

Research Article

A Reverse Hardy-Hilbert-Type Inequality

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By estimating the weight coefficient, a reverse Hardy-Hilbert-type inequality is proved. As applications, some equivalent forms and a number of particular cases are obtained.

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1. Introduction

Let $p > 1$, $1/p + 1/q = 1$, $a_n \geq 0$, $b_n \geq 0$, and $0 < \sum_{n=0}^{\infty} a_n^p < \infty$, $0 < \sum_{n=0}^{\infty} b_n^q < \infty$. Then the Hardy-Hilbert inequality is as follows:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{n=0}^{\infty} a_n^p \right)^{1/p} \left(\sum_{n=0}^{\infty} b_n^q \right)^{1/q}, \tag{1.1}$$

where the constant factor $\pi/\sin(\pi p)$ is the best possible [1].

For (1.1), Yang et al. [2–6] gave some strengthened versions and extensions as follows:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \left\{ \sum_{n=0}^{\infty} \left[\pi - \frac{7}{5(\sqrt{n+3})} \right] a_n^2 \sum_{n=0}^{\infty} \left[\pi - \frac{7}{5(\sqrt{n+3})} \right] b_n^2 \right\}^{1/2} \tag{1.2}$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \left\{ \sum_{n=0}^{\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{\ln 2 - C}{(2n+1)^{1+1/p}} \right] a_n^p \right\}^{1/p} \tag{1.3}$$

$$\times \left\{ \sum_{n=0}^{\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{\ln 2 - C}{(2n+1)^{1+1/q}} \right] b_n^q \right\}^{1/q},$$

where $\ln 2 - C = 0.1159315^+$ (C is the Euler constant),

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} < B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \left\{ \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{1-\lambda} a_n^p \right\}^{1/p} \times \left\{ \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{1-\lambda} b_n^q \right\}^{1/q}, \tag{1.4}$$

where the constant $B((p+\lambda-2)/p, (q+\lambda-2)/q)$ is the best possible ($2 - \min\{p, q\} < \lambda \leq 2$),

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{ \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{p-1-\lambda} a_n^p \right\}^{1/p} \times \left\{ \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{q-1-\lambda} b_n^q \right\}^{1/q}, \tag{1.5}$$

where the constant $B(\lambda/p, \lambda/q)$ is the best possible ($0 < \lambda \leq \min\{p, q\}$),

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} < B\left(\frac{(r-2)t+\lambda}{r}, \frac{(s-2)t+\lambda}{s}\right) \left\{ \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{p(1-t+(2t-\lambda)/r)-1} a_n^p \right\}^{1/p} \times \left\{ \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{q(1-t+(2t-\lambda)/s)-1} b_n^q \right\}^{1/q}, \tag{1.6}$$

where the constant $B(((r-2)t+\lambda)/r, ((s-2)t+\lambda)/s)$ is the best possible ($r > 1, 1/r + 1/s = 1, t \in [0, 1], 2 - \min\{r, s\}t < \lambda \leq 2 - \min\{r, s\}t + \min\{r, s\}$).

For the reverse Hardy-Hilbert inequality, recently, Yang [7] gave a reverse form of inequalities (1.4), (1.5), and (1.6) for $\lambda = 2$. The main objective of this paper is to establish an extension of the above Yang’s work for $1.5 < \lambda < 3$, by estimating the weight coefficient.

For this, we need the following expression of the β function $B(p, q)$ (see [8]):

$$B(p, q) = B(q, p) = \int_0^\infty \frac{1}{(1+u)^{p+q}} u^{p-1} du, \quad (p, q > 0) \tag{1.7}$$

and the following inequality [3]:

$$\int_0^\infty f(x) dx + \frac{1}{2} f(0) < \sum_{m=0}^{\infty} f(m) < \int_0^\infty f(x) dx + \frac{1}{2} f(0) - \frac{1}{12} f'(0), \tag{1.8}$$

where $f(x) \in C^3[0, \infty)$, and $\int_0^\infty f(x) dx < \infty, (-1)^n f^{(n)}(x) > 0, f^{(n)}(\infty) = 0 (n = 0, 1, 2, 3)$.

2. Main results

LEMMA 2.1. Let N_0 be the set of nonnegative integers, N the set of positive integers, and \mathbb{R} the set of real numbers. The weight coefficient $\omega_\lambda(n)$ is defined by

$$\omega_\lambda(n) = \sum_{m=0}^{\infty} \frac{1}{(m+n+1)^\lambda}, \quad n \in N_0, 1.5 \leq \lambda < 3. \quad (2.1)$$

Then

$$\frac{2(n+1)^{2-\lambda}}{(\lambda-1)(2n+3-\lambda)} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right] < \omega_\lambda(n) < \frac{2(n+1)^{2-\lambda}}{(\lambda-1)(2n+3-\lambda)}. \quad (2.2)$$

Proof. If $n \in N_0$, let $f(x) = 1/(m+n+1)^\lambda$, $x \in [0, \infty)$. By (1.8), we obtain

$$\begin{aligned} \omega_\lambda(n) &> \int_0^\infty \frac{dx}{(x+n+1)^\lambda} + \frac{1}{2(n+1)^\lambda} = \frac{1}{(\lambda-1)(n+1)^{\lambda-1}} + \frac{1}{2(n+1)^\lambda}, \\ \omega_\lambda(n) &< \int_0^\infty \frac{1}{(x+n+1)^\lambda} dx + \frac{1}{2(n+1)^\lambda} + \frac{\lambda}{12(n+1)^{\lambda+1}} \\ &= \frac{1}{(\lambda-1)(n+1)^{\lambda-1}} + \frac{1}{2(n+1)^\lambda} + \frac{\lambda}{12(n+1)^{\lambda+1}}. \end{aligned} \quad (2.3)$$

Since we find

$$\begin{aligned} &\left[\frac{1}{(\lambda-1)(n+1)^{\lambda-1}} + \frac{1}{2(n+1)^\lambda} \right] [2(n+1)^{\lambda-1} - (\lambda-1)(n+1)^{\lambda-2}] = \frac{2}{\lambda-1} - \frac{\lambda-1}{2(n+1)^2}, \\ &\left[\frac{1}{(\lambda-1)(n+1)^{\lambda-1}} + \frac{1}{2(n+1)^\lambda} + \frac{\lambda}{12(n+1)^{\lambda+1}} \right] [2(n+1)^{\lambda-1} - (\lambda-1)(n+1)^{\lambda-2}] \\ &= \frac{2}{\lambda-1} - \frac{2\lambda-3}{6(n+1)^2} - \frac{\lambda(\lambda-1)}{12(n+1)^3}. \end{aligned} \quad (2.4)$$

Then we obtain

$$\begin{aligned} \frac{1}{(\lambda-1)(n+1)^{\lambda-1}} + \frac{1}{2(n+1)^\lambda} &= \frac{2(n+1)^{2-\lambda}}{(\lambda-1)(2n+3-\lambda)} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right], \\ \frac{1}{(\lambda-1)(n+1)^{\lambda-1}} + \frac{1}{2(n+1)^\lambda} + \frac{\lambda}{12(n+1)^{\lambda+1}} &= \frac{2(n+1)^{2-\lambda}}{(\lambda-1)(2n+3-\lambda)} \\ &\times \left[1 - \frac{(2\lambda-3)(\lambda-1)}{12(n+1)^2} - \frac{\lambda(\lambda-1)^2}{24(n+1)^3} \right]. \end{aligned} \quad (2.5)$$

Since for $1.5 \leq \lambda < 3$, $2(n+1)^{2-\lambda}/(\lambda-1)(2n+3-\lambda) > 0$, $(2\lambda-3)(\lambda-1)/12(n+1)^2 \geq 0$, $\lambda(\lambda-1)^2/24(n+1)^3 > 0$, then we have (2.2). The lemma is proved. \square

THEOREM 2.2. *Let $0 < p < 1$, $1/p + 1/q = 1$, $1.5 \leq \lambda < 3$, and $a_n \geq 0$, $b_n > 0$, such that $0 < \sum_{n=0}^{\infty} ((n+1)^{2-\lambda} a_n^p / (2n+3-\lambda)) < \infty$, $0 < \sum_{n=0}^{\infty} ((n+1)^{2-\lambda} b_n^q / (2n+3-\lambda)) < \infty$. Then*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} > \frac{2}{\lambda-1} \left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right] a_n^p \right\}^{1/p} \times \left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda}}{2n+3-\lambda} b_n^q \right\}^{1/q}. \tag{2.6}$$

Proof. By the reverse Hölder inequality [9], we have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^{\lambda/p}} \cdot \frac{b_n}{(m+n+1)^{\lambda/q}} \\ &\geq \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m^p}{(m+n+1)^\lambda} \right\}^{1/p} \cdot \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{b_n^q}{(m+n+1)^\lambda} \right\}^{1/q} \\ &= \left\{ \sum_{m=0}^{\infty} \omega_\lambda(m) a_m^p \right\}^{1/p} \cdot \left\{ \sum_{n=0}^{\infty} \omega_\lambda(n) b_n^q \right\}^{1/q}. \end{aligned} \tag{2.7}$$

Since $0 < p < 1$ and $q < 0$, then by (2.2), we obtain (2.6). The theorem is proved. □

In Theorem 2.2, for $\lambda = 2$, we have the following corollary.

COROLLARY 2.3. *Let $0 < p < 1$, $1/p + 1/q = 1$, and $a_n \geq 0$, $b_n > 0$, such that $0 < \sum_{n=0}^{\infty} (a_n^p / (2n+1)) < \infty$, $0 < \sum_{n=0}^{\infty} (b_n^q / (2n+1)) < \infty$. Then*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^2} > 2 \left\{ \sum_{n=0}^{\infty} \left[1 - \frac{1}{4(n+1)^2} \right] \frac{a_n^p}{2n+1} \right\}^{1/p} \left\{ \sum_{n=0}^{\infty} \frac{b_n^q}{2n+1} \right\}^{1/q}. \tag{2.8}$$

Remark 2.4. Inequality (2.8) is inequality [7, Inequality (8)]. Hence, inequality (2.6) is an extension of Yang’s inequality [7, Inequality (8)] for $1 < \lambda < 3$.

THEOREM 2.5. *Let $0 < p < 1$, $1/p + 1/q = 1$, $1.5 \leq \lambda < 3$, and $a_n \geq 0$, such that $0 < \sum_{n=0}^{\infty} ((n+1)^{2-\lambda} a_n^p / (2n+3-\lambda)) < \infty$. Then*

$$\sum_{n=0}^{\infty} \left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^\lambda} \right]^p > \left(\frac{2}{\lambda-1} \right)^p \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right] a_n^p. \tag{2.9}$$

Inequalities (2.9) and (2.6) are equivalent.

Proof. Let

$$b_n = \left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^\lambda} \right]^{p-1}, \quad n \in \mathbb{N}_0. \tag{2.10}$$

By (2.6), we have

$$\begin{aligned}
\left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} b_n^q}{2n+3-\lambda} \right\}^p &= \left\{ \sum_{n=0}^{\infty} \left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^\lambda} \right]^p \right\}^p \\
&= \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} \right\}^p \\
&\geq \left(\frac{2}{\lambda-1} \right)^p \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right] a_n^p \\
&\quad \times \left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} b_n^q}{2n+3-\lambda} \right\}^{p-1}.
\end{aligned} \tag{2.11}$$

Then we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} b_n^q}{2n+3-\lambda} &= \sum_{n=0}^{\infty} \left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^\lambda} \right]^p \\
&\geq \left(\frac{2}{\lambda-1} \right)^p \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right] a_n^p.
\end{aligned} \tag{2.12}$$

If $\sum_{n=0}^{\infty} ((n+1)^{2-\lambda} b_n^q / (2n+3-\lambda)) = \infty$, then in view of

$$0 < \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} a_n^p}{2n+3-\lambda} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right] \leq \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} a_n^p}{2n+3-\lambda} < \infty \tag{2.13}$$

and (2.12), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^\lambda} \right]^p &> \left(\frac{2}{\lambda-1} \right)^p \\
&\times \sum_{n=0}^{\infty} \frac{(n+1)^{\lambda-2}}{2n+3-\lambda} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right] a_n^p;
\end{aligned} \tag{2.14}$$

if $0 < \sum_{n=0}^{\infty} ((n+1)^{\lambda-2} b_n^q / (2n+3-\lambda)) < \infty$, then by (2.6), we find

$$\begin{aligned}
\sum_{n=0}^{\infty} \left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^\lambda} \right]^p &> \left(\frac{2}{\lambda-1} \right)^p \\
&\times \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right] a_n^p.
\end{aligned} \tag{2.15}$$

Hence we obtain (2.9).

On the other hand, by the reverse Hölder inequality [9], we have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} &= \left[\sum_{n=0}^{\infty} (n+1)^{(\lambda-2)/q} (2n+3-\lambda)^{1/q} \sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^\lambda} \right] \\ &\quad \times \left[\frac{b_n}{(n+1)^{(\lambda-2)/q} (2n+3-\lambda)^{1/q}} \right] \\ &\geq \left\{ \sum_{n=0}^{\infty} \left[\frac{(n+1)^{2-\lambda}}{2n+3-\lambda} \right]^{1-p} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^\lambda} \right]^p \right\}^{1/p} \\ &\quad \times \left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} b_n^q}{2n+3-\lambda} \right\}^{1/q}. \end{aligned} \quad (2.16)$$

Hence by (2.9), it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} &> \frac{2}{\lambda-1} \left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} a_n^p}{2n+3-\lambda} \left[1 - \frac{(\lambda-1)^2}{4(n+1)^2} \right] \right\}^{1/p} \\ &\quad \times \left\{ \sum_{n=0}^{\infty} \frac{(n+1)^{2-\lambda} b_n^q}{2n+3-\lambda} \right\}^{1/q}. \end{aligned} \quad (2.17)$$

Then, (2.9) and (2.6) are equivalent. The theorem is proved. \square

In (2.9), for $\lambda = 2$, we have the following corollary.

COROLLARY 2.6. *Let $0 < p < 1$, $1/p + 1/q = 1$, $a_n \geq 0$, $0 < \sum_{n=0}^{\infty} (a_n^p / (2n+1)) < \infty$, Then*

$$\sum_{n=0}^{\infty} (2n+1)^{p-1} \left[\sum_{m=0}^{\infty} \frac{a_m}{(m+n+1)^2} \right]^p > 2^p \sum_{n=0}^{\infty} \left[1 - \frac{1}{4(n+1)^2} \right] \frac{a_n^p}{2n+1}. \quad (2.18)$$

Inequalities (2.18) and (2.8) are equivalent.

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