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Research Article

A New Subclass of Analytic Functions Defined by Generalized Ruscheweyh Differential Operator

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We investigate a new subclass of analytic functions in the open unit disk $\mathbb{U}$ which is defined by generalized Ruscheweyh differential operator. Coefficient inequalities, extreme points, and the integral means inequalities for the fractional derivatives of order $p+\eta$ $(0 \leq p \leq n,\ 0 \leq \eta < 1)$ of functions belonging to this subclass are obtained.

1. Introduction

Throughout this paper, we use the following notations:

$$\begin{aligned} \mathbb{N} &:= \{1, 2, 3, \dots\}, \\ \mathbb{N}_0 &:= \mathbb{N} \cup \{0\}, \\ \mathbb{R}_{-1} &:= \{u \in \mathbb{R} : u > -1\}, \\ \mathbb{R}_{-1}^0 &:= \mathbb{R}_{-1} \setminus \{0\}. \end{aligned} \tag{1.1}$$

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.2}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

For $f_j \in \mathcal{A}$ given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2), \tag{1.3}$$

the Hadamard product (or convolution) $f_1 * f_2$ of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \tag{1.4}$$

Using the convolution (1.4), Shaqsi and Darus [1] introduced the generalization of the Ruscheweyh derivative as follows.

For $f \in \mathcal{A}$, $\lambda \geq 0$, and $u \in \mathbb{R}_{-1}$, we consider

$$R_\lambda^u f(z) = \frac{z}{(1-z)^{u+1}} * R_\lambda f(z) \quad (z \in \mathbb{U}), \tag{1.5}$$

where $R_\lambda f(z) = (1-\lambda) f(z) + \lambda z f'(z)$, $z \in \mathbb{U}$.

If $f \in \mathcal{A}$ is of the form (1.2), then we obtain the power series expansion of the form

$$R_\lambda^u f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda] C(u,n) a_n z^n, \tag{1.6}$$

where

$$C(u,n) = \frac{(1+u)_{n-1}}{(n-1)!} \quad (n \in \mathbb{N}), \tag{1.7}$$

and where $(a)_n$ is the Pochhammer symbol (or shifted factorial) defined (in terms of the Gamma function) by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0,\ a \in \mathbb{C} \setminus \{0\}, \\ a(a+1)\cdots(a+n-1), & \text{if } n \in \mathbb{N},\ a \in \mathbb{C}. \end{cases} \tag{1.8}$$

In the case $m \in \mathbb{N}_0$, we have

$$R_\lambda^m f(z) = \frac{z\left(z^{m-1} f(z)\right)^{(m)}}{m!}, \tag{1.9}$$

and for $\lambda = 0$, we obtain uth Ruscheweyh derivative introduced in [2], $R_0^m = R^m$.

Using the generalized Ruscheweyh derivative operator R_λ^u, we define the following classes.

Definition 1.1. Let $\mathcal{S}_\lambda(u, v; \alpha)$ be the class of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left\{ \frac{R_\lambda^u f(z)}{R_\lambda^v f(z)} \right\} > \alpha \tag{1.10}$$

for some $0 \leq \alpha < 1$, $u \in \mathbb{R}_{-1}^0$, $v \in \mathbb{R}_{-1}$, $\lambda \geq 0$, and all $z \in \mathbb{U}$.

In this paper, basic properties of the class $\mathcal{S}_\lambda(u,v;\alpha)$ are studied, such as coefficient bounds, extreme points, and integral means inequalities for the fractional derivative.

2. Coefficient inequalities

Theorem 2.1. *Let* $0 \leq \alpha < 1,\ u \in \mathbb{R}^0_{-1},\ v \in \mathbb{R}_{-1}$, *and* $\lambda \geq 0$. *If* $f \in \mathcal{A}$ *satisfies*

$$\sum_{n=2}^{\infty} \mathcal{B}_n(u,v,\alpha)|a_n| \leq 2(1-\alpha), \tag{2.1}$$

where

$$\mathcal{B}_n(u,v,\alpha) := [1+(n-1)\lambda]\{|C(u,n)-(1+\alpha)C(v,n)| + C(u,n) + (1-\alpha)C(v,n)\}, \tag{2.2}$$

then $f \in \mathcal{S}_\lambda(u,v;\alpha)$.

Proof. Let (2.1) be true for $0 \leq \alpha < 1,\ u \in \mathbb{R}^0_{-1},\ v \in \mathbb{R}_{-1}$, and $\lambda \geq 0$. For $f \in \mathcal{A}$, define the function F by

$$F(z) := \frac{R^u_\lambda f(z)}{R^v_\lambda f(z)} - \alpha. \tag{2.3}$$

It is sufficient to show that

$$\left|\frac{F(z)-1}{F(z)+1}\right| < 1 \tag{2.4}$$

for $z \in \mathbb{U}$.

So, we have

$$\begin{aligned}
\left|\frac{F(z)-1}{F(z)+1}\right| &= \left|\frac{R^u_\lambda f(z) - (1+\alpha)R^v_\lambda f(z)}{R^u_\lambda f(z) + (1-\alpha)R^v_\lambda f(z)}\right| \\
&= \left|\frac{\alpha - \sum_{n=2}^{\infty}[1+(n-1)\lambda][C(u,n)-(1+\alpha)C(v,n)]a_n z^{n-1}}{(2-\alpha) + \sum_{n=2}^{\infty}[1+(n-1)\lambda][C(u,n)+(1-\alpha)C(v,n)]a_n z^{n-1}}\right| \\
&\leq \frac{\alpha + \sum_{n=2}^{\infty}[1+(n-1)\lambda]|C(u,n)-(1+\alpha)C(v,n)||a_n||z|^{n-1}}{(2-\alpha) - \sum_{n=2}^{\infty}[1+(n-1)\lambda][C(u,n)+(1-\alpha)C(v,n)]|a_n||z|^{n-1}} \\
&< \frac{\alpha + \sum_{n=2}^{\infty}[1+(n-1)\lambda]|C(u,n)-(1+\alpha)C(v,n)||a_n|}{(2-\alpha) - \sum_{n=2}^{\infty}[1+(n-1)\lambda][C(u,n)+(1-\alpha)C(v,n)]|a_n|} \\
&< 1 \quad (\text{by } (2.1)).
\end{aligned} \tag{2.5}$$

Therefore, $f \in \mathcal{S}_\lambda(u,v;\alpha)$. □

Theorem 2.2. *If $f \in \mathcal{S}_\lambda(u, v; \alpha)$, then*

$$|a_n| \le \frac{2(1-\alpha)}{[1+(n-1)\lambda]|C(u,n)-C(v,n)|}\sum_{u=1}^{n-1}[1+(n-u-1)\lambda]C(v,n-u)|a_{n-u}| \tag{2.6}$$

for $n \ge 2$, with $a_1 = 1$.

Proof. Define the function

$$G(z) := \frac{1}{1-\alpha}\left(\frac{R_\lambda^u f(z)}{R_\lambda^v f(z)} - \alpha\right) := 1 + \sum_{n=1}^{\infty}\widehat{a}_n z^n. \tag{2.7}$$

Since $\mathrm{Re}\{G(z)\} > 0$, we get

$$|\widehat{a}_n| \le 2 \tag{2.8}$$

for $n = 1, 2, \ldots$.

From the definition of $G(z)$, we obtain

$$\frac{R_\lambda^u f(z) - \alpha R_\lambda^v f(z)}{1-\alpha} = R^v f(z)\left[1 + \sum_{n=1}^{\infty}\widehat{a}_n z^n\right]. \tag{2.9}$$

So, by (1.6), we have

$$\begin{aligned} z &+ \frac{1+\lambda}{1-\alpha}[C(u,2)-\alpha C(v,2)]a_2 z^2 + \frac{1+2\lambda}{1-\alpha}[C(u,3)-\alpha C(v,3)]a_3 z^3 + \cdots \\ &= z + \widehat{a}_1 z^2 + \widehat{a}_2 z^3 + \widehat{a}_3 z^4 + \cdots \\ &\quad + (1+\lambda)C(v,2)a_2 z^2 + (1+\lambda)C(v,2)a_2\widehat{a}_1 z^3 + (1+\lambda)C(v,2)a_2\widehat{a}_2 z^4 + \cdots \\ &\quad + (1+2\lambda)C(v,3)a_3 z^3 + (1+2\lambda)C(v,3)a_3\widehat{a}_1 z^4 + \cdots \end{aligned} \tag{2.10}$$

or

$$\begin{aligned} z &+ \frac{1+\lambda}{1-\alpha}[C(u,2)-C(v,2)]a_2 z^2 + \frac{1+2\lambda}{1-\alpha}[C(u,3)-C(v,3)]a_3 z^3 + \cdots \\ &= z + \widehat{a}_1 z^2 + [(1+\lambda)C(v,2)a_2\widehat{a}_1 + \widehat{a}_2]z^3 \\ &\quad + [(1+2\lambda)C(v,3)a_3\widehat{a}_1 + (1+\lambda)C(v,2)a_2\widehat{a}_2 + \widehat{a}_3]z^4 + \cdots \end{aligned} \tag{2.11}$$

or, equivalently,

$$z+\sum_{n=2}^{\infty}\frac{1+(n-1)\lambda}{1-\alpha}[C(u,j)-C(v,j)]a_nz^n \\ = z+\sum_{n=2}^{\infty}\left(\sum_{u=1}^{n-1}[1+(n-u-1)\lambda]C(v,n-u)a_{n-u}\widehat{a}_u\right)z^n. \tag{2.12}$$

When we consider the coefficients of z^n of both series in the above equality, we have

$$a_n=\frac{1-\alpha}{[1+(n-1)\lambda][C(u,n)-C(v,n)]}\sum_{u=1}^{n-1}[1+(n-u-1)\lambda]C(v,n-u)a_{n-u}\widehat{a}_u. \tag{2.13}$$

Therefore,

$$|a_n| \le \frac{1-\alpha}{[1+(n-1)\lambda]|C(u,n)-C(v,n)|}\sum_{u=1}^{n-1}[1+(n-u-1)\lambda]C(v,n-u)|a_{n-u}||\widehat{a}_u| \\ \le \frac{2(1-\alpha)}{[1+(n-1)\lambda]|C(u,n)-C(v,n)|}\sum_{u=1}^{n-1}[1+(n-u-1)\lambda]C(v,n-u)|a_{n-u}|, \tag{2.14}$$

since $|\widehat{a}_u| \le 2,\ (u=1,2,\ldots)$. □

3. Extreme points

Definition 3.1. Let $\widetilde{\mathcal{S}}_\lambda(u,v;\alpha)$ be the subclass of $\mathcal{S}_\lambda(u,v;\alpha)$ which consists of function

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \quad (a_n \ge 0) \tag{3.1}$$

whose coefficients satisfy inequality (2.1).

Theorem 3.2. *Let* $f_1(z)=z$ *and*

$$f_k(z)=z+\frac{2(1-\alpha)}{\mathcal{B}_k(u,v,\alpha)}z^k \quad (k=2,3,\ldots), \tag{3.2}$$

where $\mathcal{B}_k(u,v,\alpha)$ *is given by* (2.2).

Then $f \in \widetilde{\mathcal{S}}_{\lambda}(u, v; \alpha)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \delta_k f_k(z), \tag{3.3}$$

where $\delta_k \geq 0$ *and* $\sum_{k=1}^{\infty} \delta_k = 1$.

Proof. Assume that

$$f(z) = \sum_{k=1}^{\infty} \delta_k f_k(z). \tag{3.4}$$

Then

$$\begin{aligned} f(z) &= \delta_1 f_1(z) + \sum_{k=2}^{\infty} \delta_k f_k(z) \\ &= \delta_1 z + \sum_{k=2}^{\infty} \delta_k \left(z + \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} z^k \right) \\ &= \left(\sum_{k=1}^{\infty} \delta_k \right) z + \sum_{k=2}^{\infty} \delta_k \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} z^k \\ &= z + \sum_{k=2}^{\infty} \delta_k \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} z^k. \end{aligned} \tag{3.5}$$

Thus

$$\sum_{k=2}^{\infty} \delta_k \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} \mathcal{B}_k(u, v, \alpha) = 2(1-\alpha) \sum_{k=2}^{\infty} \delta_k = 2(1-\alpha)(1-\delta_1) \leq 2(1-\alpha). \tag{3.6}$$

Therefore, we have $f \in \widetilde{\mathcal{S}}_{\lambda}(u, v; \alpha)$.

Conversely, suppose that $f \in \widetilde{\mathcal{S}}_{\lambda}(u, v; \alpha)$. Since

$$a_k \leq \frac{2(1-\alpha)}{\mathcal{B}_k(u, v, \alpha)} \quad (k = 2, 3, \ldots), \tag{3.7}$$

we can set

$$\begin{aligned} \delta_k &:= \frac{\mathcal{B}_k(u, v, \alpha)}{2(1-\alpha)} a_k \quad (k = 2, 3, \ldots), \\ \delta_1 &:= 1 - \sum_{k=2}^{\infty} \delta_k. \end{aligned} \tag{3.8}$$

Then

$$\begin{aligned} f(z) &= z + \sum_{k=2}^{\infty} a_k z^k \\ &= \left(\sum_{k=1}^{\infty}\delta_k\right)z + \sum_{k=2}^{\infty}\delta_k \frac{2(1-\alpha)}{\mathcal{B}_k(u,v,\alpha)} z^k \\ &= \delta_1 z + \sum_{k=2}^{\infty}\delta_k\left(z + \frac{2(1-\alpha)}{\mathcal{B}_k(u,v,\alpha)} z^k\right) \\ &= \delta_1 f_1(z) + \sum_{k=2}^{\infty}\delta_k f_k(z) \\ &= \sum_{k=1}^{\infty}\delta_k f_k(z). \end{aligned} \tag{3.9}$$

This completes the proof of Theorem 3.2. □

Corollary 3.3. *The extreme points of $\widetilde{\mathcal{S}}_\lambda(u,v;\alpha)$ are given by*

$$f_1(z) = z, \qquad f_k(z) = z + \frac{2(1-\alpha)}{\mathcal{B}_k(u,v,\alpha)} z^k \quad (k = 2,3,\ldots), \tag{3.10}$$

where $\mathcal{B}_k(u,v,\alpha)$ is given by (2.2).

4. The main integral means inequalities for the fractional derivative

We discuss the integral means inequalities for functions $f \in \widetilde{\mathcal{S}}_\lambda(u,v;\alpha)$.

The following definitions of fractional derivatives by Owa [3] (also by Srivastava and Owa [4]) will be required in our investigation.

Definition 4.1. The fractional derivative of order η is defined, for a function f, by

$$D_z^{\eta} f(z) = \frac{1}{\Gamma(1-\eta)}\frac{d}{dz}\int_0^z \frac{f(\xi)}{(z-\xi)^{\eta}}d\xi \quad (0 \le \eta < 1), \tag{4.1}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\eta}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 4.2. Under the hypothesis of Definition 4.1, the fractional derivative of order $p+\eta$ is defined, for a function f, by

$$D_z^{p+\eta} f(z) = \frac{d^p}{dz^p} D_z^{\eta} f(z), \tag{4.2}$$

where $0 \le \eta < 1$ and $p \in \mathbb{N}_0$.

It readily follows from (4.1) in Definition 4.1 that

$$D_z^\eta z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\eta)} z^{k-\eta} \quad (0 \le \eta < 1,\ k \in \mathbb{N}). \tag{4.3}$$

We will also need the concept of subordination between analytic functions and a subordination theorem of Littlewood [5] in our investigation.

Definition 4.3. Given two functions f and g, which are analytic in $\mathbb{U}$, the function f is said to be subordinate to g in $\mathbb{U}$ if there exists a function w analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in \mathbb{U}), \tag{4.4}$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \tag{4.5}$$

We denote this subordination by

$$f(z) \prec g(z). \tag{4.6}$$

Lemma 4.4. *If the functions f and g are analytic in $\mathbb{U}$ with*

$$f(z) \prec g(z), \tag{4.7}$$

then, for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\mu d\theta \le \int_0^{2\pi} |g(z)|^\mu d\theta. \tag{4.8}$$

Our main theorem is contained in the following.

Theorem 4.5. *Let $f \in \widetilde{\mathcal{S}}_\lambda(u, v; \alpha)$ and suppose that*

$$\sum_{n=2}^{\infty} (n-p)_{p+1} a_n \le \frac{2(1-\alpha)\Gamma(k+1)\Gamma(3-\eta-p)}{\mathcal{B}_k(u,v,\alpha)\Gamma(k+1-\eta-p)\Gamma(2-p)} \tag{4.9}$$

for $0 \le p \le n,\ k \ge p,\ 0 \le \eta < 1$, where $(n-p)_{p+1}$ denotes the Pochhammer symbol defined by

$$(n-p)_{p+1} = (n-p)(n-p+1)\cdots n. \tag{4.10}$$

Also let the function f_k be defined by

$$f_k(z) = z + \frac{2(1-\alpha)}{\mathcal{B}_k(u,v,\alpha)} z^k \quad (k = 2, 3, \ldots). \tag{4.11}$$

If there exists an analytic function w defined by

$$(w(z))^{k-1} := \frac{\mathcal{B}_k(u,v,\alpha)\Gamma(k+1-\eta-p)}{2(1-\alpha)\Gamma(k+1)} \sum_{n=2}^{\infty}(n-p)_{p+1}\Psi(n)a_n z^{n-1} \tag{4.12}$$

with

$$\Psi(n) = \frac{\Gamma(n-p)}{\Gamma(n+1-\eta-p)}, \quad (0 \leq \eta < 1,\ n = 2,3,\ldots), \tag{4.13}$$

then, for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left|D_z^{p+\eta} f(z)\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|D_z^{p+\eta} f_k(z)\right|^{\mu} d\theta, \quad (0 \leq \eta < 1). \tag{4.14}$$

Proof. By means of (4.3) and Definition 4.2, we find from (3.1) that

$$\begin{aligned} D_z^{p+\eta} f(z) &= \frac{z^{1-\eta-p}}{\Gamma(2-\eta-p)}\left[1 + \sum_{n=2}^{\infty}\frac{\Gamma(2-\eta-p)\Gamma(n+1)}{\Gamma(n+1-\eta-p)}a_n z^{n-1}\right] \\ &= \frac{z^{1-\eta-p}}{\Gamma(2-\eta-p)}\left[1 + \sum_{n=2}^{\infty}\Gamma(2-\eta-p)(n-p)_{p+1}\Psi(n)a_n z^{n-1}\right], \end{aligned} \tag{4.15}$$

where

$$\Psi(n) = \frac{\Gamma(n-p)}{\Gamma(n+1-\eta-p)}, \quad (0 \leq \eta < 1,\ n = 2,3,\ldots). \tag{4.16}$$

Since Ψ is a decreasing function of n, we get

$$0 < \Psi(n) \leq \Psi(2) = \frac{\Gamma(2-p)}{\Gamma(3-\eta-p)}. \tag{4.17}$$

Similarly, from (4.11), (4.3), and Definition 4.2, we have

$$D_z^{p+\eta} f_k(z) = \frac{z^{1-\eta-p}}{\Gamma(2-\eta-p)}\left[1 + \frac{2(1-\alpha)}{\mathcal{B}_k(u,v,\alpha)}\frac{\Gamma(2-\eta-p)\Gamma(k+1)}{\Gamma(k+1-\eta-p)}z^{k-1}\right]. \tag{4.18}$$

For $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$, we want to show that

$$\int_0^{2\pi} \left| 1 + \sum_{n=2}^{\infty} \Gamma(2-\eta-p)(n-p)_{p+1}\Psi(n)a_n z^{n-1} \right|^{\mu} d\theta \\ \leq \int_0^{2\pi} \left| 1 + \frac{2(1-\alpha)}{\mathcal{B}_k(u,v,\alpha)} \frac{\Gamma(2-\eta-p)\Gamma(k+1)}{\Gamma(k+1-\eta-p)} z^{k-1} \right|^{\mu} d\theta. \tag{4.19}$$

So, by applying Lemma 4.4, it is enough to show that

$$1 + \sum_{n=2}^{\infty} \Gamma(2-\eta-p)(n-p)_{p+1}\Psi(n)a_n z^{n-1} \prec 1 + \frac{2(1-\alpha)}{\mathcal{B}_k(u,v,\alpha)} \frac{\Gamma(2-\eta-p)\Gamma(k+1)}{\Gamma(k+1-\eta-p)} z^{k-1}. \tag{4.20}$$

If the above subordination holds true, then we have an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ such that

$$1 + \sum_{n=2}^{\infty} \Gamma(2-\eta-p)(n-p)_{p+1}\Psi(n)a_n z^{n-1} = 1 + \frac{2(1-\alpha)}{\mathcal{B}_k(u,v,\alpha)} \frac{\Gamma(2-\eta-p)\Gamma(k+1)}{\Gamma(k+1-\eta-p)} (w(z))^{k-1}. \tag{4.21}$$

By the condition of the theorem, we define the function w by

$$(w(z))^{k-1} = \frac{\mathcal{B}_k(u,v,\alpha)\Gamma(k+1-\eta-p)}{2(1-\alpha)\Gamma(k+1)} \sum_{n=2}^{\infty} (n-p)_{p+1}\Psi(n)a_n z^{n-1}, \tag{4.22}$$

which readily yields $w(0) = 0$. For such a function w, we have

$$\begin{aligned} |w(z)|^{k-1} &\leq \frac{\mathcal{B}_k(u,v,\alpha)\Gamma(k+1-\eta-p)}{2(1-\alpha)\Gamma(k+1)} \sum_{n=2}^{\infty} (n-p)_{p+1}\Psi(n)a_n |z|^{n-1} \\ &\leq |z| \frac{\mathcal{B}_k(u,v,\alpha)\Gamma(k+1-\eta-p)}{2(1-\alpha)\Gamma(k+1)} \Psi(2) \sum_{n=2}^{\infty} (n-p)_{p+1}a_n \\ &= |z| \frac{\mathcal{B}_k(u,v,\alpha)\Gamma(k+1-\eta-p)}{2(1-\alpha)\Gamma(k+1)} \frac{\Gamma(2-p)}{\Gamma(3-\eta-p)} \sum_{n=2}^{\infty} (n-p)_{p+1}a_n \\ &\leq |z| < 1 \end{aligned} \tag{4.23}$$

by means of the hypothesis of the theorem.

Thus the theorem is proved. □

As a special case $p = 0$, we have the following result from Theorem 4.5.

Corollary 4.6. *Let $f \in \widetilde{\mathcal{S}}_\lambda(u, v; \alpha)$ and suppose that*

$$\sum_{n=2}^{\infty} n a_n \leq \frac{2(1-\alpha)\Gamma(k+1)\Gamma(3-\eta)}{\mathcal{B}_k(u, v, \alpha)\Gamma(k+1-\eta)} \quad (k = 2, 3, \ldots). \tag{4.24}$$

If there exists an analytic function w defined by

$$(w(z))^{k-1} = \frac{\mathcal{B}_k(u, v, \alpha)\Gamma(k+1-\eta)}{2(1-\alpha)\Gamma(k+1)} \sum_{n=2}^{\infty} n\Psi(n) a_n z^{n-1} \tag{4.25}$$

with

$$\Psi(n) = \frac{\Gamma(n)}{\Gamma(n+1-\eta)}, \quad (0 \leq \eta < 1, \ n = 2, 3, \ldots), \tag{4.26}$$

then, for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| D_z^\eta f(z) \right|^\mu d\theta \leq \int_0^{2\pi} \left| D_z^\eta f_k(z) \right|^\mu d\theta, \quad (0 \leq \eta < 1). \tag{4.27}$$

Letting $p = 1$ in Theorem 4.5, we have the following.

Corollary 4.7. *Let $f \in \widetilde{\mathcal{S}}_\lambda(u, v; \alpha)$ and suppose that*

$$\sum_{n=2}^{\infty} n(n-1) a_n \leq \frac{2(1-\alpha)\Gamma(k+1)\Gamma(2-\eta)}{\mathcal{B}_k(u, v, \alpha)\Gamma(k-\eta)} \quad (k = 2, 3, \ldots). \tag{4.28}$$

If there exists an analytic function w defined by

$$(w(z))^{k-1} = \frac{\mathcal{B}_k(u, v, \alpha)\Gamma(k-\eta)}{2(1-\alpha)\Gamma(k+1)} \sum_{n=2}^{\infty} n(n-1)\Psi(n) a_n z^{n-1} \tag{4.29}$$

with

$$\Psi(n) = \frac{\Gamma(n-1)}{\Gamma(n-\eta)}, \quad (0 \leq \eta < 1, \ n = 2, 3, \ldots), \tag{4.30}$$

then, for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| D_z^{1+\eta} f(z) \right|^\mu d\theta \leq \int_0^{2\pi} \left| D_z^{1+\eta} f_k(z) \right|^\mu d\theta, \quad (0 \leq \eta < 1). \tag{4.31}$$

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