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Research Article

On the Generalized Favard-Kantorovich and Favard-Durrmeyer Operators in Exponential Function Spaces

Grzegorz Nowak and Aneta Sikorska-Nowak

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We consider the Kantorovich- and the Durrmeyer-type modifications of the generalized Favard operators and we prove an inverse approximation theorem for functions f such that $w_\sigma f \in L_p(R)$, where $1 \le p \le \infty$ and $w_\sigma(x) = \exp(-\sigma x^2)$, $\sigma > 0$.

1. Preliminaries

Let

$$L_{p,\sigma}(R) = \{f : \|w_\sigma f\|_p < \infty\} \quad \text{for } 1 \le p \le \infty \tag{1.1}$$

be the weighted function space, where $w_\sigma(x) = \exp(-\sigma x^2)$, $\sigma > 0$,

$$\begin{aligned} \|g\|_p &= \left(\int_{-\infty}^{\infty} |g(x)|^p dx\right)^{1/p} \quad \text{if } 1 \le p < \infty, \\ \|g\|_\infty &= \operatorname*{essup}_{x\in R} |g(x)|. \end{aligned} \tag{1.2}$$

We define the generalized Favard operators F_n for functions $f : R \to R$ by

$$F_n f(x) = \sum_{k=-\infty}^{\infty} f(k/n) p_{n,k}(x;\gamma) \quad (x \in R,\ n \in N), \tag{1.3}$$

where $N = \{1, 2, \ldots\}$,

$$p_{n,k}(x;\gamma) = \frac{1}{n\gamma_n\sqrt{2\pi}} \exp\left(-\frac{1}{2\gamma_n^2}\left(\frac{k}{n} - x\right)^2\right) \tag{1.4}$$

and $\gamma = (\gamma_n)_{n=1}^{\infty}$ is a positive sequence convergent to zero (see [1]). In the case where $\gamma_n^2 = \vartheta/(2n)$ with a positive constant ϑ, F_n become the known Favard operators introduced by Favard [2]. Some approximation properties of the classical Favard operators for continuous functions f on R are presented in [3, 4]. Some approximation properties of their generalization can be found, for example, in [1, 5]. Denote by F_n^* the Kantorovich-type modification of operators F_n, defined by

$$F_n^* f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{k/n}^{(k+1)/n} f(t)dt \quad (x \in R,\ n \in N), \tag{1.5}$$

and by $\widetilde{F}_n$ the Durrmeyer-type modification of operators F_n

$$\widetilde{F}_n f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t)dt \qquad (x \in R,\ n \in N), \tag{1.6}$$

where $f \in L_{p,\sigma}(R)$. Some estimates concerning the rates of pointwise convergence of the operators $F_n^* f$ and $\widetilde{F}_n f$ can be found in [6, 7].

Recently, several autors investigated the conditions under which global smoothness of a function f, as measured by its modulus of continuity $\omega(f;\circ)$, is retained by the elements of approximating sequences $(L_n f)$ (see, e.g., [8, 9]). For example, Kratz and Stadtmüller considered in [10] a wide class of discrete operators L_n and derived estimates of the form

$$\omega(L_n f;t) \leq K\omega(f;t) \quad (t > 0), \tag{1.7}$$

with a positive constant K independent of f, n, and t. For bounded functions $f \in C(R)$ and operators F_n satisfying

$$\gamma_1^2 \geq \frac{1}{2}\pi^{-2}\log 2, \qquad n^2\gamma_n^2 \geq \frac{1}{2}\pi^{-2}\log n \quad \text{if } n \geq 2, \tag{1.8}$$

they obtained the inequality

$$\omega(F_n f;t) \leq 140\omega(f;t) + 16\pi \cdot t\|f\| \quad (t > 0), \tag{1.9}$$

where $\|f\| = \sup\{|f(x)| : x \in R\}$.

Forbounded functions $f \in C_m(R) = \{f : \|w_m f\|_\infty < \infty\}$, $w_m(x) = (1 + x^{2m})^{-1}$, $m \in N$ and for operators F_n satisfying $n\gamma_n^2 \geq c > 0$ for all $n \in N$, Pych-Taberska [5] obtained the inequality

$$\omega_2(F_n f;t)_m \leq K\{(1 + t_0^2)\omega_2(f;t)_m + t^2\|f\|_m\} \quad (0 < t \leq t_0) \tag{1.10}$$

for all $n \in N$, $n \geq n_c$ where $n_c \in N$ and K is a constant.

In this paper, we obtained an analogous inequality for the rth weighted modulus of smoothness of the function $f \in L_{p,\sigma}(R), \sigma > 0, 1 \le p \le \infty$,

$$\omega_r(f;t)_{\sigma,p} = \sup_{0<h\le t} \|w_\sigma \Delta_h^r f\|_p \quad (r \in N), \tag{1.11}$$

where

$$\Delta_h^r f(x) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i f(x + h(r/2 - i)). \tag{1.12}$$

Namely, suppose that (γ_n) is a positive null sequence satisfying $n\gamma_n^{r/2+1} \ge c \max_{n\in N}\{\gamma_n^{r/2-1}\} > 0$ for all $n \in N$ and $\sigma_1 > \sigma > 0$. Then there exist positive constants, K, K_1, such that for all $n \ge K_1$ and for arbitrary positive number t_0

$$\omega_r(L_n f,t)_{\sigma_1,p} \le K\Big\{(1+t_0^2)\omega_r(f,t)_{\sigma,p} + t^r\|w_\sigma f\|_p\Big\} \quad (0 < t \le t_0), \tag{1.13}$$

where L_n denotes the Favard-Kantorovich operator or the Favard-Durrmeyer operator.

Throughout the paper, the symbols $K(\sigma,\sigma_1,\dots)$, $K_j(\sigma,\sigma_1,\dots)$ $(j = 1,2,\dots)$ will mean some positive constants, not necessarily the same at each occurrence, depending only on the parameters indicated in parentheses.

2. Preliminary results

Let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be a positive sequence and let $n\gamma_n^2 \ge c$ for all $n \in N$, with a positive absolute constant c. As is known [5], for $\nu \in N_0 = \{0\} \cup N$, $n \in N$, $x \in R$,

$$\sum_{k=-\infty}^{\infty} \left|\frac{k}{n} - x\right|^{\nu} p_{n,k}(x;\gamma) \le 15A_c\left(\frac{2}{e}\right)^{\nu/2}\sqrt{(2\nu)!}\,\gamma_n^{\nu}, \tag{2.1}$$

where $A_c = \max\{1,(2c\pi^2)^{-1}\}$. A simple calculation and the known Schwarz inequality lead to

$$\int_{-\infty}^{\infty} \left|\frac{k}{n} - t\right|^{\nu} p_{n,k}(t;\gamma)dt \le \sqrt{(2\nu)!!}\,\frac{\gamma_n^{\nu}}{n} \quad (k \in Z = \{0,\pm1,\pm2,\dots\}). \tag{2.2}$$

Let us choose $n \in N$, $j \in N_0$ and let us write

$$G_{n,j}^{*} f(x) = n\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\left(\frac{k}{n} - x\right)^{j} \int_{k/n}^{(k+1)/n} f(t)dt, \tag{2.3}$$

$$\widetilde{G}_{n,j} f(x) = n\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\left(\frac{k}{n} - x\right)^{j} \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t)dt, \tag{2.4}$$

where $f \in L_{p,\sigma}(R)$, $1 \le p \le \infty$, $\sigma > 0$. Obviously, $G_{n,0}^{*} f(x) = F_n^{*} f(x)$ and $\widetilde{G}_{n,0} f(x) = \widetilde{F}_n f(x)$.

Lemma 2.1. *Let $\gamma=(\gamma_n)_{n=1}^{\infty}$ be a positive sequence convergent to 0 and let $n\gamma_n^2\geq c$ for all $n\in N$, with a positive absolute constant c. Then for $j\in N_0$, $f\in L_{p,\sigma}(R)$, $\sigma>0$, $1\leq p\leq\infty$, and $\sigma_1>\sigma$,*

$$\|w_{\sigma_1}G^*_{n,j}f\|_p\leq 15A_c\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\sqrt{(2j)!}2^{j/2}\gamma_n^j\|w_\sigma f\|_p \tag{2.5}$$

for all $n\in N$ such that $\gamma_n^2\leq(\sigma_1-\sigma)/(4\sigma(\sigma+\sigma_1))$,

$$\|w_{\sigma_1}\widetilde{G}_{n,j}f\|_p\leq 30A_c\sqrt{(2j)!}2^{j/2}\gamma_n^j\|w_\sigma f\|_p \tag{2.6}$$

for all $n\in N$ such that $\gamma_n\leq\max\{(\sigma_1-\sigma)/(2\sqrt{\sigma}(\sigma+\sigma_1));(\sqrt{\sigma_1-\sigma})/(\sqrt{2}(\sigma+\sigma_1))\}$.

Proof. In view of definition (2.3),

$$\begin{aligned}\exp(-\sigma_1x^2)|G^*_{n,j}f(x)|\leq n\sum_{k=-\infty}^{\infty}\exp(-\sigma_1x^2)p_{n,k}(x;\gamma)\left|\frac{k}{n}-x\right|^j\\ \times\exp\left(\sigma(|k|+1)^2/n^2\right)\int_{k/n}^{(k+1)/n}\exp(-\sigma t^2)|f(t)|dt.\end{aligned} \tag{2.7}$$

Using the inequality

$$(u+v)^2\leq\frac{\sigma+\sigma_1}{2\sigma}u^2+\frac{\sigma+\sigma_1}{\sigma_1-\sigma}v^2\quad(u\in R,\ v\in R), \tag{2.8}$$

we can easily observe, that

$$\begin{aligned}p_{n,k}(x;\gamma)\exp(-\sigma_1x^2)\exp\left(\sigma\left(\frac{k+1}{n}\right)^2\right)&\leq\sqrt{2}\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)p_{n,k}(x;\sqrt{2}\gamma),\\ p_{n,k}(x;\gamma)\exp(-\sigma_1x^2)\exp\left(\sigma\left(\frac{k}{n}\right)^2\right)&\leq\sqrt{2}p_{n,k}(x;\sqrt{2}\gamma),\end{aligned} \tag{2.9}$$

for $n\in N$ such that $\gamma_n^2\leq(\sigma_1-\sigma)/(4\sigma(\sigma+\sigma_1))$ (see [9]), where the symbol $\sqrt{2}\gamma$ means the sequence $(\sqrt{2}\gamma_n)_{n=1}^{\infty}$. Therefore,

$$\begin{aligned}\exp(-\sigma_1x^2)|G^*_{n,j}f(x)|\leq\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\sqrt{2}n\sum_{k=-\infty}^{\infty}p_{n,k}(x;\sqrt{2}\gamma)\\ \times\left|\frac{k}{n}-x\right|^j\int_{k/n}^{(k+1)/n}\exp(-\sigma t^2)|f(t)|dt.\end{aligned} \tag{2.10}$$

From (2.2), we have

$$\|w_{\sigma_1}G^*_{n,j}f\|_1\leq\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\sqrt{(2j)!!}(\sqrt{2})^{j+1}\gamma_n^j\|w_\sigma f\|_1. \tag{2.11}$$

Instead, for $p = \infty$, from (2.1) it follows that

$$\begin{aligned}||w_{\sigma_1} G^{*}_{n,j} f||_\infty &\le \sqrt{2}\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)||w_\sigma f||_\infty \operatorname*{essup}_{x\in R}\left(\sum_{k=-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma)\left|\frac{k}{n}-x\right|^j\right)\\ &\le 15\sqrt{2}A_c\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\sqrt{2^j(2j)!}\gamma_n^j\|w_\sigma f\|_\infty.\end{aligned} \tag{2.12}$$

Finally, by Riesz-Thorin theorem, we have (2.5).

In view of definition (2.4) and the inequality

$$p_{n,k}(x;\gamma)p_{n,k}(t;\gamma)\exp(-\sigma_1 x^2)\exp(\sigma t^2) \le 2p_{n,k}(x;\sqrt{2}\gamma)p_{n,k}(t;\sqrt{2}\gamma), \tag{2.13}$$

for $n \in N$ such that $\gamma_n \le \max\{(\sigma_1-\sigma)/(2\sqrt{\sigma}(\sigma+\sigma_1)); \sqrt{\sigma_1-\sigma}/(\sqrt{2}(\sigma+\sigma_1))\}$ (see [6]), we have

$$\begin{aligned}&\exp(-\sigma_1 x^2)\,|\widetilde{G}_{n,j}f(x)|\\ &\quad\le 2n\sum_{k=-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma)\left|\frac{k}{n}-x\right|^j\int_{-\infty}^{\infty}\exp(-\sigma t^2)p_{n,k}(\sqrt{2}t;\gamma)\,|f(t)|\,dt.\end{aligned} \tag{2.14}$$

Applying (2.1) and (2.2), we get

$$\begin{aligned}\|w_{\sigma_1}\widetilde{G}_{n,j}f\|_1 &\le 30A_c\sqrt{(2j)!!}\gamma_n^j 2^{j/2}\|w_\sigma f\|_1,\\ \|w_{\sigma_1}\widetilde{G}_{n,j}f\|_\infty &\le 30A_c\sqrt{(2j)!}\gamma_n^j 2^{j/2}\|w_\sigma f\|_\infty.\end{aligned} \tag{2.15}$$

Finally, by Riesz-Thorin theorem, we have (2.6).

Further, for $\delta > 0$, $x \in R$, and $r \in N$ we define Stieklov function of f

$$f_{(\delta,2r)}(x) = \frac{1}{\delta^{2r}}\frac{2}{\binom{2r}{r}}\int_{-\delta/2}^{\delta/2}\cdots\int_{-\delta/2}^{\delta/2}\sum_{i=1}^{r}\binom{2r}{r-i}(-1)^{i-1}f(x+i(t_1+\cdots+t_{2r}))\,dt_1\cdots dt_{2r}. \tag{2.16}$$

□

LEMMA 2.2. *For all $r = 1,2,\dots$, $0 < \delta \le 1$, $\sigma_1 > \sigma > 0$, $1 \le p \le \infty$, and $x \in R$,*

$$||w_{\sigma_1} f^{(r)}_{(\delta,2r)}||_p \le K(r,\sigma,\sigma_1)\frac{1}{\delta^r}\omega_r(f;\delta)_{\sigma,p}, \tag{2.17}$$

$$||w_{\sigma_1}(f_{(\delta,2r)}-f)||_p \le K(r,\sigma,\sigma_1)\omega_r(f;\delta)_{\sigma,p}. \tag{2.18}$$

Proof. It is easy to see by induction that

$$\begin{aligned}f^{(r)}_{(\delta,2r)}(x) &= \frac{2}{\binom{2r}{r}}\sum_{i=1}^{r}(-1)^{i-1}\binom{2r}{r-i}\frac{1}{(i\delta)^{2r}}\\ &\quad\times\int_{-i\delta/2}^{i\delta/2}\cdots\int_{-i\delta/2}^{i\delta/2}\Delta^r_{i\delta}f(x+u_1+\cdots+u_r)\,du_1\cdots du_r.\end{aligned} \tag{2.19}$$

Let $\sigma_2 = (2\sigma_1 + \sigma)/3$. In view of the inequality

$$\exp\Big(-\sigma_1 x^2 + \sigma_2 (x+u)^2\Big) \le \exp\left(\frac{\sigma_2\sigma_1}{\sigma_1 - \sigma_2} u^2\right), \tag{2.20}$$

where $0 < \delta \le 1$ and $u = u_1 + \cdots + u_r$, $(u \le r^2/2)$, we have

$$\|w_{\sigma_1} f^{(r)}_{(\delta,2r)}\|_p \le \frac{2}{\binom{2r}{r}} \exp\left(\frac{\sigma_1\sigma_2}{\sigma_1 - \sigma_2}\frac{r^4}{4}\right) \sum_{i=1}^{r} \binom{2r}{r-i} \frac{1}{(i\delta)^r} \|w_{\sigma_2} \Delta^r_{i\delta} f\|_p. \tag{2.21}$$

Applying the Minkowski inequality and the fact that for $0 \le l_i \le i-1$ $(0 \le i \le r)$, $0 < h \le 1$,

$$\exp\left(-\sigma_2 x^2 + \sigma\left(x + h\left(l_1 + \cdots + l_r - \frac{r(i-1)}{2}\right)\right)^2\right) \le \exp\left(\frac{\sigma\sigma_2}{\sigma_2 - \sigma}\left(\frac{r(i-1)}{2}\right)^2\right), \tag{2.22}$$

we obtain

$$\begin{aligned} &\|w_{\sigma_2} \Delta^r_{i\delta} f\|_p \\ &\quad = \sup_{|h|\le\delta} \left\{ \int_{-\infty}^{\infty} \left| \exp(-\sigma_2 x^2) \sum_{l_1=0}^{i-1} \cdots \sum_{l_r=0}^{i-1} \Delta^r_h f\left(x + h\left(l_1 + \cdots + l_r - \frac{r(i-1)}{2}\right)\right)\right|^p dx \right\}^{1/p} \\ &\quad \le \exp\left(\frac{\sigma\sigma_2}{\sigma_2 - \sigma}\frac{r^2(i-1)^2}{4}\right) i^r \omega_r(f;\delta)_{\sigma,p}. \end{aligned} \tag{2.23}$$

So (2.17) is evident. It is easy to see that

$$f_{(\delta,2r)}(x) - f(x) = \frac{(-1)^{r-1}}{\delta^{2r}} \frac{1}{\binom{2r}{r}} \int_{-\delta/2}^{\delta/2} \cdots \int_{-\delta/2}^{\delta/2} \Delta^{2r}_{t_1 + \cdots + t_{2r}} f(x)\, dt_1 \cdots dt_{2r}. \tag{2.24}$$

By Minkowski inequality, for $1 \le p \le \infty$, we have (2.18). □

Lemma 2.3. *Suppose that $\gamma = (\gamma_n)_{n=1}^{\infty}$ is a positive sequence convergent to 0 and that $n\gamma_n^{r/2+1} \ge cK(r)$, where $r \in N$, $r \ge 2$, $K(r) = \max_{n\in N}\{\gamma_n^{r/2-1}\}$, c is a positive absolute constant and let $a_r = 1$ for even r and $a_r = 2$ for odd r. Then for $f \in L_{p,\sigma}(R)$, $\sigma > 0$, $1 \le p \le \infty$ and $\sigma_1 > \sigma$, we have*

$$\|w_{\sigma_1}((F_n^* f)^{(r)} - (n/a_r)^r F_n^* \Delta^r_{a_r/n} f)\|_p \le K(\sigma,\sigma_1,c,r)\|w_\sigma f\|_p \tag{2.25}$$

for all $n \in N$ such that $\gamma_n^2 \le (\sigma_1 - \sigma)/(4\sigma(\sigma + \sigma_1))$ and $n\gamma_n > 4a_r^2 r^2$, and

$$\|w_{\sigma_1}((\widetilde{F}_n f)^{(r)} - (n/a_r)^r \widetilde{F}_n \Delta^r_{a_r/n} f)\|_p \le K(\sigma,\sigma_1,c,r)\|w_\sigma f\|_p \tag{2.26}$$

for all $n \in N$ such that $\gamma_n \le \max\{(\sigma_1 - \sigma)/(2\sqrt{\sigma}(\sigma + \sigma_1)); \sqrt{\sigma_1 - \sigma}/(\sqrt{2}(\sigma + \sigma_1))\}$ and $n\gamma_n > r^2/4$.

Proof. We consider an even r. Let $r = 2r_1$, $r_1 \in N$, $x \in R$. Then

$$\begin{aligned}
n^r F_n^*\left(\Delta_{1/n}^r f(x)\right) &= n^{2r_1+1} \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \sum_{i=0}^{2r_1} \binom{2r_1}{i} (-1)^i \int_{(k+r_1-i)/n}^{(k+r_1-i+1)/n} f(t)dt \\
&= n^{2r_1+1} \sum_{i=0}^{r_1-1} \binom{2r_1}{i} (-1)^i \\
&\quad \times \sum_{k=-\infty}^{\infty} \left(p_{n,k-(r_1-i)}(x;\gamma) + p_{n,k+(r_1-i)}(x;\gamma)\right) \int_{k/n}^{(k+1)/n} f(t)dt \\
&\quad + n^{2r_1+1} \sum_{k=-\infty}^{\infty} \binom{2r_1}{r_1} (-1)^{r_1} p_{n,k}(x;\gamma) \int_{k/n}^{(k+1)/n} f(t)dt.
\end{aligned} \tag{2.27}$$

It is easy to see that

$$\begin{aligned}
&p_{n,k-(r_1-i)}(x;\gamma) + p_{n,k+(r_1-i)}(x;\gamma) \\
&\quad = p_{n,k}(x;\gamma) \left\{ \exp\left(\frac{r_1-i}{n\gamma_n^2}\left(\frac{k}{n} - x\right) - \frac{(r_1-i)^2}{2n^2\gamma_n^2} \right) + \exp\left(-\frac{r_1-i}{n\gamma_n^2}\left(\frac{k}{n} - x\right) - \frac{(r_1-i)^2}{2n^2\gamma_n^2} \right) \right\} \\
&\quad = p_{n,k}(x;\gamma) \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \sum_{j=0}^{[l/2]} \binom{l}{2j} 2^{2j+1-l} \left(\frac{k}{n} - x\right)^{2j} n^{2j-2l} \gamma_n^{-2l} (r_1 - i)^{2l-2j} + 2p_{n,k}(x;\gamma).
\end{aligned} \tag{2.28}$$

Consequently, using definition (2.3), we get

$$\begin{aligned}
n^r F_n^*\left(\Delta_{1/n}^r f(x)\right) &= \sum_{l=1}^{2r_1} \sum_{j=0}^{[l/2]} n^{2(r_1+j-l)} \gamma_n^{-2l} \frac{(-1)^l 2^{2j+1-l}}{(2j)!(l-2j)!} \\
&\quad \times \sum_{i=0}^{r_1-1} \binom{2r_1}{i} (-1)^i (r_1-i)^{2l-2j} G_{n,2j}^* f(x) \\
&\quad + \sum_{l=2r_1+1}^{\infty} \sum_{j=0}^{[l/2]} n^{2(r_1+j-l)} \gamma_n^{-2l} \frac{(-1)^l 2^{2j+1-l}}{(2j)!(l-2j)!} \\
&\quad \times \sum_{i=0}^{r_1-1} \binom{2r_1}{i} (-1)^i (r_1-i)^{2l-2j} G_{n,2j}^* f(x) \\
&= S_{n,1} f(x) + S_{n,2} f(x).
\end{aligned} \tag{2.29}$$

In view of (2.5) and using Stirling formula, we obtain

$$\begin{aligned}
||w_{\sigma_1}S_{n,2}f||_p &\le K_1(\sigma,\sigma_1,c)||w_\sigma f||_p 4^{r_1}n^{2r_1}\sum_{l=2r_1+1}^{\infty}\frac{r_1^{2l}}{n^{2l}\gamma_n^{2l}2^l}\sum_{j=0}^{[l/2]}\frac{\sqrt{(4j)!}2^j}{(2j)!(l-2j)!}n^{2j}\gamma_n^{2j}4^j r_1^{-2j}\\
&\le K_2(\sigma,\sigma_1,c,r)||w_\sigma f||_p n^{2r_1}\sum_{l=2r_1+1}^{\infty}\frac{(r_1^2/2)^l}{(n^2\gamma_n^2)^l}\sum_{j=0}^{[l/2]}(n^2\gamma_n^2)^j 64^j\\
&\le K_3(\sigma,\sigma_1,c,r)||w_\sigma f||_p\left\{\frac{(16r_1^2)^{2r_1+1}}{n^2\gamma_n^{2r_1+2}}+n^{2r_1}\sum_{l=2r_1+2}^{\infty}\left(\frac{16r_1^2}{n\gamma_n}\right)^l\right\}.
\end{aligned} \tag{2.30}$$

Assuming $(16r_1^2)/(n\gamma_n) < 1$ and using the condition $n\gamma_n^{r_1+1} \ge cK(r)$, we get

$$\|w_{\sigma_1}S_{n,2}f\|_p \le K_4(\sigma,\sigma_1,c,r)\|w_\sigma f\|_p. \tag{2.31}$$

Now observe that

$$\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2s} = \begin{cases}0 & \text{if } 0<s<r_1,\\ (2r_1)!/2 & \text{if } s=r_1.\end{cases} \tag{2.32}$$

The equality follows simply from properties of finite differences since the left-hand side of the equation is a half of the finite difference of the polynomial $(r_1-x)^{2s}$. Therefore,

$$\begin{aligned}
S_{n,1}f(x) &= \sum_{l=r_1}^{2r_1}\frac{(-1)^l 2^{2j+1-l}}{l!n^{2l-2j-2r_1}\gamma_n^{2l}}\binom{l}{2j}\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2l-2j}G^*_{n,2j}f(x)\\
&= \sum_{l=0}^{r_1}\sum_{j=0}^{l-1}\frac{(-1)^{r_1+l}2^{2j+1-l-r_1}}{(r_1+l)!n^{2l-2j}\gamma_n^{2l+2r_1}}\binom{r_1+l}{2j}\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2r_1+2l-2j}G^*_{n,2j}f(x)\\
&\quad+\sum_{l=0}^{r_1}\frac{(-1)^{2r_1-l}}{\gamma_n^{4r_1-2l}}\frac{(2r_1)!}{2^l l!(2r_1-2l)!}G^*_{n,2j}f(x).
\end{aligned} \tag{2.33}$$

It is easy to see, by the method of induction, that

$$p_{n,k}^{(\nu)}(x;\gamma) = p_{n,k}(x;\gamma)\sum_{i=0}^{[\nu/2]}\frac{\nu!(-1)^i}{(\nu-2i)!(2i)!!}\frac{1}{\gamma_n^{2\nu-2i}}\left(\frac{k}{n}-x\right)^{\nu-2i}, \quad \nu\in N. \tag{2.34}$$

Therefore,

$$\begin{aligned}
S_{n,1}f(x) &= \sum_{l=0}^{r_1}\sum_{j=0}^{l-1}\frac{(-1)^{r_1+l}2^{2j+1-l-r_1}}{(r_1+l)!n^{2l-2j}\gamma_n^{2l+2r_1}}\binom{r_1+l}{2j}\sum_{i=0}^{r_1-1}\binom{2r_1}{i}(-1)^i(r_1-i)^{2r_1+2l-2j}G^*_{n,2j}f(x)\\
&\quad+\left(F_n^*f(x)\right)^{(2r_1)}.
\end{aligned} \tag{2.35}$$

Consequently, from (2.29)

$$\begin{aligned}
&\left|(F_n^* f)^{(2r_1)}(x)-n^{2r_1}F_n^*\Delta_{1/n}^{2r_1}f(x)\right|\\
&\quad\le K_5(r)\sum_{j=0}^{r_1-1}\sum_{l=j+1}^{r_1}\frac{n^{2j}}{(n\gamma_n)^{2l}\gamma_n^{2r_1}}\left|G_{n,2j}^* f(x)\right|+\left|S_{n,2}f(x)\right|.
\end{aligned}\tag{2.36}$$

The condition $n\gamma_n^{r_1+1}\ge cK(r)$ and the boundedness of the sequence (γ_n) lead to

$$\left|(F_n^* f)^{(2r_1)}(x)-n^{2r_1}F_n^*\Delta_{1/n}^{2r_1}f(x)\right|\le K_6(r,c)\sum_{j=0}^{r_1-1}\gamma_n^{-2j}\left|G_{n,2j}^* f(x)\right|+\left|S_{n,2}f(x)\right|.\tag{2.37}$$

Collecting the results we get estimate (2.25) for even r, immediately.

Now, we will prove inequality (2.25) for odd r. Namely, let $r=2r_2+1$, $r_2\in N$, $x\in R$. Then

$$\begin{aligned}
n^r F_n^*\left(\Delta_{2/n}^r f(x)\right)&=n^{2r_2+2}\sum_{i=0}^{r_2}\sum_{k=-\infty}^{\infty}\binom{2r_2+1}{i}(-1)^i\\
&\quad\times\left(p_{n,k-(2r_2+1-2i)}(x;\gamma)-p_{n,k+(2r_2+1-2i)}(x;\gamma)\right)\int_{k/n}^{(k+1)/n}f(t)dt.
\end{aligned}\tag{2.38}$$

It is easy to see that

$$\begin{aligned}
&p_{n,k-(2r_2+1-2i)}(x;\gamma)-p_{n,k+(2r_2+1-2i)}(x;\gamma)\\
&\quad=p_{n,k}(x;\gamma)\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l!}\sum_{j=0}^{[(l-1)/2]}\binom{l}{2j+1}2^{2j+2-l}\left(\frac{k}{n}-x\right)^{2j+1}\frac{n^{2j+1-2l}}{\gamma_n^{2l}(2r_2+1-2i)^{2j-2l+1}}.
\end{aligned}\tag{2.39}$$

Consequently,

$$\begin{aligned}
n^r F_n^*\left(\Delta_{2/n}^r f(x)\right)&=\sum_{l=1}^{2r_2+1}n^{2r_2+2}\sum_{j=0}^{[(l-1)/2]}n^{2j-2l}\gamma_n^{-2l}\frac{(-1)^{l+1}2^{2j+2-l}}{(2j+1)!(l-2j-1)!}\\
&\quad\times\sum_{i=0}^{r_2}\binom{2r_2+1}{i}(-1)^i(2r_2+1-2i)^{2l-2j-1}G_{n,2j+1}^* f(x)\\
&\quad+\sum_{l=2r_2+2}^{\infty}n^{2r_2+2}\sum_{j=0}^{[(l-1)/2]}n^{2j-2l}\gamma_n^{-2l}\frac{(-1)^{l+1}2^{2j+2-l}}{(2j+1)!(l-2j-1)!}\\
&\quad\times\sum_{i=0}^{r_2}\binom{2r_2+1}{i}(-1)^i(2r_2+1-2i)^{2l-2j-1}G_{n,2j+1}^* f(x)\\
&=S_{n,1}^* f(x)+S_{n,2}^* f(x).
\end{aligned}\tag{2.40}$$

Some simple calculation, Stirling formula and (2.5) give

$$\left\|w_{\sigma_1}S_{n,2}^* f\right\|_p\le K_7(\sigma,\sigma_1,c,r)\left\|w_\sigma f\right\|_p\tag{2.41}$$

for $n \in N$ such that $(16r^2)/(n\gamma_n) < 1$. Next, in view of (2.25) and the equality

$$\sum_{i=0}^{r_2} \binom{2r_2+1}{i} (-1)^i (r_2 - i + 1/2)^{2s-1} = \begin{cases} 0 & \text{if } 0 < s < r_2+1, \\ (2r_2+1)!/2 & \text{if } s = r_2+1 \end{cases} \tag{2.42}$$

we obtain

$$\begin{aligned} S_{n,1}^{*} f(x) = {} & \sum_{l=0}^{r_2} \sum_{j=0}^{l-1} \frac{(-1)^{r_2+l} 2^{2j+1-l-r_2}}{(2j+1)!(l+r_2-2j)!} n^{2j-2l} \gamma_n^{-2l-2r_2-2} \\ & \times \sum_{i=0}^{r_2} \binom{2r_2+1}{i} (-1)^i (2r_2+1-2i)^{2r_2+2l-2j+1} \\ & \times G_{n,2j+1}^{*} f(x) + 2^{2r_2+1} (F_n^{*} f)^{(2r_2+1)}(x). \end{aligned} \tag{2.43}$$

Using (2.40) and the condition $n\gamma_n^{r_2+3/2} \geq cK(r)$, we have

$$\begin{aligned} & \left| (F_n^{*} f)^{(2r_2+1)}(x) - (n/2)^{2r_2+1} F_n^{*} \Delta_{2/n}^{2r_2+1} f(x) \right| \\ & \quad \leq K_8(r,c) \sum_{j=0}^{r_2-1} \frac{1}{\gamma_n^{2j+1}} \left| G_{n,2j+1}^{*} f(x) \right| + \left| S_{n,2}^{*} f(x) \right|. \end{aligned} \tag{2.44}$$

Applying (2.5), we get (2.25) for odd r. Therefore, inequality (2.25) is proved.

Now we will prove (2.26). Let $r = 2r_1$, $r_1 \in N$. A simple calculation and the equality $p_{n,k}(t - (r_1 - i)/n; \gamma) = p_{n,k+r_1-i}(t;\gamma)$ give

$$\begin{aligned} n^r \widetilde{F}_n(\Delta_{1/n}^r f(x)) = {} & n^{2r_1+1} \sum_{i=0}^{r_1-1} \sum_{k=-\infty}^{\infty} \binom{2r_1}{i} (-1)^i (p_{n,k-(r_1-i)}(x;\gamma) + p_{n,k+(r_1-i)}(x;\gamma)) \\ & \times \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t)\,dt + n^{2r_1+1} \sum_{k=-\infty}^{\infty} \binom{2r_1}{r_1} (-1)^i p_{n,k}(x;\gamma) \\ & \times \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t)\,dt. \end{aligned} \tag{2.45}$$

The estimate (2.26) follows now the same way as (2.25). □

3. Main result

Theorem 3.1. *Suppose that $r \in N$, (γ_n) is a positive null sequence satisfying $n\gamma_n^{r/2+1} \geq cK(r)$ for all $n \in N$ with some $c > 0$ where $K(r) = \max_{n\in N}\{\gamma_n^{r/2-1}\}$. Then there exists a constant $K > 0$, such that for all $f \in L_{p,\sigma}(R)$, $\sigma_1 > \sigma > 0$, $1 \leq p \leq \infty$, and for an arbitrary positive number t_0,*

$$\omega_r(F_n^{*} f, t)_{\sigma_1,p} \leq K(\sigma,\sigma_1,r,c)\{(1+t_0^2)\omega_r(f,t)_{\sigma,p} + t^r \|w_\sigma f\|_p\} \quad (0 < t \leq t_0) \tag{3.1}$$

for all $n \in N$ such that $\gamma_n^2 \leq (\sigma_1 - \sigma)/(4\sigma(\sigma+\sigma_1))$ and $n\gamma_n > 16r^2$, and

$$\omega_r(\widetilde{F}_n f, t)_{\sigma_1,p} \leq K(\sigma,\sigma_1,r,c)\{(1+t_0^2)\omega_r(f,t)_{\sigma,p} + t^r \|w_\sigma f\|_p\} \quad (0 < t \leq t_0) \tag{3.2}$$

for all $n \in N$ *such that* $\gamma_n \le \max\{(\sigma_1 - \sigma)/(2\sqrt{\sigma}(\sigma+\sigma_1)); \sqrt{\sigma_1-\sigma}/(\sqrt{2}(\sigma+\sigma_1))\}$ *and* $n\gamma_n > r^2/4$.

Proof. Let $\sigma_2 = (3\sigma_1+\sigma)/4$. In view of the inequality

$$\exp\left(-\sigma_1 x^2 + \sigma_2(x+u)^2\right) \le \exp\left(\frac{\sigma_2\sigma_1}{\sigma_1-\sigma_2}u^2\right) \quad (u \in R) \tag{3.3}$$

and the generalized Minkowski inequality it is easy to see that for $0 < h \le 1$

$$\begin{aligned} ||w_{\sigma_1}\Delta_h^r f||_p &= \Bigg|\Bigg| w_{\sigma_1}\int_{-h/2}^{h/2}\cdots\int_{-h/2}^{h/2} f^{(r)}(\circ + s_1+\cdots+s_r)\exp\left(\sigma_2(\circ+s_1+\cdots+s_r)^2\right) \\ &\qquad \times \exp\left(-\sigma_2(\circ+s_1+\cdots+s_r)^2\right)ds_1\cdots ds_r\Bigg|\Bigg|_p \\ &\le \exp\left(\frac{r^2}{4}\frac{\sigma_2\sigma_1}{\sigma_1-\sigma_2}\right)h^r||w_{\sigma_2}f^{(r)}||_p, \end{aligned} \tag{3.4}$$

$$||w_{\sigma_1}\Delta_h^r f||_p \le 2^r\exp\left(\frac{r^2}{4}\frac{\sigma_2\sigma_1}{\sigma_1-\sigma_2}\right)||w_{\sigma_2}f||_p. \tag{3.5}$$

Applying these inequalities, we get

$$\begin{aligned} ||w_{\sigma_1}\Delta_h^r f||_p &\le ||w_{\sigma_1}\Delta_h^r(f - f_{(\delta,2r)})||_p + ||w_{\sigma_1}\Delta_h^r f_{(\delta,2r)}||_p \\ &\le 2^r\exp\left(\frac{r^2}{4}\frac{\sigma_2\sigma_1}{\sigma_1-\sigma_2}\right)\left(||w_{\sigma_2}(f-f_{(\delta,2r)})||_p + h^r||w_{\sigma_2}f_{(\delta,2r)}^{(r)}||_p\right), \end{aligned} \tag{3.6}$$

where $f_{(\delta,2r)}(x)$ ($\delta > 0$, $x \in R$, $r \in N$) is defined by (2.16).

Hence, applying this inequality for $F_n^* f$ we have

$$\omega_r(F_n^* f, t)_{\sigma_1,p} \le 2^r\exp\left(\frac{r^2}{4}\frac{\sigma_2\sigma_1}{\sigma_1-\sigma_2}\right)\left\{||w_{\sigma_2}F_n^*(f-f_{(\delta,2r)})||_p + t^r||w_{\sigma_2}(F_n^* f_{(\delta,2r)})^{(r)}||_p\right\}. \tag{3.7}$$

Hence, $||w_{\sigma_2}(F_n^* f_{(\delta,2r)})^{(r)}||_p$ can be estimated by (2.5) for $j=0$, (2.25), and (3.4). Let $\sigma_3 = (2\sigma_1+\sigma)/3$, then

$$\begin{aligned} &||w_{\sigma_2}(F_n^* f_{(\delta,2r)})^{(r)}||_p \\ &\quad \le ||w_{\sigma_2}((F_n^* f_{(\delta,2r)})^{(r)} - (n/a_r)F_n^*\Delta_{a_r/n}^r f_{(\delta,2r)})||_p + n^r||w_{\sigma_2}F_n^*\Delta_{a_r/n}^r f_{(\delta,2r)}||_p \\ &\quad \le K(\sigma_2,\sigma_3,r,c)\left(||w_{\sigma_3}f_{(\delta,2r)}||_p + ||w_{\sigma_3}f_{(\delta,2r)}^{(r)}||_p\right). \end{aligned} \tag{3.8}$$

Using (2.5) for $j=0$ and (3.7) we have

$$\omega_r(F_n^* f, t)_{\sigma_1,p} \le K(\sigma,\sigma_1,r,c)\left\{||w_{\sigma_3}(f-f_{(\delta,2r)})||_p + t^r||w_{\sigma_3}f_{(\delta,2r)}^{(r)}||_p + t^r||w_{\sigma_3}f_{(\delta,2r)}||_p\right\}. \tag{3.9}$$

Consequently by (2.17), (2.18) and assuming now $0 < t \le t_0$, we have

$$\omega_r(F_n^* f, t)_{\sigma_1,p} \le K(\sigma,\sigma_1,r,c)\left((1+t_0^r)\omega_r(f;t)_{\sigma,p} + t^r \|w_\sigma f\|_p\right). \tag{3.10}$$

On the same way we can prove (3.2) for $\widetilde{F}_n f$, using (2.6) and (2.26). □

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Grzegorz Nowak: Higher School of Marketing and Management, Ostroroga 9a,
64-100 Leszno, Poland
Email address: grzegnow@amu.edu.pl

Aneta Sikorska-Nowak: Faculty of Mathematics and Computer Science,
Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
Email address: anetas@amu.edu.pl