3}C_{32})^{(1,3,2)} = C_{12}\star_{\mathbf{C}_2}C_{23} = (C_{21}\star_{\mathbf{C}_1}C_{13})^{(2,1,3)}. \tag{3.7}$$

Since $W_2 \preceq C \preceq M_2$ for every $C \in \mathcal{C}_2$, Proposition 2.3 ensures that, for each (i,j,k) in $\mathcal{P}$,

$$(C_{ij}\star_{W_2}C_{jk})^{(i,j,k)} \preceq \widetilde{C} \preceq (C_{ij}\star_{M_2}C_{jk})^{(i,j,k)}. \tag{3.8}$$

By definition of concordance order, for each (i,j,k) in $\mathcal{P}$ and $\mathbf{u} = (u_1,u_2,u_3) \in [0,1]^3$, we have that

$$(C_{ij}\star_{W_2}C_{jk})(u_i,u_j,u_k) \le \widetilde{C}(\mathbf{u}) \le (C_{ij}\star_{M_2}C_{jk})(u_i,u_j,u_k), \tag{3.9}$$

$$\overline{(C_{ij}\star_{W_2}C_{jk})}(u_i,u_j,u_k) \le \overline{\widetilde{C}}(\mathbf{u}) \le \overline{(C_{ij}\star_{M_2}C_{jk})}(u_i,u_j,u_k). \tag{3.10}$$

The first inequality in (3.10) is equivalent to:

$$\begin{aligned}&1-u_1-u_2-u_3+C_{ij}(u_i,u_j)+C_{jk}(u_j,u_k)+(C_{ij}*_{W_2}C_{jk})(u_i,u_k)-(C_{ij}\star_{W_2}C_{jk})(u_i,u_j,u_k)\\ &\quad\le 1-u_1-u_2-u_3+C_{ij}(u_i,u_j)+C_{jk}(u_j,u_k)+C_{ik}(u_i,u_k)-\widetilde{C}(u_i,u_j,u_k).\end{aligned} \tag{3.11}$$

The second inequality in (3.10) is equivalent to:

$$\begin{aligned}&1-u_1-u_2-u_3+C_{ij}(u_i,u_j)+C_{jk}(u_j,u_k)+C_{ik}(u_i,u_k)-\widetilde{C}(u_i,u_j,u_k)\\ &\quad\le 1-u_1-u_2-u_3+C_{ij}(u_i,u_j)+C_{jk}(u_j,u_k)+(C_{ij}*_{M_2}C_{jk})(u_i,u_k)-(C_{ij}\star_{M_2}C_{jk})(u_i,u_j,u_k).\end{aligned} \tag{3.12}$$

Easy calculations show that these inequalities are equivalent to:

$$\begin{aligned}\widetilde{C}(\mathbf{u}) &\le (C_{ij}\star_{W_2}C_{jk})(u_i,u_j,u_k)+C_{ik}(u_i,u_k)-(C_{ij}*_{W_2}C_{jk})(u_i,u_k),\\ \widetilde{C}(\mathbf{u}) &\ge (C_{ij}\star_{M_2}C_{jk})(u_i,u_j,u_k)+C_{ik}(u_i,u_k)-(C_{ij}*_{M_2}C_{jk})(u_i,u_k).\end{aligned} \tag{3.13}$$

Using these inequalities and (3.9), we directly get (3.1). □

Bounds of the above type are based on the so-called "method of conditioning", formulated for the first time by Rüschendorf [2] in a more general framework. Later, the same method was adopted in [5, Theorem 3.11], where it was provided an upper bound F_U and a lower bound F_L for $\mathcal{F}(C_{12}, C_{13}, C_{23})$ given by

$$\begin{aligned} F_U(u_1,u_2,u_3) &= \min\{C_{12}(u_1,u_2), C_{13}(u_1,u_3), C_{23}(u_2,u_3), 1-u_1-u_2-u_3 \\ &\quad + C_{12}(u_1,u_2)+C_{13}(u_1,u_3)+C_{23}(u_2,u_3)\} \\ F_L(u_1,u_2,u_3) &= \max\{0, C_{12}(u_1,u_2)+C_{13}(u_1,u_3)-u_1, C_{12}(u_1,u_2) \\ &\quad + C_{23}(u_2,u_3)-u_2, C_{13}(u_1,u_3)+C_{23}(u_2,u_3)-u_3\}. \end{aligned} \tag{3.14}$$

Here, a comparison with our bounds is presented.

Proposition 3.2. *Let C_{12}, C_{13} and C_{23} be three compatible 2-copulas. Then, for every $\mathbf{u} = (u_1,u_2,u_3) \in [0,1]^3$, one has that $C_L(\mathbf{u}) \geq F_L(\mathbf{u})$ and $C_U(\mathbf{u}) \leq F_U(\mathbf{u})$.*

Proof. Let $\mathbf{u}$ be in $[0,1]^3$. We have that

$$\begin{aligned} C_L(\mathbf{u}) &\geq (C_{13} \star_{W_2} C_{32})(u_1,u_3,u_2) \\ &= \int_0^{u_3} W_2\left(\frac{\partial}{\partial t}C_{13}(u_1,t), \frac{\partial}{\partial t}C_{32}(t,u_2)\right)\mathrm{d}t \\ &\geq C_{13}(u_1,u_3)+C_{23}(u_2,u_3)-u_3, \end{aligned} \tag{3.15}$$

and, analogously,

$$\begin{aligned} C_L(\mathbf{u}) &\geq C_{12}(u_1,u_2)+C_{13}(u_1,u_3)-u_1, \\ C_L(\mathbf{u}) &\geq C_{12}(u_1,u_2)+C_{23}(u_2,u_3)-u_2. \end{aligned} \tag{3.16}$$

Therefore, since $C_L(\mathbf{u}) \geq 0$, it follows that $C_L(\mathbf{u}) \geq F_L(\mathbf{u})$ for every $\mathbf{u}$ in $[0,1]^3$.

On the other hand, we have that

$$\begin{aligned} C_U(\mathbf{u}) &\leq (C_{13} \star_{M_2} C_{32})(u_1,u_3,u_2) \\ &= \int_0^{u_3} \min\left(\frac{\partial}{\partial t}C_{13}(u_1,t), \frac{\partial}{\partial t}C_{32}(t,u_2)\right)\mathrm{d}t \\ &\leq \min\left(C_{13}(u_1,u_3), C_{23}(u_2,u_3)\right), \end{aligned} \tag{3.17}$$

and, analogously, $C_U(\mathbf{u}) \leq C_{12}(u_1,u_2)$. Moreover, for every $\mathbf{u} \in [0,1]^3$, we have that

$$\begin{aligned} &(C_{12} \star_{W_2} C_{23})(u_1,u_2,u_3) + C_{13}(u_1,u_3) - (C_{12} * _{W_2} C_{23})(u_1,u_3) \\ &\quad \leq 1-u_1-u_2-u_3+C_{12}(u_1,u_2)+C_{13}(u_1,u_3)+C_{23}(u_2,u_3), \end{aligned} \tag{3.18}$$

as a consequence of the fact that $\overline{(C_{12} \star_{W_2} C_{23})}(\mathbf{u}) \geq 0$. Thus $C_U(\mathbf{u}) \leq F_U(\mathbf{u})$ for every $\mathbf{u}$ in $[0,1]^3$. $\square$

While the bounds F_L and F_U come from inequalities involving three random variables, the bounds C_L and C_U come from inequalities involving sets of two random variables, applied over each value of the third variable. These last bounds can be considered, in fact, as conditional Fréchet lower and upper bounds for the d.f.'s and the survival d.f.'s from each of the three permutations $(U_1, U_2) \mid U_3$, $(U_1, U_3) \mid U_2$ and $(U_2, U_3) \mid U_1$.

In general, C_U is strictly less than F_U (resp., C_L is strictly greater than F_L).

Example 3.3. Let us consider the copula $C(u_1, u_2) = u_1 u_2 (1 + (1 - u_1)(1 - u_2))$. We want to determine the bounds for $\mathcal{F}(C, C, C)$. First of all, note that $\mathcal{F}(C, C, C) \neq \varnothing$, because it contains the copula

$$\widetilde{C}(u_1, u_2, u_3) = u_1 u_2 u_3 (1 + (1 - u_1)(1 - u_2) + (1 - u_1)(1 - u_3) + (1 - u_2)(1 - u_3)) \tag{3.19}$$

(you can check that $\widetilde{C}$ is a copula just by computing that its density is positive). Now, it is easy to calculate that, for every $u \in [0, 1]$,

$$\begin{gathered} F_U(u, u, u) = \min\{C(u, u), 1 - 3u + 3C(u, u)\}, \\ C_U(u, u, u) = \min\left\{(C \star_{M_2} C)(u, u, u), (C \star_{W_2} C)(u, u, u) + C(u, u) - (C \ast_{W_2} C)(u, u)\right\}. \end{gathered} \tag{3.20}$$

When $u = 1/3$, we obtain

$$F_U(u, u, u) = C(u, u) = \frac{13}{81} > \frac{17}{243} = (C \star_{M_2} C)(u, u, u) \geq C_U(u, u, u). \tag{3.21}$$

Moreover, one has

$$\begin{gathered} F_L(u, u, u) = \max\{0, 2C(u, u) - u\}, \\ C_L(u, u, u) = \max\left\{(C \star_{W_2} C)(u, u, u), (C \star_{M_2} C)(u, u, u) + C(u, u) - (C \ast_{M_2} C)(u, u)\right\}. \end{gathered} \tag{3.22}$$

When $u = 3/5$, $F_L(u, u, u) = 147/625$ and $C_L(u, u, u) \geq (C \star_{W_2} C)(u, u, u) = 1/3 > F_L(u, u, u)$.

In the case of pairwise independence, C_U and F_U (resp., F_L and C_L) coincide.

Example 3.4. From Theorem 3.1, if $\widetilde{C}$ is in $\mathcal{F}(\Pi_2, \Pi_2, \Pi_2)$, then, for every u_1, u_2 and u_3 in $[0, 1]$, we have

$$C_L(u_1, u_2, u_3) \leq \widetilde{C}(u_1, u_2, u_3) \leq C_U(u_1, u_2, u_3), \tag{3.23}$$

where

$$\begin{aligned} C_L(u_1, u_2, u_3) &= \max\left\{u_1 W_2(u_2, u_3), u_2 W_2(u_1, u_3), u_3 W_2(u_1, u_2)\right\}, \\ C_U(u_1, u_2, u_3) &= \min\left\{u_1 u_2, u_1 u_3, u_2 u_3, (1 - u_1)(1 - u_2)(1 - u_3) + u_1 u_2 u_3\right\}. \end{aligned} \tag{3.24}$$

It is easy to check that, in this case, $C_L = F_L$ and $C_U = F_U$. These bounds were also obtained by Deheuvels [12] and Rodríguez-Lallena and Úbeda-Flores [13] (compare also with [5, Section 3.4.1]). Moreover, C_L and C_U may not be copulas, as noted in [13].

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References

[1] A. J. McNeil, R. Frey, and P. Embrechts, *Quantitative Risk Management. Concepts, Techniques and Tool*, Princeton Series in Finance, Princeton University Press, Princeton, NJ, USA, 2005.
[2] L. Rüschendorf, "Bounds for distributions with multivariate marginals," in *Stochastic Orders and Decision under Risk (Hamburg, 1989)*, vol. 19 of *IMS Lecture Notes—Monograph Series*, pp. 285–310, Institute of Mathematical Statistics, Hayward, Calif, USA, 1991.
[3] L. Rüschendorf, "Fréchet-bounds and their applications," in *Advances in Probability Distributions with Given Marginals (Rome, 1990)*, vol. 67 of *Mathematics and Its Applications*, pp. 151–187, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.
[4] H. Joe, "Families of m-variate distributions with given margins and $m(m-1)/2$ bivariate dependence parameters," in *Distributions with Fixed Marginals and Related Topics (Seattle, WA, 1993)*, vol. 28 of *IMS Lecture Notes—Monograph Series*, pp. 120–141, Institute of Mathematical Statistics, Hayward, Calif, USA, 1996.
[5] H. Joe, *Multivariate Models and Dependence Concepts*, vol. 73 of *Monographs on Statistics and Applied Probability*, Chapman & Hall, London, UK, 1997.
[6] R. B. Nelsen, *An Introduction to Copulas*, Springer Series in Statistics, Springer, New York, NY, USA, 2nd edition, 2006.
[7] M. Sklar, "Fonctions de répartition à n dimensions et leurs marges," *Publications de l'Institut de Statistique de l'Université de Paris*, vol. 8, pp. 229–231, 1959.
[8] W. F. Darsow, B. Nguyen, and E. T. Olsen, "Copulas and Markov processes," *Illinois Journal of Mathematics*, vol. 36, no. 4, pp. 600–642, 1992.
[9] A. Kolesárová, R. Mesiar, and C. Sempi, "Measure-preserving transformations, copulæ and compatibility," *Mediterranean Journal of Mathematics*, vol. 5, no. 3, pp. 325–338, 2008.
[10] F. Durante, E. P. Klement, J. J. Quesada-Molina, and P. Sarkoci, "Remarks on two product-like constructions for copulas," *Kybernetika*, vol. 43, no. 2, pp. 235–244, 2007.
[11] A. J. Patton, "Modelling asymmetric exchange rate dependence," *International Economic Review*, vol. 47, no. 2, pp. 527–556, 2006.
[12] P. Deheuvels, "Indépendance multivariée partielle et inégalités de Fréchet," in *Studies in Probability and Related Topics*, pp. 145–155, Nagard, Rome, Italy, 1983.
[13] J. A. Rodríguez-Lallena and M. Úbeda-Flores, "Compatibility of three bivariate quasi-copulas: applications to copulas," in *Soft Methodology and Random Information Systems*, Advances in Soft Computing, pp. 173–180, Springer, Berlin, Germany, 2004.