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Research Article

A Note on $|A|_k$ Summability Factors for Infinite Series

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We obtain sufficient conditions on a nonnegative lower triangular matrix A and a sequence λ_n for the series $\sum a_n\lambda_n/na_{nn}$ to be absolutely summable of order $k \geq 1$ by A.

A weighted mean matrix, denoted by $(\overline{N}, p_n)$, is a lower triangular matrix with entries p_k/P_n, where $\{p_k\}$ is a nonnegative sequence with $p_0 > 0$, and $P_n := \sum_{k=0}^{n} p_k$.

Mishra and Srivastava [1] obtained sufficient conditions on a sequence $\{p_k\}$ and a sequence $\{\lambda_n\}$ for the series $\sum a_n P_n \lambda_n / np_n$ to be absolutely summable by the weighted mean matrix $(\overline{N}, p_n)$. Bor [2] extended this result to absolute summability of order $k \geq 1$. Unfortunately, an incorrect definition of absolute summability was used.

In this note, we establish the corresponding result for a nonnegative triangle, using the correct definition of absolute summability of order $k \geq 1$, (see [3]). As a corollary, we obtain the corrected version of Bor's result.

Let A be an infinite lower triangular matrix. We may associate with A two lower triangular matrices $\overline{A}$ and $\hat{A}$, whose entries are defined by

$$\overline{a}_{nk} = \sum_{i=k}^{n} a_{ni}, \qquad \hat{a}_{nk} = \overline{a}_{nk} - \overline{a}_{n-1,k}, \tag{1}$$

respectively. The motivation for these definitions will become clear as we proceed.

Let A be an infinite matrix. The series $\sum a_k$ is said to be absolutely summable by A, of order $k \geq 1$, written as $|A|_k$, if

$$\sum_{k=0}^{\infty} n^{k-1} \left| \Delta t_{n-1} \right|^{k} < \infty, \tag{2}$$

where Δ is the forward difference operator and t_n denotes the nth term of the matrix transform of the sequence $\{s_n\}$, where $s_n := \sum_{k=0}^{n} a_k$.

Thus

$$t_n = \sum_{k=0}^{n} a_{nk} s_k = \sum_{k=0}^{n} a_{nk} \sum_{\nu=0}^{k} a_\nu = \sum_{\nu=0}^{n} a_\nu \sum_{k=\nu}^{n} a_{nk} = \sum_{\nu=0}^{n} \overline{a}_{n\nu} a_\nu,$$
$$t_n - t_{n-1} = \sum_{\nu=0}^{n} \overline{a}_{n\nu} a_\nu - \sum_{\nu=0}^{n-1} \overline{a}_{n-1,\nu} a_\nu = \sum_{\nu=0}^{n} \hat{a}_{n\nu} a_\nu, \tag{3}$$

since $\overline{a}_{n-1,n} = 0$.

The result to be proved is the following.

Theorem 1. *Let A be a triangle with nonnegative entries satisfying*

(i) $\overline{a}_{n0} = 1$, $n = 0, 1, \ldots$,
(ii) $a_{n-1,\nu} \geq a_{n\nu}$ *for* $n \geq \nu + 1$,
(iii) $na_{nn} \asymp O(1)$,
(iv) $\Delta(1/a_{nn}) = O(1)$,
(v) $\sum_{\nu=0}^{n} a_{\nu\nu} |a_{n,\nu+1}| = O(a_{nn})$.

If $\{X_n\}$ is a positive nondecreasing sequence and the sequences $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy

(vi) $|\Delta\lambda_n| \leq \beta_n$,
(vii) $\lim \beta_n = 0$,
(viii) $|\lambda_n| X_n = O(1)$,
(ix) $\sum_{n=1}^{\infty} nX_n |\Delta\beta_n| < \infty$,
(x) $T_n := \sum_{\nu=1}^{n} (|s_\nu|^k/\nu) = O(X_n)$,

then the series $\sum_{\nu=1}^{\infty} a_n\lambda_n/na_{nn}$ is summable $|A|_k$, $k \geq 1$.

The proof of the theorem requires the following lemma.

Lemma 2 (see Mishra and Srivastava [1]). *Let $\{X_n\}$ be a positive nondecreasing sequence and the sequences $\{\beta_n\}$, $\{\lambda_n\}$ satisfy conditions (vi)–(ix) of Theorem 1. Then*

$$nX_n\beta_n = O(1), \tag{4}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{5}$$

Since $\{X_n\}$ is nondecreasing, $X_n \geq X_0$, which is a positive constant. Hence condition (viii) implies that λ_n is bounded. It also follows from (4) that $\beta_n = O(1/n)$, and hence that $\Delta\lambda_n = O(1/n)$ by condition (iv).

Proof. Let T_n denote the nth term of the A-transform of the series $\sum (a_n\lambda_n)/(na_{nn})$. Then we may write

$$T_n = \sum_{\nu=0}^{n} a_{n\nu} \sum_{i=0}^{\nu} \frac{a_i\lambda_i}{a_{ii} i} = \sum_{i=0}^{m} \frac{a_i\lambda_i}{a_{ii} i} \sum_{\nu=i}^{n} a_{n\nu} = \sum_{i=0}^{n} \overline{a}_{ni} \frac{a_i\lambda_i}{a_{ii} i}. \tag{6}$$

Thus,

$$\begin{aligned} T_n - T_{n-1} &= \sum_{i=0}^{n} \overline{a}_{ni} \frac{a_i\lambda_i}{a_{ii}i} - \sum_{i=0}^{n-1} \overline{a}_{n-1,i} \frac{a_i\lambda_i}{a_{ii}i} = \sum_{i=0}^{n} \left(\overline{a}_{ni} - \overline{a}_{n-1,i}\right) \frac{a_i\lambda_i}{a_{ii}i} = \sum_{i=0}^{n} \widehat{a}_{ni} \frac{a_i\lambda_i}{a_{ii}i} \\ &= \sum_{i=0}^{n} \widehat{a}_{ni} \frac{\lambda_i}{a_{ii}i} (s_i - s_{i-1}) = \sum_{i=0}^{n-1} \widehat{a}_{ni} \frac{\lambda_i}{a_{ii}i} s_i + a_{nn} \frac{\lambda_n}{a_{nn}n} s_n - \sum_{i=0}^{n} \widehat{a}_{ni} \frac{\lambda_i s_{i-1}}{a_{ii}i} \\ &= \sum_{i=0}^{n-1} \widehat{a}_{ni} \frac{\lambda_i}{a_{ii}i} s_i + a_{nn} \frac{\lambda_n}{a_{nn}n} s_n - \sum_{i=0}^{n-1} \widehat{a}_{n,i+1} \frac{\lambda_{i+1} s_i}{(i+1)a_{i+1,i+1}} \\ &= \sum_{i=0}^{n} \left(\widehat{a}_{ni} \frac{\lambda_i}{a_{ii}i} - \widehat{a}_{n,i+1} \frac{\lambda_{i+1}}{(i+1)a_{i+1,i+1}} \right) s_i + a_{nn} \frac{\lambda_n}{n a_{nn}}. \end{aligned} \tag{7}$$

We may write

$$\begin{aligned} \frac{\widehat{a}_{ni}\lambda_i}{ia_{ii}} - \frac{\widehat{a}_{n,i+1}\lambda_{i+1}}{(i+1)a_{i+1,i+1}} &= \frac{\widehat{a}_{ni}\lambda_i}{ia_{ii}} - \frac{\widehat{a}_{n,i+1}\lambda_{i+1}}{(i+1)a_{i+1,i+1}} + \frac{\widehat{a}_{n,i+1}\lambda_i}{(i+1)a_{i+1,i+1}} - \frac{\widehat{a}_{n,i+1}\lambda_i}{(i+1)a_{i+1,i+1}} \\ &= \Delta_i \left(\frac{\widehat{a}_{ni}}{ia_{ii}} \right) \lambda_i + \frac{\widehat{a}_{n,i+1}}{(i+1)a_{i+1,i+1}} \Delta(\lambda_i). \end{aligned} \tag{8}$$

Also we may write

$$\begin{aligned} \Delta_i \left(\frac{\widehat{a}_{ni}}{ia_{ii}} \right) \lambda_i &= \frac{\widehat{a}_{ni}}{ia_{ii}} \lambda_i - \frac{\widehat{a}_{n,i+1}}{(i+1)a_{i+1,i+1}} \lambda_i - \frac{\widehat{a}_{n,i+1}}{ia_{ii}} \lambda_i + \frac{\widehat{a}_{n,i+1}}{ia_{ii}} \lambda_i \\ &= \frac{\Delta_i(\widehat{a}_{ni})\lambda_i}{ia_{ii}} + a_{n,i+1}\lambda_i \left(\frac{1}{ia_{ii}} - \frac{1}{(i+1)a_{i+1,i+1}} \right). \end{aligned} \tag{9}$$

Hence,

$$\begin{aligned} T_n - T_{n-1} = \sum_{i=0}^{n-1} \frac{\Delta_i(\widehat{a}_{ni})}{ia_{ii}} \lambda_i s_i + \sum_{i=0}^{n-1} \widehat{a}_{n,i+1} \lambda_i \left(\frac{1}{ia_{ii}} - \frac{1}{(i+1)a_{i+1,i+1}} \right) s_i \\ + \sum_{i=0}^{n-1} \frac{\widehat{a}_{n,i+1}}{(i+1)a_{i+1,i+1}} \Delta_i(\lambda_i) s_i + \frac{\lambda_n}{n} s_n \\ = T_{n1} + T_{n2} + T_{n3} + T_{n4}, \quad \text{say}. \end{aligned} \tag{10}$$

To finish the proof of the theorem, it will be sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} \left| T_{nr} \right|^k < \infty, \quad \text{for } r = 1,2,3,4. \tag{11}$$

Using Hölder's inequality and (iii),

$$
\begin{aligned}
I_1 &= \sum_{n=1}^{m+1} n^{k-1} \left| T_{n1} \right|^k \leq \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} \left| \frac{\Delta_i(\hat{a}_{ni})}{i a_{ii}} \lambda_i s_i \right| \right)^k \\
&= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} \left| \Delta_i(\hat{a}_{ni}) \lambda_i s_i \right| \right)^k \\
&= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} \left| \Delta_i(\hat{a}_{ni}) \right| \left| \lambda_i \right|^k \left| s_i \right|^k \right) \left(\sum_{i=0}^{n-1} \left| \Delta_i(\hat{a}_{ni}) \right| \right)^{k-1}.
\end{aligned}
\tag{12}
$$

But using (ii),

$$
\Delta_i(\hat{a}_{ni}) = \hat{a}_{ni} - \hat{a}_{n,i+1} = \bar{a}_{ni} - \bar{a}_{n-1,i} - \bar{a}_{n,i+1} + \bar{a}_{n-1,i+1} = a_{ni} - a_{n-1,i} \leq 0. \tag{13}
$$

Thus using (i),

$$
\sum_{i=0}^{n-1} \left| \Delta_i(\hat{a}_{ni}) \right| = \sum_{i=0}^{n-1} \left| a_{n-1,i} - a_{ni} \right| = 1 - 1 + a_{nn} = a_{nn}. \tag{14}
$$

From (viii), it follows that $\lambda_n = O(1)$. Using (iii), (vi), (x), and property (5) of Lemma 2,

$$
\begin{aligned}
I_1 &= O(1) \sum_{n=1}^{m+1} (n a_{nn})^{k-1} \sum_{i=0}^{n-1} \left| \lambda_i \right|^k \left| s_i \right|^k \left| \Delta_i(\hat{a}_{ni}) \right| \\
&= O(1) \sum_{n=1}^{m+1} (n a_{nn})^{k-1} \left(\sum_{i=0}^{n-1} \left| \lambda_i \right|^{k-1} \left| \lambda_i \right| \left| \Delta_i(\hat{a}_{ni}) \right| \left| s_i \right|^k \right) \\
&= O(1) \sum_{i=0}^{m} \left| \lambda_i \right| \left| s_i \right|^k \sum_{n=i+1}^{m+1} (n a_{nn})^{k-1} \left| \Delta_i(\hat{a}_{ni}) \right| \\
&= O(1) \sum_{i=0}^{m} \left| \lambda_i \right| \left| s_i \right|^k a_{ii} = \left| \lambda_0 \right| \left| s_0 \right|^k a_{00} + O(1) \sum_{i=1}^{m} \frac{\left| \lambda_i \right| \left| s_i \right|^k}{i} \\
&= O(1) + O(1) \sum_{i=1}^{m} \left| \lambda_i \right| \left[\sum_{r=1}^{i} \frac{\left| s_r \right|^k}{r} - \sum_{r=1}^{i-1} \frac{\left| s_r \right|^k}{r} \right] \\
&= O(1) \left[\sum_{i=1}^{m} \left| \lambda_i \right| \sum_{r=1}^{i} \frac{\left| s_r \right|^k}{r} - \sum_{j=0}^{m-1} \left| \lambda_{j+1} \right| \sum_{r=1}^{j} \frac{\left| s_r \right|^k}{r} \right] \\
&= O(1) \sum_{i=1}^{m-1} \Delta(\left| \lambda_i \right|) \sum_{r=1}^{i} \frac{1}{r} \left| s_r \right|^k + O(1) \left| \lambda_m \right| \sum_{i=1}^{m} \frac{\left| s_i \right|^k}{i}
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{i=1}^{m-1}\Delta(|\lambda_i|)X_i + O(1)|\lambda_m|X_m \\
&= O(1)\sum_{i=1}^{m}\beta_i X_i + O(1)|\lambda_m|X_m = O(1), \\
I_2 &= \sum_{n=1}^{m+1} n^{k-1}|T_{n2}|^k = \sum_{n=1}^{m+1} n^{k-1}\left|\sum_{i=0}^{n-1}\hat{a}_{n,i+1}\lambda_i\Delta\left(\frac{1}{ia_{ii}}\right)s_i\right|^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\left\{\sum_{i=0}^{n-1}|\hat{a}_{n,i+1}||\lambda_i|\Delta\left(\frac{1}{ia_{ii}}\right)|s_i|\right\}^k.
\end{aligned}
\tag{15}
$$

Now

$$
\begin{aligned}
\Delta\left(\frac{1}{ia_{ii}}\right) &= \frac{1}{ia_{ii}} - \frac{1}{(i+1)a_{i+1,i+1}} \\
&= \frac{1}{ia_{ii}} - \frac{1}{(i+1)a_{i+1,i+1}} + \frac{1}{(i+1)a_{ii}} - \frac{1}{(i+1)a_{ii}} \\
&= \frac{1}{(i+1)}\left(\frac{1}{a_{ii}} - \frac{1}{a_{i+1,i+1}}\right) + \frac{1}{a_{ii}}\left(\frac{1}{i} - \frac{1}{i+1}\right) \\
&= \frac{1}{(i+1)}\left[\Delta\left(\frac{1}{a_{ii}}\right) + \frac{1}{ia_{ii}}\right].
\end{aligned}
\tag{16}
$$

Thus using (iv) and (ii),

$$
\begin{aligned}
\left|\Delta\left(\frac{1}{ia_{ii}}\right)\right| &= \left|\frac{1}{i+1}\left[\Delta\left(\frac{1}{a_{ii}}\right) + \frac{1}{ia_{ii}}\right]\right| \le \frac{1}{i+1}\left\{\frac{|a_{i+1,i+1}-a_{ii}|}{|a_{ii}a_{i+1,i+1}|} + \frac{1}{ia_{ii}}\right\} \\
&= \frac{1}{i+1}[O(1)+O(1)].
\end{aligned}
\tag{17}
$$

Hence, using Hölder's inequality, (v) and (iii),

$$
\begin{aligned}
I_2 &= O(1)\sum_{n=1}^{m+1} n^{k-1}\left\{\sum_{i=0}^{n-1}|\hat{a}_{n,i+1}||\lambda_i|\frac{1}{i+1}|s_i|\right\}^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\left\{\sum_{i=0}^{n-1}|\hat{a}_{n,i+1}|a_{ii}|\lambda_i||s_i|\right\}^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{i=0}^{n-1}|\hat{a}_{n,i+1}|a_{ii}|\lambda_i|^k|s_i|^k\right)\left(\sum_{i=0}^{n-1}a_{ii}|\hat{a}_{n,i+1}|\right)^{k-1} \\
&= O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\sum_{i=0}^{n-1}|\hat{a}_{n,i+1}|a_{ii}|\lambda_i|^k|s_i|^k
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{i=0}^{m} |\lambda_i|^k |s_i|^k a_{ii} \sum_{n=i+1}^{m+1} (na_{nn})^{k-1} |\hat{a}_{n,i+1}| \\
&= O(1)\sum_{i=0}^{m} |\lambda_i|^k |s_i|^k a_{ii} \sum_{n=i+1}^{m+1} |\hat{a}_{n,i+1}|.
\end{aligned}
\tag{18}
$$

From [4],

$$
\sum_{n=i+1}^{m+1} |\hat{a}_{n,i+1}| \le 1. \tag{19}
$$

Hence,

$$
I_2 = O(1)\sum_{i=1}^{m} |\lambda_i|^k |s_i|^k a_{ii} = O(1)\sum_{i=1}^{m} |\lambda_i| |\lambda_i|^{k-1} |s_i|^k \frac{1}{i} = \sum_{i=1}^{m} |\lambda_i| \frac{|s_i|^k}{i} = O(1), \tag{20}
$$

as in the proof of I_1.

Using (iii), Hölder's inequality, and (v),

$$
\begin{aligned}
I_3 &= \sum_{n=1}^{m+1} n^{k-1} |T_{n3}|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{i=0}^{n-1} \frac{\hat{a}_{n,i+1}(\Delta\lambda_i)s_i}{(i+1)a_{i+1,i+1}} \right|^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\Delta\lambda_i| |s_i| \right)^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=0}^{n-1} \frac{a_{ii}}{a_{ii}} |\hat{a}_{n,i+1}| |\Delta\lambda_i| |s_i| \right\}^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=0}^{n-1} a_{ii} \frac{|\hat{a}_{n,i+1}|}{a_{ii}^k} |\Delta\lambda_i|^k |s_i|^k \right\} \left\{ \sum_{i=0}^{n-1} a_{ii} |\hat{a}_{n,i+1}| \right\}^{k-1} \\
&= O(1)\sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=0}^{n-1} a_{ii} \frac{|\hat{a}_{n,i+1}|}{a_{ii}^k} |\Delta\lambda_i|^k |s_i|^k \\
&= O(1)\sum_{n=1}^{m+1} \sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\Delta\lambda_i|^k |s_i|^k \frac{1}{a_{ii}^k} a_{ii} \\
&= O(1)\sum_{i=0}^{m} \frac{a_{ii}}{a_{ii}^k} |\Delta\lambda_i|^k |s_i|^k \sum_{n=i+1}^{m+1} |\hat{a}_{n,i+1}| \\
&= O(1)\sum_{i=0}^{m} \left(\frac{|\Delta\lambda_i|}{a_{ii}} \right)^{k-1} |\Delta\lambda_i| |s_i|^k \\
&= O(1)\sum_{i=0}^{m} |\Delta\lambda_i| |s_i|^k = O(1)\sum_{i=0}^{m} |s_i|^k \beta_i.
\end{aligned}
\tag{21}
$$

Since $|s_i|^k = i(T_i - T_{i-1})$ by (x), we have

$$I_3 = O(1)\sum_{i=1}^{m} i(T_i - T_{i-1})\beta_i. \tag{22}$$

Using Abel's transformation, (vi), and (5),

$$\begin{aligned} I_3 &= O(1)\sum_{i=1}^{m-1} T_i\Delta(i\beta_i) + O(1)mT_n\beta_n \\ &= O(1)\sum_{i=1}^{m-1} i|\Delta\beta_i|X_i + O(1)\sum_{i=1}^{m-1} X_i\beta_i + O(1)mX_n\beta_n = O(1). \end{aligned} \tag{23}$$

Using (viii) and (x),

$$\begin{aligned} I_4 &= \sum_{n=1}^{m+1} n^{k-1}|T_{n4}|^k = \sum_{n=1}^{m+1} n^{k-1}\left|\frac{s_n\lambda_n}{n}\right|^k = \sum_{n=1}^{m+1} |s_n|^k|\lambda_n|^k\frac{1}{n} \\ &= \sum_{n=1}^{m+1} \frac{|s_n|^k}{n}|\lambda_n||\lambda_n|^{k-1} = O(1), \end{aligned} \tag{24}$$

as in the proof of I_1. □

Corollary 3. *Let $\{p_n\}$ be a positive sequence such that $P_n = \sum_{k=0}^{n} p_k \to \infty$ and satisfies*
(i) $np_n \asymp O(P_n)$;
(ii) $\Delta(P_n/p_n) = O(1)$.
If $\{X_n\}$ is a positive nondecreasing sequence and the sequences $\{\lambda_n\}$ and $\{\beta_n\}$ are such that
(iii) $|\Delta\lambda_n| \le \beta_n$,
(iv) $\beta_n \to 0$ *as* $n \to \infty$,
(v) $|\lambda_n|X_n = O(1)$ *as* $n \to \infty$,
(vi) $\sum_{n=1}^{\infty} nX_n|\Delta\beta_n| < \infty$,
(vii) $T_n = \sum_{\nu=1}^{n} |s_\nu|^k/\nu = O(X_n)$,
then the series $\sum(a_nP_n\lambda_n)/(np_n)$ is summable $|\overline{N}, p_n|_k$, $k \ge 1$.

Proof. Conditions (iii)–(vii) of Corollary 3 are, respectively, conditions (vi)–(x) of Theorem 1 .

Conditions (i), (ii), and (v) of Theorem 1 are automatically satisfied for any weighted mean method. Conditions (iii) and (iv) of Theorem 1 become, respectively, conditions (i) and (ii) of Corollary 3. □

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