We establish Schur-convexities of two types of one-parameter mean values in \( n \) variables. As applications, Schur-convexities of some well-known functions involving the complete elementary symmetric functions are obtained.

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1. Introduction

Throughout the paper, \( \mathbb{R} \) denotes the set of real numbers and \( \mathbb{R}_+ \) denotes the set of strictly positive real numbers. Let \( n \geq 2, n \in \mathbb{N}, x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^n \), and \( x^{1/r} = (x_1^{1/r}, x_2^{1/r}, \ldots, x_n^{1/r}) \), where \( r \in \mathbb{R}, r \neq 0 \); let \( E_{n-1} \subset \mathbb{R}_{n-1}^{n-1} \) be the simplex

\[
E_{n-1} = \left\{ (u_1, \ldots, u_{n-1}) : u_i > 0 (1 \leq i \leq n-1), \sum_{i=1}^{n-1} u_i \leq 1 \right\},
\]

and let \( d\mu = du_1, \ldots, du_{n-1} \) be the differential of the volume in \( E_{n-1} \).

The weighted arithmetic mean \( A(x, u) \) and the power mean \( M_r(x, u) \) of order \( r \) with respect to the numbers \( x_1, x_2, \ldots, x_n \) and the positive weights \( u_1, u_2, \ldots, u_n \) with \( \sum_{i=1}^{n} u_i = 1 \) are defined, respectively, as \( A(x, u) = \sum_{i=1}^{n} u_i x_i, M_r(x, u) = (\sum_{i=1}^{n} u_i x_i^r)^{1/r} \) for \( r \neq 0 \), and \( M_0(x, u) = \prod_{i=1}^{n} x_i^{u_i} \). For \( u = (1/n, 1/n, \ldots, 1/n) \), we denote \( A(x, u) \overset{\Delta}{=} A(x), M_r(x, u) \overset{\Delta}{=} M_r(x) \).

The well-known logarithmic mean \( L(x_1, x_2) \) of two positive numbers \( x_1 \) and \( x_2 \) is

\[
L(x_1, x_2) = \begin{cases} 
\frac{x_1 - x_2}{\ln x_1 - \ln x_2}, & x_1 \neq x_2, \\
x_1, & x_1 = x_2.
\end{cases}
\]
As further generalization of $L(x_1, x_2)$, Stolarsky [1] studied the one-parameter mean, that is,

$$
L_r(x_1, x_2) = \begin{cases} 
\left( \frac{x_1^{r+1} - x_2^{r+1}}{(r + 1)(x_1 - x_2)} \right)^{1/r}, & r \neq -1, 0, x_1 \neq x_2, \\
\frac{x_1 - x_2}{\ln x_1 - \ln x_2}, & r = -1, x_1 \neq x_2, \\
\frac{1}{e} \left( \frac{x_1}{x_2} \right)^{1/(x_1 - x_2)}, & r = 0, x_1 \neq x_2, \\
x_1, & x_1 = x_2.
\end{cases}
$$

(1.3)

Alzer [2, 3] obtained another form of one-parameter mean, that is,

$$
F_r(x_1, x_2) = \begin{cases} 
\frac{r}{r + 1} \cdot \frac{x_1^{r+1} - x_2^{r+1}}{x_1^{r} - x_2^{r}}, & r \neq -1, 0, x_1 \neq x_2, \\
x_1 x_2 \cdot \frac{\ln x_1 - \ln x_2}{x_1 - x_2}, & r = -1, x_1 \neq x_2, \\
\frac{x_1 - x_2}{\ln x_1 - \ln x_2}, & r = 0, x_1 \neq x_2, \\
x_1, & x_1 = x_2.
\end{cases}
$$

(1.4)

These two means can be written also as

$$
L_r(x_1, x_2) = \begin{cases} 
\left( \int_0^1 (x_1 u + x_2 (1 - u))^r du \right)^{1/r}, & r \neq 0, \\
\exp \left( \int_0^1 \ln(x_1 u + x_2 (1 - u)) du \right), & r = 0,
\end{cases}
$$

(1.5)

$$
F_r(x_1, x_2) = \begin{cases} 
\int_0^1 (x_1^r u + x_2^r (1 - u))^{1/r} du, & r \neq 0, \\
\int_0^1 x_1^{r-1} u du, & r = 0.
\end{cases}
$$

Correspondingly, Pittenger [4] and Pearce et al. [5] investigated the means above in $n$ variables, respectively,

$$
L_r(x) = \begin{cases} 
\left( (n - 1)! \int_{E_{n-1}} (A(x, u))^r d\mu \right)^{1/r}, & r \neq 0, \\
\exp \left( (n - 1)! \int_{E_{n-1}} \ln A(x, u) d\mu \right), & r = 0,
\end{cases}
$$

(1.6)

$$
F_r(x) = (n - 1)! \int_{E_{n-1}} M_r(x, u) d\mu,
$$

where $u_n = 1 - \sum_{i=1}^{n-1} u_i$. 
In this paper, we establish the Schur-convexities of two types of one-parameter mean.

Let $\Omega \subset \mathbb{R}^n$. A function $\varphi : \Omega \rightarrow \mathbb{R}$ is said to be a Schur-convex (Schur-concave) function if $u \prec v$ implies $\varphi(u) \leq (\geq) \varphi(v)$.

**Expressions (1.3) and (1.4) can be also written by using 2-order determinants, that is,**

\[
L_r(x_1, x_2) = \left( \begin{array}{c|c}
1 & x_2^r \\
1 & x_1^r \\
\end{array} \right)^{1/r}, \quad r \neq -1, 0, x_1 \neq x_2,
\]

\[
\exp \left\{ \left( \begin{array}{c|c}
1 & x_2 \\
1 & x_1 \\
\end{array} \right) \ln \left( \begin{array}{c|c}
1 & x_2 \\
1 & x_1 \\
\end{array} \right) - 1 \right\}, \quad r = 0, x_1 \neq x_2,
\]

\[
F_r(x_1, x_2) = \left( \begin{array}{c|c}
1 & x_2 \\
1 & x_1 \\
\end{array} \right)^{(r+1)1/r} \left( \begin{array}{c|c}
1 & x_2^r \\
1 & x_1^r \\
\end{array} \right), \quad r \neq -1, 0, x_1 \neq x_2,
\]

\[
\exp \left( \begin{array}{c|c}
1 & x_2 \\
1 & x_1 \\
\end{array} \right) \ln \left( \begin{array}{c|c}
1 & x_2 \\
1 & x_1 \\
\end{array} \right), \quad r = 0, x_1 \neq x_2,
\]

Utilizing higher-order generalized Vandermonde determinants, Xiao et al. [8, 7, 6, 9] gave the analogous definitions of $L_r(x)$ and $F_r(x)$.

Obviously, $L_r(x)$ and $F_r(x)$ are symmetric with respect to $x_1, x_2, \ldots, x_n$, $r \rightarrow L_r(x)$ and $r \rightarrow F_r(x)$ are continuous for any $x \in \mathbb{R}^n$.

In [4, 5, 10, 11], the authors studied the Schur-convexities of $L_r(x_1, x_2)$ and $F_r(x_1, x_2)$. In this paper, we establish the Schur-convexities of two types of one-parameter mean values $L_r(x)$ and $F_r(x)$ for several positive numbers. As applications, Schur-convexities of some well-known functions involving the complete elementary symmetric functions are obtained.

**2. Some definitions and lemmas**

The Schur-convex function was introduced by Schur [12] in 1923, and has many important applications in analytic inequalities. The following definitions can be found in many references such as [12–17].

**Definition 2.1.** For $u = (u_1, u_2, \ldots, u_n), v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, without loss of generality, it is assumed that $u_1 \geq u_2 \geq \cdots \geq u_n$ and $v_1 \geq v_2 \geq \cdots \geq v_n$. Then $u$ is said to be majorized by $v$ (in symbols $u \prec v$) if $\sum_{i=1}^{k} u_i \leq \sum_{i=1}^{k} v_i$ for $k = 1, 2, \ldots, n - 1$ and $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i$.

**Definition 2.2.** Let $\Omega \subset \mathbb{R}^n$. A function $\varphi : \Omega \rightarrow \mathbb{R}$ is said to be a Schur-convex (Schur-concave) function if $u \prec v$ implies $\varphi(u) \leq (\geq) \varphi(v)$.
Every Schur-convex function is a symmetric function [18]. But it is not hard to see that not every symmetric function can be a Schur-convex function [15, page 258]. However, we have the following so-called Schur condition.

**Lemma 2.3** [12, page 57]. Suppose that $\Omega \subset \mathbb{R}^n$ is symmetric with respect to permutations and convexset, and has a nonempty interior set $\Omega^0$. Let $\phi : \Omega \to \mathbb{R}$ be continuous on $\Omega$ and continuously differentiable in $\Omega^0$. Then, $\phi$ is a Schur-convex (Schur-concave) function if and only if it is symmetric and if

$$
(u_1 - u_2) \left( \frac{\partial \phi}{\partial u_1} - \frac{\partial \phi}{\partial u_2} \right) \geq (\leq) 0
$$

holds for any $u = (u_1, u_2, \ldots, u_n) \in \Omega^0$.

**Lemma 2.4.** Let $m \geq 1, n \geq 2, m, n \in \mathbb{N}$, $\Lambda \subset \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$, $\phi : \Lambda \times \Omega \to \mathbb{R}$, $\phi(v, x)$ be continuous with respect to $v \in \Lambda$ for any $x \in \Omega$. Let $\Delta$ be a set of all $v \in \Lambda$ such that the function $x \mapsto \phi(v, x)$ is a Schur-convex (Schur-concave) function. Then $\Delta$ is a closed set of $\Lambda$.

**Proof.** Let $l \geq 1$, $l \in \mathbb{N}$, $v_1 \in \Delta$, $v_0 \in \Lambda$, $v_1 \rightarrow v_0$ if $l \rightarrow +\infty$. According to Definition 2.2, $\phi(v_i, y) \geq (\leq) \phi(v_i, z)$ holds for any $y, z \in \Omega$ and $y > z$. Let $l \rightarrow +\infty$, then we have $\phi(v_0, y) \geq (\leq) \phi(v_0, z)$. Hence $v_0 \in \Delta$, so $\Delta$ is a closed set of $\Lambda$. □

**3. Main results**

**Theorem 3.1.** Given $r \in \mathbb{R}$, $L_r(x)$ is Schur-convex if $r \geq 1$ and Schur-concave if $r \leq 1$.

**Proof.** Denote $\tilde{u} = (u_2, u_1, u_3, \ldots, u_n)$.

If $r \neq 0$, owing to the symmetry of $L_r(x)$ with respect to $x_1, x_2, \ldots, x_n$, we have

$$
g_r(x) \triangleq \int_{E_{n-1}} (A(x, u))^r \, d\mu = \int_{E_{n-1}} (A(x, \tilde{u}))^r \, d\mu. \tag{3.1}
$$

Therefore,

$$
\frac{\partial g_r}{\partial x_1} = r \int_{E_{n-1}} u_1 (A(x, u))^{r-1} \, d\mu = r \int_{E_{n-1}} u_2 (A(x, \tilde{u}))^{r-1} \, d\mu,
$$

$$
\frac{\partial g_r}{\partial x_2} = r \int_{E_{n-1}} u_1 (A(x, \tilde{u}))^{r-1} \, d\mu = r \int_{E_{n-1}} u_2 (A(x, u))^{r-1} \, d\mu. \tag{3.2}
$$

It follows that

$$
\frac{\partial g_r}{\partial x_1} - \frac{\partial g_r}{\partial x_2} = r \int_{E_{n-1}} u_1 \left[ (A(x, u))^{r-1} - (A(x, \tilde{u}))^{r-1} \right] \, d\mu, \tag{3.3}
$$

$$
\frac{\partial g_r}{\partial x_1} - \frac{\partial g_r}{\partial x_2} = r \int_{E_{n-1}} u_2 \left[ (A(x, \tilde{u}))^{r-1} - (A(x, u))^{r-1} \right] \, d\mu.
$$

By combining (3.3) with (3.2), we have

$$
\frac{\partial g_r}{\partial x_1} - \frac{\partial g_r}{\partial x_2} = r \int_{E_{n-1}} (u_1 - u_2) \left[ (A(x, u))^{r-1} - (A(x, \tilde{u}))^{r-1} \right] \, d\mu. \tag{3.4}
$$
By Lagrange’s mean value theorem, we find that

\[
(A(x, u))^{r-1} - (A(x, \tilde{u}))^{r-1} = (r - 1)(x_1u_1 + x_2u_2 - x_1u_2 - x_2u_1) \left( \xi + \sum_{i=3}^{n} u_i x_i \right)^{r-2}
\]

\[
= (r - 1)(u_1 - u_2)(x_1 - x_2) \left( \xi + \sum_{i=3}^{n} u_i x_i \right)^{r-2},
\]

where \(\xi\) is between \(x_1u_1 + x_2u_2\) and \(x_2u_1 + x_1u_2\).

From (3.4) and (3.5), we have

\[
(x_1 - x_2) \left( \frac{\partial g_r}{\partial x_1} - \frac{\partial g_r}{\partial x_2} \right) = \frac{r(r-1)}{2} (x_1 - x_2)^2 S_r(x),
\]

where

\[
S_r(x) = \int_{E_{n-1}} (u_1 - u_2)^2 \left( \xi + \sum_{i=3}^{n} u_i x_i \right)^{r-2} \, d\mu \geq 0.
\]

Hence, for \(r \neq 0\), we get

\[
(x_1 - x_2) \left( \frac{\partial L_r}{\partial x_1} - \frac{\partial L_r}{\partial x_2} \right) = (n - 1)! \cdot \frac{1}{r} \cdot (L_r)^{1-r} \cdot (x_1 - x_2) \left( \frac{\partial g_r}{\partial x_1} - \frac{\partial g_r}{\partial x_2} \right)
\]

\[
= (n - 1)! \cdot \frac{r-1}{2} \cdot (L_r)^{1-r} \cdot (x_1 - x_2)^2 S_r(x).
\]

From Lemma 2.3, it is clear that \(L_r\) is Schur-convex for \(r > 1\) and Schur-concave for \(r < 1\) and \(r \neq 0\).

According to Lemma 2.4 and the continuity of \(r \mapsto L_r(x)\), let \(r \to 0, 1-, \) or \(1+ \) in \(L_r(x)\), we know that \(L_0(x)\) is a Schur-concave function, and \(L_1(x)\) is both a Schur-concave function and a Schur-convex function.

\[\square\]

**Theorem 3.2.** Given \(r \in \mathbb{R}\), \(F_r(x)\) is Schur-convex if \(r \geq 1\) and Schur-concave if \(r \leq 1\).

**Proof.** Denote \(\tilde{u} = (u_2, u_1, u_3, \ldots, u_n)\). For \(r \neq 0\),

\[
F_r(x) = (n - 1)! \int_{E_{n-1}} M_r(x, u) \, d\mu = (n - 1)! \int_{E_{n-1}} M_r(x, \tilde{u}) \, d\mu,
\]

\[
\frac{\partial F_r}{\partial x_1} = (n - 1)! \int_{E_{n-1}} x_1^{r-1} u_1 (M_r(x, u))^{1-r} \, d\mu = (n - 1)! \int_{E_{n-1}} u_1 \left[ \frac{M_r(x, u)}{x_1} \right]^{1-r} \, d\mu,
\]

\[
\frac{\partial F_r}{\partial x_2} = (n - 1)! \int_{E_{n-1}} x_2^{r-1} u_1 (M_r(x, \tilde{u}))^{1-r} \, d\mu = (n - 1)! \int_{E_{n-1}} u_1 \left[ \frac{M_r(x, \tilde{u})}{x_2} \right]^{1-r} \, d\mu.
\]
Combination of (3.10) with (3.11) yields

\[
\frac{\partial F_r}{\partial x_1} - \frac{\partial F_r}{\partial x_2} = (n-1)! \int_{E_{n-1}} u_1 \left\{ \left( \frac{M_r(x, u)}{x_1} \right)^{1-r} - \left( \frac{M_r(x, \hat{u})}{x_2} \right)^{1-r} \right\} d\mu. \tag{3.12}
\]

By using the mean value theorem, we find

\[
\left( \frac{M_r(x, u)}{x_1} \right)^{1-r} - \left( \frac{M_r(x, \hat{u})}{x_2} \right)^{1-r} = \\
\left( u_1 + \frac{u_2 x_2^r + \sum_{i=3}^{n} u_i x_i^r}{x_1^r} \right)^{(1-r)/r} - \left( u_1 + \frac{u_2 x_1^r + \sum_{i=3}^{n} u_i x_i^r}{x_2^r} \right)^{(1-r)/r} \\
= \frac{1-r}{r} \left( \frac{u_2 x_2^r + \sum_{i=3}^{n} u_i x_i^r}{x_1^r} - \frac{u_2 x_1^r + \sum_{i=3}^{n} u_i x_i^r}{x_2^r} \right) (u_1 + \theta_1)^{(1-2r)/r} \\
= \frac{1-r}{r} \frac{u_2 x_2^r + \sum_{i=3}^{n} u_i x_i^r}{x_1^r} - \frac{u_2 x_1^r + \sum_{i=3}^{n} u_i x_i^r}{x_2^r} (u_1 + \theta_1)^{(1-2r)/r} \\
= (1-r)(x_2-x_1) (u_1 + \theta_1)^{(1-2r)/r} T(x, u; \theta_2),
\tag{3.13}
\]

where \( \theta_1 \) is between \( (u_2 x_2^r + \sum_{i=3}^{n} u_i x_i^r)/x_1^r \) and \( (u_2 x_1^r + \sum_{i=3}^{n} u_i x_i^r)/x_2^r \), \( \theta_2 \) is between \( x_1 \) and \( x_2 \), and \( T(x, u; \theta_2) = (2u_2 \theta_2^{2r-1} + \theta_2^{r-1} \sum_{i=3}^{n} u_i x_i^r)/x_1^r x_2^r \geq 0 \).

From (3.12) and (3.13), we have

\[
(x_1 - x_2) \left( \frac{\partial F_r}{\partial x_1} - \frac{\partial F_r}{\partial x_2} \right) = (r-1)(x_1 - x_2)^2 (n-1)! \int_{E_{n-1}} u_1 (u_1 + \theta_1)^{(1-2r)/r} T(x, u; \theta_2) d\mu.
\tag{3.14}
\]

It follows that \( F_r \) is Schur-convex for \( r > 1 \) and Schur-concave for \( r < 1 \) and \( r \neq 0 \) by Lemma 2.3.

According to Lemma 2.4 and the continuity of \( r \mapsto F_r(x) \), let \( r \mapsto 0, 1-, \) or \( 1+ \) in \( F_r(x) \). We know that \( F_0(x) \) is a Schur-concave function, and \( F_1(x) \) is both a Schur-concave function and a Schur-convex function. \( \square \)

**Theorem 3.3.** \( L_r(x^{1/r}) \) and \( F_r(x^{1/r}) \) are Schur-concave functions if \( r \geq 1 \), and Schur-convex functions if \( r \leq 1 \) and \( r \neq 0 \).
Proof. We can easily obtain that

\[
L_r(x^{1/r}) = \left[ (n-1)! \int_{E_{n-1}} M_{1/r}(x,u)\,d\mu \right]^{1/r} = F_{1/r}^{1/r}(x),
\]

(3.15)

\[
F_r(x^{1/r}) = (n-1)! \int_{E_{n-1}} \left[ A(x,u) \right]^{1/r} \,d\mu = L_{1/r}^{1/r}(x),
\]

\[
(x_1 - x_2) \left( \frac{\partial L_r(x^{1/r})}{\partial x_1} - \frac{\partial L_r(x^{1/r})}{\partial x_2} \right) = \frac{1}{r} (x_1 - x_2) \left( \frac{\partial F_{1/r}(x)}{\partial x_1} - \frac{\partial F_{1/r}(x)}{\partial x_2} \right) \cdot F_{1/r}^{(1-r)/r}(x),
\]

(3.15)

\[
(x_1 - x_2) \left( \frac{\partial F_r(x^{1/r})}{\partial x_1} - \frac{\partial F_r(x^{1/r})}{\partial x_2} \right) = r (x_1 - x_2) \left( \frac{\partial L_{1/r}(x)}{\partial x_1} - \frac{\partial L_{1/r}(x)}{\partial x_2} \right) \cdot L_{1/r}^{r-1}(x).
\]

(3.16)

From Theorems 3.1 and 3.2, we know that both \( L_{1/r}(x) \) and \( F_{1/r}(x) \) are Schur-concave functions if \( r \geq 1 \) and Schur-convex functions if \( 0 < r \leq 1 \) or \( r < 0 \). According to Lemma 2.3 and (3.16), the required result of Theorem 3.3 is proved. \( \square \)

4. Applications

As applications of the theorems above, we have the following corollaries.

**Corollary 4.1** (See [19, Theorem 3.1] and [12, page 82]). For \( r \geq 1, r \in \mathbb{N} \), the complete elementary symmetric function

\[
C_r(x) = \sum_{i_1 + i_2 + \ldots + i_n = r, i_1, \ldots, i_n \geq 0 \text{ are integers}} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}
\]

(4.1)

is Schur-convex.

**Proof.** If \( r \geq 1, r \in \mathbb{N} \), then (see [20, page 164])

\[
C_r(x) = \binom{n-1+r}{r} L_{1/r}^r(x). \quad (4.2)
\]

By Theorem 3.1 and Lemma 2.3, it is easy to see that \( L_{1/r}^r(x) \) is a Schur-convex function. Therefore, \( C_r(x) \) is a Schur-convex function. \( \square \)

**Corollary 4.2.** The complete symmetric function of the first degree:

\[
D_r(x) = \sum_{i_1 + i_2 + \ldots + i_n = r, i_1, \ldots, i_n \geq 0 \text{ are integers}} \left( x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \right)^{1/r}
\]

(see [6, Theorem 5] and [9]), is Schur-concave for \( r \geq 1, r \in \mathbb{N} \).
Proof. If \( r \geq 1, r \in \mathbb{N} \), then we have (see [6, Theorem 5])

\[
D_r(x) = \left( \frac{n-1+r}{r} \right) F_{1/r}(x). \tag{4.4}
\]

By considering Theorem 3.2, we prove the required result. \qed

Corollary 4.3. Let \( r \neq 0, x, y \in \mathbb{R}^n_+ \), \( x' > y' \). Then \( L_r(x) \leq L_r(y) \) and \( F_r(x) \leq F_r(y) \) if \( r \geq 1 \). They are reversed if \( r \leq 1 \) and \( r \neq 0 \).

Proof. Suppose \( r \geq 1 (r \leq 1, r \neq 0) \). \( L_r(x^{1/r}) \) is a Schur-concave (Schur-convex) function by Theorem 3.3. Then

\[
L_r\left( (x')^{1/r} \right) \leq (\geq )L_r\left( (y')^{1/r} \right), \quad L_r(x) \leq (\geq )L_r(y). \tag{4.5}
\]

For \( F_r(x^{1/r}) \), the proof is similar; we omit the details. \qed

Corollary 4.4. If \( r \geq 1 \), then

\[
A(x) \leq L_r(x) \leq M_r(x), \\
A(x) \leq F_r(x) \leq M_r(x). \tag{4.6}
\]

Inequalities (4.6) are reversed if \( r \leq 1 \).

Proof. If \( r \geq 1 \), owing to Theorem 3.1 and

\[
(x_1, x_2, \ldots, x_n) \succ (A(x), A(x), \ldots, A(x)) \triangleq \overline{A}(x), \tag{4.7}
\]

we have

\[
L_r(x) \geq L_r(\overline{A}(x)) = \left( (n-1)! \int_{E_{n-1}} \left( \sum_{i=1}^{n} A(x) u_i \right)^r d\mu \right)^{1/r} = A(x) \left( (n-1)! \int_{E_{n-1}} \left( \sum_{i=1}^{n} u_i \right)^r d\mu \right)^{1/r} = A(x). \tag{4.8}
\]

Obviously, if \( r \leq 1, r \neq 0 \), inequality (4.8) is reversed by Theorem 3.1. For \( r = 0 \), because of the continuity of \( r \mapsto L_r(x) \), we have \( L_0(x) \leq A(x) \).

By the same way, we find that \( F_r(x) \geq A(x) \) if \( r \geq 1 \), and \( F_r(x) \leq A(x) \) if \( r \leq 1 \). In addition,

\[
x' = (x'_1, x'_2, \ldots, x'_n) \succ (M'_r(x), M'_r(x), \ldots, M'_r(x)) \triangleq (M_r(x))^r, \tag{4.9}
\]
If $r \geq 1$, according to Corollary 4.3, we get

$$L_r(x) \leq L_r(\mathcal{M}_r(x)) = \left( (n-1)! \int_{E_{n-1}} \left( \sum_{i=1}^{n} M_r(x) u_i \right)^r d\mu \right)^{1/r} = M_r(x). \quad (4.10)$$

If $r \leq 1$, inequality (4.10) is obviously reversed by Corollary 4.3 again.

Similarly, we have $F_r(x) \leq M_r(x)$ if $r \geq 1$, and $F_r(x) \geq M_r(x)$ if $r \leq 1$. □

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References

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