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Research Article

Convergence for Hyperbolic Singular Perturbation of Integrodifferential Equations

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By virtue of an operator-theoretical approach, we deal with hyperbolic singular perturbation problems for integrodifferential equations. New convergence theorems for such singular perturbation problems are obtained, which generalize some previous results by Fattorini (1987) and Liu (1993).

1. Introduction

Let A and B be linear unbounded operators in a Banach space X, let $K(t)$ be a linear bounded operator for each $t \geq 0$ in X, and let $f(t;\varepsilon)$ and $f(t)$ be X-valued functions. We study the convergence of derivatives of solutions of

$$\begin{gathered} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) = (\varepsilon^2 A + B)u(t;\varepsilon) + \int_0^t K(t-s)(\varepsilon^2 A + B)u(s;\varepsilon)ds + f(t;\varepsilon), \quad t \geq 0, \\ u(0;\varepsilon) = u_0(\varepsilon), \qquad u'(0;\varepsilon) = u_1(\varepsilon), \end{gathered} \tag{1.1}$$

to derivatives of solutions of

$$\begin{gathered} w'(t) = Bw(t) + \int_0^t K(t-s)Bw(s)ds + f(t), \quad t \geq 0, \\ w(0) = w_0, \end{gathered} \tag{1.2}$$

as $\varepsilon \to 0$.

The notion of *hyperbolic singular perturbation problem* comes from the work of Fattorini [1], where the inhomogeneous hyperbolic singular perturbation problem

$$\begin{gathered} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) = (\varepsilon^2 A + B)u(t;\varepsilon) + f(t;\varepsilon), \quad t \ge 0, \\ u(0;\varepsilon) = u_0(\varepsilon), \qquad u'(0;\varepsilon) = u_1(\varepsilon), \end{gathered} \tag{1.3}$$

arising from problems of traffic flow, is studied. It was shown in [1], under some conditions on A, B, and f, that as $\varepsilon \to 0$, if $u_0(\varepsilon) \to w_0$, $u_1(\varepsilon) \to Bw_0$, $Bu_0(\varepsilon) \to Bw_0$, $f(\cdot;\varepsilon) \to f(\cdot)$, and $f'(\cdot;\varepsilon) \to f'(\cdot)$, then $u(t;\varepsilon) \to w(t)$ and $u'(t;\varepsilon) \to w'(t)$ uniformly on compact subsets of $t \ge 0$, where $u(t;\varepsilon)$ is the solution of the Cauchy problem (1.3) and w is the solution of the Cauchy problem

$$\begin{gathered} w'(t) = Bw(t) + f(t), \quad t \ge 0, \\ w(0) = w_0. \end{gathered} \tag{1.4}$$

This generalizes his earlier result in [3] about the parabolic singular perturbation problem

$$\begin{gathered} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) = Au(t;\varepsilon) + f(t;\varepsilon), \quad t \ge 0, \\ u(0;\varepsilon) = u_0(\varepsilon), \qquad u'(0;\varepsilon) = u_1(\varepsilon), \\ w'(t) = Aw(t) + f(t), \quad t \ge 0, \\ w(0) = w_0, \end{gathered} \tag{1.5}$$

where the same result mentioned above holds.

Stimulated by the work of Fattorini [1] and some models in physics, such as viscoelasticity, we studied in [4] the convergence of solutions of the problem (1.1) to solutions of the Cauchy problem (1.2). We proved in [4], with some suitable assumptions, that as $\varepsilon \to 0$, if $u_0(\varepsilon) \to w_0$, $\varepsilon^2 u_1(\varepsilon) \to 0$, and $f(\cdot;\varepsilon) \to f(\cdot)$, then $u(t;\varepsilon) \to w(t)$ uniformly on compact subsets of $t \ge 0$ for the solution $u(t;\varepsilon)$ of (1.1) and the solution $w(t)$ of (1.2).

In this paper, we will continue these studies and investigate the convergence of derivatives of solutions for the problem (1.1) and the problem (1.2). Under those conditions of Fattorini [1] and some conditions on $K(\cdot)$, we will prove that we also have $u'(t;\varepsilon) \to w'(t)$ uniformly on compact subsets of $t \ge 0$ for the problem (1.1) and the problem (1.2). This result includes the corresponding result [1, Theorem 3.4] as a special case for equations without the integral term (i.e., $K(\cdot) \equiv 0$). This result also covers [2, Theorem 2.1].

For references in this area and related topics, we refer the reader to, for example, the monographs [3, 5–7] and the papers [1, 2, 4, 8–11], and the references therein.

2. Preliminaries

Here, we follow [1, 4]. Throughout this paper, $\varepsilon > 0$, X is a Banach space, $\mathbf{L}(X)$ denotes the space of all continuous linear operators from X to itself, and $D(A)$ stands for the domain of an operator A.

We recall some basic assumptions and results of Fattorini [1] that will be used in this work (see [1] for details).

(A1) $\varepsilon^2 A + B$ is the generator of a strongly continuous cosine function on X. This is equivalent to the following:

(1) $D(\varepsilon^2 A + B) = D(A) \cap D(B)$ is dense in X;

(2) the homogeneous version of (1.3) ($f(\cdot;\varepsilon) = 0$) has a solution for $u_0(\varepsilon)$, $u_1(\varepsilon)$ in a dense subspace D of X;

(3) the solutions of the homogeneous version of (1.3) depend continuously on their initial data uniformly on compacts of $t \geq 0$

(cf. [3, 1]; see also [12, 13]).

With (A1), one can define two propagators of the homogeneous version of (1.3) by

$$Q(t;\varepsilon)u := u(t;\varepsilon), \quad G(t;\varepsilon)u := v(t;\varepsilon), \quad u \in D,\ t \geq 0, \tag{2.1}$$

where $u(t;\varepsilon)$ (resp., $v(t;\varepsilon)$) is the solution of the homogeneous version of (1.3) with $u(0;\varepsilon) = u$, $u'(0;\varepsilon) = 0$ (resp., with $v(0;\varepsilon) = 0$, $v'(0;\varepsilon) = \varepsilon^{-2}u$); these propagators can be extended to all of X as bounded operators, which we denote by the same symbol; and these operator-valued functions are strongly continuous in $t \geq 0$. Moreover, it follows from [1] that the solutions of (1.3) are given by

$$u(t;\varepsilon) = Q(t;\varepsilon)u_0(\varepsilon) + G(t;\varepsilon)[\varepsilon^2 u_1(\varepsilon)] \int_0^t G(t-s;\varepsilon) f(s;\varepsilon)\,ds, \tag{2.2}$$

and that for $u \in X$,

$$\varepsilon^2 G'(t;\varepsilon)u = Q(t;\varepsilon)u - G(t;\varepsilon)u. \tag{2.3}$$

Following Fattorini [1], we also make the following assumptions.

(A2) There exist constants C, ω, ε_0 independent of t and ε such that for $t \geq 0$ and $0 \leq \varepsilon \leq \varepsilon_0$,

$$||Q(t;\varepsilon)||, ||G(t;\varepsilon)|| \leq Ce^{\omega t}. \tag{2.4}$$

(A3) The restriction B_0 of B to $D(A)$ is closable and there is a ν such that $(\lambda - B_0)D(B_0)$ is dense in X for $\operatorname{Re}\lambda > \nu$.

Theorems 3.2 and 8.3 in [1] tell us that under these assumptions, the closure $\overline{B_0}$ of B_0 generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ satisfying

$$||S(t)|| \leq Me^{\mu t}, \quad t \geq 0 \tag{2.5}$$

for constants M and μ; and

$$\lim_{\varepsilon \to 0} Q(t,\varepsilon)u = S(t)u, \quad u \in X, \tag{2.6}$$

$$\lim_{\varepsilon \to 0} [G(t,\varepsilon) + e^{-t/\varepsilon^2} I]u = S(t)u, \quad u \in X, \tag{2.7}$$

uniformly on compact subset of $t \geq 0$, where I is the identity operator.

To link the semigroup $\{S(t)\}_{t\geq 0}$ and the problem (1.4), we assume

(A4) $\overline{B_0} = B$.

Therefore, under the assumption (A4), the solutions of (1.4) are given by

$$w(t) = S(t)w_0 + \int_0^t S(t-s)f(s)ds, \quad w_0 \in D(\overline{B_0}). \tag{2.8}$$

The following assumption is made especially for (1.1) and (1.2).

(A5) $\{K(t)\}_{t\geq 0} \subset \mathbf{L}(X)$. For each $x \in X$, $K(\cdot)x \in W^{2,1}_{\text{loc}}([0,\infty);X)$. $\|K''(\cdot)\|$ is locally bounded on $[0,\infty)$. Here K'' is the strong derivative.

Definition 2.1. An X-valued function $u(\cdot;\varepsilon)$ on $[0,\infty)$ is called a solution of the problem (1.1) if $u(\cdot;\varepsilon)$ is twice continuously differentiable, $u(t;\varepsilon) \in D(A) \cap D(B)$ for $t \geq 0$ and the problem (1.1) is satisfied. Similarly, an X-valued function $w(\cdot)$ on $[0,\infty)$ is called a solution of the problem (1.2) if $w(\cdot)$ is continuously differentiable, $w(t) \in D(B)$ for $t \geq 0$ and the problem (1.2) is satisfied.

Let $u(t;\varepsilon)$ be a solution of (1.1), and as in [1, 3, 10], we write

$$v\left(\frac{t}{\varepsilon}\right) := e^{t/\varepsilon^2}u(t;\varepsilon), \quad \widetilde{K}(t;\varepsilon) := \varepsilon K(\varepsilon t)e^{t/2\varepsilon}, \quad \widetilde{f}(t;\varepsilon) := f(\varepsilon t;\varepsilon)e^{t/2\varepsilon}, \quad t \geq 0. \tag{2.9}$$

Then, by (1.1) we have

$$\begin{gathered} v''(t) = \left(\varepsilon^2 A + B + \frac{1}{4\varepsilon^2}\right)v(t) + \int_0^t \widetilde{K}(t-s;\varepsilon)(\varepsilon^2 A + B)v(s)ds + \widetilde{f}(t;\varepsilon), \\ v(0;\varepsilon) = u_0(\varepsilon), v'(0;\varepsilon) = \frac{1}{2\varepsilon}u_0(\varepsilon) + \varepsilon u_1(\varepsilon). \end{gathered} \tag{2.10}$$

Since the singular perturbations is what we are concerned in this paper, we assume that the problem (1.1) (i.e., the problem (2.10) for every $\varepsilon > 0$ and the problem (1.2) have unique solutions, respectively. For the existence and uniqueness theorems for solutions of the problem (2.10) and the problem (1.2), we refer the reader to [14–16].

3. Convergence theorems

Now, we state and prove our main result of the paper concerning the convergence of derivatives of solutions for the problem (1.1) and the problem (1.2).

Theorem 3.1. *Let $T > 0$ be fixed, (A1)–(A5) hold, and*

(A6) $u_0(\varepsilon) \to w_0$, $u_1(\varepsilon) \to Bw_0$, $Bu_0(\varepsilon) \to Bw_0$, as $\varepsilon \to 0$,

(A7) $f(\cdot;\varepsilon) \to f(\cdot)$ and $f'(\cdot;\varepsilon) \to f'(\cdot)$ in $L^1([0,T];X)$; $f(0;\varepsilon) \to f(0) = 0$ in X, as $\varepsilon \to 0$.

Let $u(t;\varepsilon)$ and $w(t)$ be the solution of the problem (1.1) and the problem (1.2) on $[0,T]$, respectively. Then,

$$u'(t;\varepsilon) \longrightarrow w'(t) \quad \textit{uniformly for } t \in [0,T] \textit{ as } \varepsilon \longrightarrow 0. \tag{3.1}$$

Proof. Using (A5) and a standard fixed point argument, one can deduce that there exists an $\mathbf{L}(X)$-valued function $F(\cdot)$ such that

$$\begin{gathered} F(t)+K(t)+\int_0^t K(t-s)F(s)ds=0, \\ F(\cdot)x\in W^{2,1}_{\text{loc}}([0,\infty);X) \quad \text{for each } x\in X, \\ \|F'(\cdot)\| \text{ and } \|F''(\cdot)\| \text{ are locally bounded on } [0,\infty), \end{gathered} \tag{3.2}$$

where F' and F'' are strong derivatives (cf. [17, 18]).

Let $\delta(\cdot)$ be the Dirac measure. Then,

$$(\delta+F)*(\delta+K)=\delta. \tag{3.3}$$

Since $u(t;\varepsilon)$ satisfies the problem (1.1), we get

$$\varepsilon^2u''(t;\varepsilon)+u'(t;\varepsilon)=(\delta+K)*(\varepsilon^2A+B)u(t;\varepsilon)+f(t;\varepsilon), \tag{3.4}$$

then by (3.3), we obtain

$$(\delta+F)*\left[\varepsilon^2u''(t;\varepsilon)+u'(t;\varepsilon)\right]=(\varepsilon^2A+B)u(t;\varepsilon)+(\delta+F)*f(t;\varepsilon). \tag{3.5}$$

This means that $u(t;\varepsilon)$ satisfies

$$\begin{gathered} \varepsilon^2u''(t;\varepsilon)+u'(t;\varepsilon)=(\varepsilon^2A+B)u(t;\varepsilon)+\hat{f}(t;\varepsilon), \\ u(0;\varepsilon)=u_0(\varepsilon), \qquad u'(0;\varepsilon)=u_1(\varepsilon), \end{gathered} \tag{3.6}$$

where

$$\hat{f}(t;\varepsilon)=(\delta+F)*f(t;\varepsilon)-F*\left[\varepsilon^2u''(t;\varepsilon)+u'(t;\varepsilon)\right]. \tag{3.7}$$

Similarly, we have

$$\begin{gathered} w'(t)=Bw(t)+\hat{f}(t), \quad t\geq 0, \\ w(0)=w_0, \end{gathered} \tag{3.8}$$

where

$$\hat{f}(t)=(\delta+F)*f(t)-F*w'(t). \tag{3.9}$$

By linearity, we view the solution of the problem (3.6) (resp., the problem (3.8)) as the addition of two solutions such that the first one, $\mathbf{u}_1$ (resp., $\mathbf{w}_1$), is with $\hat{f}(t;\varepsilon)$ (resp., $\hat{f}(t)$) being zero and the second one, $\mathbf{u}_2$ (resp., $\mathbf{w}_2$), is with zero initial data, so that we have

$$\mathbf{u}_2(t;\varepsilon)=\int_0^t G(t-s;\varepsilon)\hat{f}(s;\varepsilon)ds, \qquad \mathbf{w}_2(t)=\int_0^t S(t-s)\hat{f}(s)ds. \tag{3.10}$$

For the first solutions $\mathbf{u}_1$ and $\mathbf{w}_1$ for the problem (3.6) and the problem (3.8), it was shown in Fattorini [1], with these conditions, that $\mathbf{u}_1'(t;\varepsilon) - \mathbf{w}_1'(t) \to 0$ in X uniformly for $t \in [0,T]$ as $\varepsilon \to 0$. Therefore,

$$\begin{aligned} \|u'(t;\varepsilon) - w'(t)\| &\leq \|\mathbf{u}_1'(t;\varepsilon) - \mathbf{w}_1'(t)\| + \|\mathbf{u}_2'(t;\varepsilon) - \mathbf{w}_2'(t)\| \\ &\leq 0(\varepsilon,[0,T]) + \|\mathbf{u}_2'(t;\varepsilon) - \mathbf{w}_2'(t)\|, \end{aligned} \tag{3.11}$$

where $0(\varepsilon,[0,T])$ satisfies

$$0(\varepsilon,[0,T]) \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0, \text{ uniformly for } t \in [0,T]. \tag{3.12}$$

As $G(0;\varepsilon) = 0$, $S(0) = \text{Identity}$, and $f(0) = 0$, we obtain

$$\begin{aligned} \mathbf{u}_2'(t;\varepsilon) - \mathbf{w}_2'(t) &= \int_0^t G'(t-s;\varepsilon)\hat{f}(s;\varepsilon) - \int_0^t S'(t-s)\hat{f}(s)ds - \hat{f}(t) \\ &= \int_0^t G'(t-s;\varepsilon)[\hat{f}(s;\varepsilon) - \hat{f}(s)]ds \\ &\quad + \int_0^t [G'(t-s;\varepsilon) - S'(t-s)]\hat{f}(s)ds - \hat{f}(t) \\ &= \int_0^t G'(t-s;\varepsilon)[\hat{f}(s;\varepsilon) - \hat{f}(s)]ds + \int_0^t [G(t-s;\varepsilon) - S(t-s)]\hat{f}'(s)ds \\ &\quad + [G(t;\varepsilon) - S(t)]f(0) = \int_0^t G'(t-s;\varepsilon)[\hat{f}(s;\varepsilon) - \hat{f}(s)]ds \\ &\quad + \int_0^t [G(t-s;\varepsilon) - S(t-s)]\hat{f}'(s)ds. \end{aligned} \tag{3.13}$$

Note that

$$\begin{aligned} \hat{f}'(t) = f'(t) + F(0)f(t) + \int_0^t F'(t-s)f(s)ds \\ - F(0)w'(t) + F'(t)w_0 - F'(0)w(t) - \int_0^t F''(t-s)w(s)ds, \end{aligned} \tag{3.14}$$

so, from (2.7), we obtain (similar to [4])

$$\begin{aligned} &\left\| \int_0^t [G(t-s;\varepsilon) - S(t-s)]\hat{f}'(s)ds \right\| \\ &\quad \leq \left\| \int_0^t [G(t-s;\varepsilon) + e^{-(t-s)/\varepsilon^2}I - S(t-s)]\hat{f}'(s)ds\| + \| = \int_0^t e^{-(t-s)/\varepsilon^2}\hat{f}'(s)ds \right\| \\ &\quad = 0(\varepsilon,[0,T]). \end{aligned} \tag{3.15}$$

Next, we have

$$\begin{aligned}\int_0^t G'(t-s;\varepsilon)[\hat{f}(s;\varepsilon)-\hat{f}(s)]ds &= \int_0^t G'(t-s;\varepsilon)[\hat{f}(s;\varepsilon)-\hat{f}(s)+\varepsilon^2F(0)u'(s;\varepsilon)]ds \\ &\quad -\int_0^t G'(t-s;\varepsilon)\varepsilon^2F(0)u'(s;\varepsilon)ds,\end{aligned} \tag{3.16}$$

$$\begin{aligned}\int_0^t G'(t-s;\varepsilon)\varepsilon^2F(0)u'(s;\varepsilon)ds &= \int_0^t G'(t-s;\varepsilon)\varepsilon^2F(0)[u'(s;\varepsilon)-w'(s)]ds \\ &\quad +\int_0^t G'(t-s;\varepsilon)\varepsilon^2F(0)w'(s)ds.\end{aligned} \tag{3.17}$$

From (2.3), (2.6), and (2.7), and similar to (3.15), we obtain

$$\begin{aligned}\left\|\int_0^t G'(t-s;\varepsilon)\varepsilon^2F(0)w'(s)ds\right\| &= \left\|\int_0^t [Q(t-s;\varepsilon)-G(t-s;\varepsilon)]F(0)w'(s)ds\right\| \\ &\le \left\|\int_0^t [Q(t-s;\varepsilon)-S(t-s)]F(0)w'(s)ds\right\| \\ &\quad +\left\|\int_0^t [G(t-s;\varepsilon)-S(t-s)]F(0)w'(s)ds\right\| \\ &= 0(\varepsilon,[0,T]),\end{aligned} \tag{3.18}$$

and from (2.3) and (2.4), we obtain

$$\left\|\int_0^t G'(t-s;\varepsilon)\varepsilon^2F(0)[u'(s;\varepsilon)-w'(s)]ds\right\| \le (\text{const})\int_0^t \|u'(s;\varepsilon)-w'(s)\|ds. \tag{3.19}$$

Therefore, from (3.17)–(3.19), we obtain

$$\left\|\int_0^t G'(t-s;\varepsilon)\varepsilon^2F(0)u'(s;\varepsilon)ds\right\| \le 0(\varepsilon,[0,T])+(\text{const})\int_0^t \|u'(s;\varepsilon)-w'(s)\|ds. \tag{3.20}$$

Next,

$$\begin{aligned}&\int_0^t G'(t-s;\varepsilon)[\hat{f}(s;\varepsilon)-\hat{f}(s)+\varepsilon^2F(0)u'(s;\varepsilon)]ds \\ &\quad = G(t;\varepsilon)[f(0;\varepsilon)-f(0)+\varepsilon^2F(0)u_1(\varepsilon)] \\ &\quad\quad +\int_0^t G(t-s;\varepsilon)[\hat{f}(s;\varepsilon)-\hat{f}(s)+\varepsilon^2F(0)u'(s;\varepsilon)]'ds,\end{aligned} \tag{3.21}$$

and from (2.4), (A6), and (A7),

$$||G(t;\varepsilon)[f(0;\varepsilon)-f(0)+\varepsilon^2F(0)u_1(\varepsilon)]||=0(\varepsilon,[0,T]). \tag{3.22}$$

Moreover,

$$\begin{aligned}
&\int_0^t G(t-s;\varepsilon)[\hat{f}(s;\varepsilon)-\hat{f}(s)+\varepsilon^2F(0)u'(s;\varepsilon)]'ds\\
&\quad=\int_0^t G(t-s;\varepsilon)\bigg\{[f(s;\varepsilon)-f(s)]+\int_0^s F(s-h)[f(h;\varepsilon)-f(h)]dh\\
&\qquad\qquad-\int_0^s F'(s-h)[u(h;\varepsilon)-w(h)]dh\\
&\qquad\qquad-[\varepsilon^2F'(0)+F(0)][u(s;\varepsilon)-w(s)]-\varepsilon^2F'(0)w(s)\\
&\qquad\qquad-\varepsilon^2\int_0^s F''(s-h)[u(h;\varepsilon)-w(h)]dh-\varepsilon^2\int_0^s F''(s-h)w(h)dh\\
&\qquad\qquad+F(s)[u_0(\varepsilon)-w_0]+\varepsilon^2F'(s)u_0(\varepsilon)+\varepsilon^2F(s)u_1(\varepsilon)\bigg\}'ds\\
&\quad=\int_0^t G(t-s;\varepsilon)\bigg\{[f'(s;\varepsilon)-f'(s)]+F(0)[f(s;\varepsilon)-f(s)]\\
&\qquad\qquad+\int_0^s F'(s-h)[f(h;\varepsilon)-f(h)]dh-F'(0)[u(s;\varepsilon)-w(s)]\\
&\qquad\qquad-\int_0^s F''(s-h)[u(h;\varepsilon)-w(h)]dh\\
&\qquad\qquad-[\varepsilon^2F'(0)+F(0)][u'(s;\varepsilon)-w'(s)]\\
&\qquad\qquad-\varepsilon^2F'(0)w'(s)-\varepsilon^2\int_0^s F''(s-h)[u'(h;\varepsilon)-w'(h)]dh\\
&\qquad\qquad-\varepsilon^2\int_0^s F''(s-h)w'(h)dh+F'(s)[u_0(\varepsilon)-w_0]\\
&\qquad\qquad+\varepsilon^2F''(s)u_0(\varepsilon)+\varepsilon^2F'(s)u_1(\varepsilon)\bigg\}ds.
\end{aligned} \tag{3.23}$$

Note that it is proved in [4] that $u(t;\varepsilon)\to w(t)$ uniformly for $t\in[0,T]$ as $\varepsilon\to 0$, therefore, from (3.23), (A6), and (A7), we obtain

$$\begin{aligned}
&\left|\left|\int_0^t G(t-s;\varepsilon)[\hat{f}(s;\varepsilon)-\hat{f}(s)+\varepsilon^2F(0)u'(s;\varepsilon)]'ds\right|\right|\\
&\qquad\le 0(\varepsilon,[0,T])+(\text{const})\int_0^t||u'(s;\varepsilon)-w'(s)||ds,\quad t\in[0,T].
\end{aligned} \tag{3.24}$$

Now, from (3.11)–(3.16), (3.20)–(3.22), and (3.24), we obtain

$$||u'(t;\varepsilon)-w'(t)|| \le 0(\varepsilon,[0,T])+(\text{const})\int_0^t ||u'(s;\varepsilon)-w'(s)||ds, \quad t\in[0,T]. \tag{3.25}$$

Therefore, from Gronwall's inequality, we obtain

$$||u'(t;\varepsilon)-w'(t)|| \le 0(\varepsilon,[0,T]), \quad t\in[0,T]. \tag{3.26}$$

This completes the proof. □

Theorem 3.2. *Let $T>0$ be fixed, and let (A1), (A2), (A5), (A6), and (A7) hold. Also, assume that B generates a strongly continuous semigroup on X and $D(A)\cap D(B)$ is a core of B. Let $u(t;\varepsilon)$ and $w(t)$ be the solutions of (1.1) and (1.2) on $[0,T]$, respectively. Then*

$$u'(t;\varepsilon)\longrightarrow w'(t) \quad \textit{uniformly for } t\in[0,T] \ \textit{as } \varepsilon\longrightarrow 0. \tag{3.27}$$

Proof. Since B generates a strongly continuous semigroup on X, and $D(A)\cap D(B)$ is a core of B, we see that (A3) and (A4) hold. Thus, we get the conclusion by Theorem 3.1. □

In the case that the assumption (A4) is not satisfied, then instead of (1.2), we can consider

$$\begin{gathered} w'(t)=\overline{B_0}w(t)+\int_0^t K(t-s)\overline{B_0}w(s)ds+f(t), \quad t\ge 0, \\ w(0)=w_0, \end{gathered} \tag{3.28}$$

whose solution is defined in a way similar to that of (1.2). Now, under the assumption (A3), we know from [1] that $\overline{B_0}$ generates a semigroup $\{S(t)\}_{t\ge 0}$ satisfying (2.5)–(2.7), and the solutions of (3.28) are given by

$$w(t)=S(t)w_0+\int_0^t S(t-s)f(s)ds, \quad w_0\in D(\overline{B_0}). \tag{3.29}$$

That is, we have the same settings as before, thus, the arguments made above for solutions of (1.1) and (1.2) can also be made for solutions of (1.1) and (3.28). Therefore, we have the following.

Theorem 3.3. *Let $T>0$ be fixed, and (A1), (A2), (A3), (A5), (A6), and (A7) hold. Let $u(t;\varepsilon)$ and $w(t)$ be the solutions of (1.1) and (3.28) on $[0,T]$, respectively. Then,*

$$u'(t;\varepsilon)\longrightarrow w'(t) \quad \textit{uniformly for } t\in[0,T] \ \textit{as } \varepsilon\longrightarrow 0. \tag{3.30}$$

Remark 3.4. Clearly, if $K(\cdot)\equiv 0$, then $F(\cdot)\equiv 0$, and hence $\widehat{f}(t;\varepsilon)=f(t;\varepsilon)$, $\widehat{f}(t)=f(t)$. Therefore, when $K(\cdot)\equiv 0$, Theorem 3.3 goes back to [1, Theorem 3.4] for equations without the integral term. Furthermore, it is easy to see that if $A=0$, then $D(A)=X$, so that $B_0=B$. Thus, (A1) implies (A3) and (A4), therefore, Theorems 3.1 and 3.3 cover [2, Theorem 2.1].

Remark 3.5. It is pointed out in [3] (for equations without the integral term) that $f(0) = 0$ is almost necessary to obtain the convergence in derivative at $t = 0$. For equations with the integral term, we also need this condition in [2] and here. If $f(0) \neq 0$, then, from (3.13) and

$$[G(t;\varepsilon) - S(t)]f(0) = [G(t;\varepsilon) + e^{-t/\varepsilon^2}I - S(t)]f(0) - e^{-t/\varepsilon^2}f(0), \tag{3.31}$$

we can obtain the convergence in derivatives for $t > 0$.

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