Research Article

On $\bar{\lambda}$-Statistically Convergent Double Sequences of Fuzzy Numbers

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1. Introduction

Nanda [1] studied sequence of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. Nuray [2] proved the inclusion relations between the set of statistically convergent and lacunary statistically convergent sequences of fuzzy numbers. Kwon and Shim [3] studied statistical convergence and lacunary statistical convergence of sequences of fuzzy numbers, and they showed that Nuray’s conditions are sufficient as well as necessary. Savaş [4] introduced and discussed double convergent sequence of fuzzy numbers and showed that the set of all double convergent sequences of fuzzy numbers is complete. In [5], Savaş generalized the statistical convergence by using de la Vallee-Poussin mean. Quite recently, Savaş and Mursaleen [6] introduced of statistically convergent and statistically Cauchy for double sequence of fuzzy numbers.

In this paper, we continue to study the concepts of strongly double $[V,\bar{\lambda}]$-summable and double $S_{\bar{\lambda}}$-convergent for double sequence of fuzzy numbers.

2. Preliminaries

Before continuing with the discussion, we pause to establish some notation. Let $C(R^n) = \{ A \subset R^n : A$ compact and convex $\}$. The spaces $C(R^n)$ have a linear structure induced by the operations

\[ A + B = \{ a + b, \ a \in A, \ b \in B \}, \]
\[ \lambda A = \{ \lambda a, \ \lambda \in A \} \]  \hspace{1cm} (2.1)
for \( A, B \in C(R^n) \), and \( \lambda \in R \). The Hausdorff distance between \( A \) and \( B \) of \( C(R^n) \) is defined as

\[
\delta_\infty(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \| a - b \|, \sup_{b \in B} \inf_{a \in A} \| a - b \| \right\}.
\]

(2.2)

It is well known that \((C(R^n), \delta_\infty)\) is a complete (not separable) metric space.

A fuzzy number is a function \( X \) from \( R^n \) to \([0, 1]\) satisfying

1. \( X \) is normal, that is, there exists an \( x_0 \in R^n \) such that \( X(x_0) = 1 \);
2. \( X \) is fuzzy convex, that is, for any \( x, y \in R^n \) and \( 0 \leq \lambda \leq 1 \),

\[
X(\lambda x + (1 - \lambda)y) \geq \min\{X(x), X(y)\};
\]

(2.3)

3. \( X \) is upper semicontinuous;
4. the closure of \([ x \in R^n : X(x) > 0 \])\), denoted by \( X^0 \), is compact.

These properties imply that for each \( 0 < \alpha \leq 1 \), the \( \alpha \)-level set

\[
X^\alpha = \{ x \in R^n : X(x) \geq \alpha \}
\]

(2.4)

is a nonempty compact convex, subset of \( R^n \), as is the support \( X^0 \). Let \( L(R^n) \) denote the set of all fuzzy numbers. The linear structure of \( L(R^n) \) induces addition \( X + Y \) and scalar multiplication \( \lambda X, \lambda \in R \), in terms of \( \alpha \)-level sets by

\[
[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha,
\]

\[
[\lambda X]^\alpha = \lambda [X]^\alpha
\]

(2.5)

for each \( 0 \leq \alpha \leq 1 \).

Define for each \( 1 \leq q < \infty \),

\[
d_q(X, Y) = \left\{ \int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q \, d\alpha \right\}^{1/q}
\]

(2.6)

and \( d_\infty = \sup_{0 \leq \alpha \leq 1} \delta_\infty(X^\alpha, Y^\alpha) \). Clearly, \( d_\infty(X, Y) = \lim_{q \to \infty} d_q(X, Y) \) with \( d_q \leq d_r \) if \( q \leq r \). Moreover, \( d_q \) is a complete, separable, and locally compact metric space [7].

Throughout the paper, \( d \) will denote \( d^q \) with \( 1 \leq q \leq \infty \).

We will need the following definitions.

**Definition 2.1.** A double sequence \( X = (X_{kl}) \) of fuzzy numbers is said to be convergent in the Pringsheim’s sense or \( P \)-convergent to a fuzzy number \( X_0 \) if for every \( \varepsilon > 0 \), there exists \( N \in \mathcal{N} \) such that

\[
d(X_{kl}, X_0) < \varepsilon \quad \text{for} \quad k, l > N,
\]

(2.7)

and we denote \( P - \lim X = X_0 \). The number \( X_0 \) is called the Pringsheim limit of \( X_{kl} \).

More exactly, we say that a double sequence \( (X_{kl}) \) converges to a finite number \( X_0 \) if \( X_{kl} \) tend to \( X_0 \) as both \( k \) and \( l \) tends to \( \infty \) independently of one another.
Let $c^2(F)$ denote the set of all double convergent sequences of fuzzy numbers.

**Definition 2.2.** A double sequence $X = (X_{kl})$ of fuzzy numbers is bounded if there exists a positive number $M$ such that $d(X_{kl}, X_0) < M$ for all $k$ and $l$,

$$\|x\|_{(\infty,2)} = \sup_{k,l} d(X_{kl}, X_0) < \infty. \quad (2.8)$$

We will denote the set of all bounded double sequences by $l^2_\infty(F)$.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K_{m,n}$ be the numbers of $(i, j)$ in $K$ such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density of $K$ is defined as

$$P - \liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K). \quad (2.9)$$

In the case when the sequence $(K_{m,n}/mn)_{m,n=1,1}^{\infty,\infty}$ has a limit, then we say that $K$ has a natural density and is defined as

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K). \quad (2.10)$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of natural numbers. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \leq P - \lim_{m,n} \sqrt{m} \sqrt{n} \frac{\sqrt{m} \sqrt{n}}{mn} = 0 \quad (2.11)$$

(i.e., the set $K$ has double natural density zero).

**Definition 2.3.** A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be statistically convergent to $X_0$ provided that for each $\epsilon > 0$,

$$P - \lim_{m,n} \frac{1}{mn} \sum_{k \in I_n} \sum_{l \in I_m} d(X_{kl}, X_0) = 0. \quad (2.12)$$

In this case, we write $\text{st}_2 - \lim_{k,l} X_{k,l} = X_0$ and we denote the set of all double statistically convergent sequences of fuzzy numbers by $\text{st}_2^2(F)$.

**Definition 2.4.** $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ could be two nondecreasing sequences of positive real numbers such that each tends to $\infty$ and

$$\lambda_{n+1} \leq \lambda_n + 1, \quad \lambda_1 = 1, \quad (2.13)$$

$$\mu_{m+1} \leq \mu_m + 1, \quad \mu_1 = 1.$$

A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be $\bar{\lambda}$-summable if there is fuzzy number $X_0$ such that

$$P - \lim_{n,m} \frac{1}{\lambda_{nm}} \sum_{k \in I_n} \sum_{l \in I_m} d(X_{kl}, X_0) = 0, \quad (2.14)$$

where $I_n = [n - \lambda_n + 1, n], I_m = [m - \mu_m + 1, m]$, and $\bar{\lambda}_{nm} = \lambda_n \mu_m$. 
In this case, we say that $X$ is strongly double $\lambda$-summable to $X_0$ and we denote the set of all strongly double $\lambda$-summable sequences by $[V,\lambda](F)$. If $\lambda_{nm} = nm$, then strongly double $\lambda$-summable reduces to $[C,1,1](F)$, the space of strongly double Cesàro summable sequences defined as follows:

$$
P - \lim_{nm} \frac{1}{nm} \sum_{k,l=1}^{\infty} d(X_{kl},X_0) = 0.
$$

(2.15)

Definition 2.5. A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be double $\lambda$-statistically convergent or $S_\lambda$-convergent to $X_0$ if for every $\epsilon > 0$,

$$
P - \lim_{n,m} \frac{1}{\lambda_{nm}} \left| \left\{ k \in I_n, l \in I_m : d(X_{kl},X_0) \geq \epsilon \right\} \right| = 0.
$$

(2.16)

In this case, we write $S_\lambda - \lim X = X_0$ or $X_{kl} \overset{P}{\rightarrow} X_0(S_\lambda)$ and we denote the set of all double $S_\lambda$-statistically convergent sequences of fuzzy numbers by $(S_\lambda)(F)$.

If $\lambda_{nm} = nm$, for all $n,m$, then the set $S_\lambda(F)$ of $S_\lambda$-convergent sequences reduces to the space $s^2(F)$.

We need the following proposition in future. A metric $d$ on $L(\mathbb{R})$ is said to be a translation invariant if $d(X + Z,Y + Z) = d(X,Y)$ for $X,Y,Z \in L(\mathbb{R})$.

Proposition 2.6. If $d$ is a translation invariant metric on $L(\mathbb{R})$, then

$$
d(X + Y,0) \leq d(X,0) + d(Y,0).
$$

(2.17)

Proof is clear so we omitted it.

In the next theorem, we give some connections between strongly double $\lambda$-summable and double $\lambda$-statistical convergences.

3. Main results

Theorem 3.1. A double sequence $X = (X_{kl})$ of fuzzy numbers is strongly double $\lambda$-summable $X_0$, then it is double $\lambda$-statistically convergent to $X_0$.

Proof. Let $\epsilon > 0$ and since

$$
\sum_{k \in I_n, l \in I_m} d(X_{kl},X_0) \geq \sum_{k \in I_n, l \in I_m, d(X_{kl},X_0) \geq \epsilon} d(X_{kl},X_0) \geq \epsilon \left| \left\{ k \in I_n, l \in I_m : d(X_{kl},X_0) \geq \epsilon \right\} \right|.
$$

(3.1)

This implies that if a sequence $X = (X_{kl})$ is strongly double $\lambda$-summable $X_0$, then $X$ is double $\lambda$-statistically convergent to $X_0$.

This completes the proof.

We have the following theorem.

Theorem 3.2. If a bounded $(X_{kl})$ is double $\lambda$-statistically convergent to $X_0$, then it is strongly double $\lambda$-summable $X_0$.  

Proof. Suppose that \((X_{kl})\) is bounded and double \(\lambda\)-statistically convergent to \(X_0\). Since \(X\) is bounded we write \(d(X_{kl}, X_0) \leq M\) for all \(k, l\). Also for given \(\varepsilon > 0\) and \(n\) and \(m\) large we obtain

\[
\frac{1}{\lambda_{nm}} \sum_{k \in I_n, l \in I_m} d(X_{kl}, X_0) = \frac{1}{\lambda_{nm}} \sum_{k \in I_n, l \in I_m, d(X_{kl}, X_0) \geq \varepsilon} d(X_{kl}, X_0) + \frac{1}{\lambda_{nm}} \sum_{k \in I_n, l \in I_m, d(X_{kl}, X_0) < \varepsilon} d(X_{kl}, X_0)
\]

\[
\leq M \frac{1}{\lambda_{nm}} \left| \{k \in I_n, l \in I_m : d(X_{kl}, X_0) \geq \varepsilon \} \right| + \varepsilon,
\]

which implies that \(X\) is strongly double \(\lambda\)-summable \(X_0\).

This completes the proof. \(\square\)

**Theorem 3.3.** If a sequence \(X = (X_{kl})\) of fuzzy numbers is double statistically convergent to \(X_0\), then it is double \(\lambda\)-statistically convergent to \(X_0\) if and only if

\[
P - \lim \inf_{nm} \frac{\lambda_{nm}}{nm} > 0.
\]

**Proof.** For given \(\varepsilon > 0\), we have

\[
\{ k \leq n, l \leq m : d(X_{kl}, X_0) \geq \varepsilon \} \supset \{ k \in I_n, l \in I_m : d(X_{kl}, X_0) \geq \varepsilon \}.
\]

Therefore,

\[
\frac{1}{nm} \left| \{ k \leq n, l \leq m : d(X_{kl}, X_0) \geq \varepsilon \} \right| \geq \frac{1}{nm} \left| \{ k \in I_n, l \in I_m : d((X_{kl}, X_0) \geq \varepsilon) \} \right|
\]

\[
\geq \frac{\lambda_{nm}}{nm} \frac{1}{\lambda_{nm}} \left| \{ k \in I_n, l \in I_m : d((X_{kl}, X_0) \geq \varepsilon) \} \right|.
\]

Taking the limit as \(n, m \to \infty\) and using hypothesis, we get \(X\) is double \(\lambda\)-statistically convergent to \(X_0\).

Conversely, suppose that \(X \in \text{st}_2(F)\) and since \(\lambda_{nm} = \lambda_n \mu_m\), either \(P - \lim \inf \lambda_n / n = 0\) or \(P - \lim \inf (\mu_n / m) = 0\) or both are zero. Then we can choose subsequences \((n(p))_{p=1}^{\infty}\) and \((m(q))_{q=1}^{\infty}\) such that \(\lambda_{n(p)} / n(p) < 1/p\) and \(\mu_{m(q)} / m(q) < 1/q\). Define a sequence \(X = (X_{kl})\) by

\[
X_{kl} = \begin{cases} 
1 & \text{if } k \in I_{n(p)}, l \in I_{m(q)} \ (p, q = 1, 2, \ldots), \\
0 & \text{otherwise.} 
\end{cases}
\]

Then \(X \in [C, 1, 1](F)\) and hence, by [6, Theorem 6(a)], \(X \in \text{st}_2(F)\). But on the other hand, \(X \notin [V, \overline{\lambda}](F)\) and from Theorem 3.1, \(X \notin (S_{\overline{\lambda}})(F)\); a contradiction and hence (3.3) must hold. \(\square\)

Finally, we conclude this paper by stating a definition which generalizes Definition 2.4.
**Definition 3.4.** Let $X = (X_{kl})$ be a double sequence of fuzzy numbers and let $p$ be positive real numbers. The sequence $X$ is said to be strongly double $\lambda_p$-summable if there is fuzzy number $X_0$ such that

$$P - \lim_{nm} \frac{1}{\lambda_{nm}} \sum_{k \in I_n} \sum_{l \in I_m} d(X_{kl}, X_0)^p = 0. \tag{3.7}$$

In this case, we say that $X$ is strongly double $\lambda_p$-summable to $X_0$. If $\lambda_{nm} = nm$, then strongly double $\lambda_p$-summable reduces to strongly double $p$-Cesàro summable to $X_0$.

**Theorem 3.5.** (1) Let $p \in (0, \infty)$. If a double sequence $X = (X_{kl})$ of fuzzy numbers is strongly double $\lambda_p$-summable $X_0$, then it is double $\lambda$-statistically convergent to $X_0$.

(2) Let $p \in (0, \infty)$. If a bounded $(X_{kl})$ is double $\lambda$-statistically convergent to $X_0$, then it is strongly double $\lambda_p$-summable $X_0$.

**Proof.** The proof of theorem is similar to that of Theorems 3.1 and 3.2 so we omitted it. 

**References**


