

## Research Article

# Asymptotic Behavior of Solutions to Some Homogeneous Second-Order Evolution Equations of Monotone Type

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We study the asymptotic behavior of solutions to the second-order evolution equation  $p(t)u''(t) + r(t)u'(t) \in Au(t)$  a.e.  $t \in (0, +\infty)$ ,  $u(0) = u_0$ ,  $\sup_{t \geq 0} |u(t)| < +\infty$ , where  $A$  is a maximal monotone operator in a real Hilbert space  $H$  with  $A^{-1}(0)$  nonempty, and  $p(t)$  and  $r(t)$  are real-valued functions with appropriate conditions that guarantee the existence of a solution. We prove a weak ergodic theorem when  $A$  is the subdifferential of a convex, proper, and lower semicontinuous function. We also establish some weak and strong convergence theorems for solutions to the above equation, under additional assumptions on the operator  $A$  or the function  $r(t)$ .

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## 1. Introduction

Let  $H$  be a real Hilbert space with inner product  $(\cdot, \cdot)$  and norm  $|\cdot|$ . We denote weak convergence in  $H$  by  $\rightharpoonup$  and strong convergence by  $\rightarrow$ . We will refer to a nonempty subset  $A$  of  $H \times H$  as a (nonlinear) possibly multivalued operator in  $H$ .  $A$  is called monotone (resp., strongly monotone) if  $(y_2 - y_1, x_2 - x_1) \geq 0$  (resp.,  $(y_2 - y_1, x_2 - x_1) \geq \beta|x_1 - x_2|^2$  for some  $\beta > 0$ ) for all  $[x_i, y_i] \in A$ ,  $i = 1, 2$ .  $A$  is called maximal monotone if  $A$  is monotone and  $R(I + A) = H$ , where  $I$  is the identity operator on  $H$ .

Existence, as well as asymptotic behavior of solutions to second-order evolution equations of the form

$$\begin{aligned} p(t)u''(t) + r(t)u'(t) &\in Au(t) \quad \text{a.e. on } \mathbb{R}^+, \\ u(0) &= u_0, \quad \sup_{t \geq 0} |u(t)| < +\infty, \end{aligned} \tag{1.1}$$

in the special case  $p(t) \equiv 1$  and  $r(t) \equiv 0$ , were studied by many authors, see, for example, Barbu [1], Moroşanu [2, 3], and the references therein, Mitidieri [4, 5], Poffald and Reich [6], and Véron [7].

Véron [8, 9] studied the existence and uniqueness of solutions to (1.1) with the following assumptions on  $p(t)$  and  $r(t)$ :

$$p \in W^{2,\infty}(0, +\infty), \quad r \in W^{1,\infty}(0, +\infty), \tag{1.2}$$

$$\exists \alpha > 0 \quad \text{such that } \forall t \geq 0, p(t) \geq \alpha,$$

$$\int_0^{+\infty} e^{-\int_0^t (r(s)/p(s)) ds} dt = +\infty. \tag{1.3}$$

The following theorem is proved in [9].

**THEOREM 1.1.** *Assume that  $A$  is a maximal monotone,  $0 \in A(0)$ , and (1.2) and (1.3) are satisfied. Then for each  $u_0 \in D(A)$ , there exists a continuously differentiable function  $u \in H^2((0, +\infty); H)$ , satisfying*

$$p(t)u''(t) + r(t)u'(t) \in Au(t) \quad \text{a.e. on } \mathbb{R}^+, \tag{1.4}$$

$$u(0) = u_0, \quad u(t) \in D(A) \quad \text{a.e. on } \mathbb{R}^+.$$

If  $u$  (resp.,  $v$ ) are solutions to (1.1) with initial conditions  $u_0$  (resp.,  $v_0$ ), then for each  $t \geq 0$ ,

$$|u(t) - v(t)| \leq |u_0 - v_0|. \tag{1.5}$$

In addition,  $|u(t)|$  is nonincreasing.

Véron [8, 9] also proved another existence theorem by assuming  $A$  to be strongly monotone, instead of (1.3).

It is easy to show that without loss of generality, the condition  $0 \in A(0)$  in Theorem 1.1 can be replaced by the more general assumption  $A^{-1}(0) \neq \emptyset$ .

In Section 2, we present our main results on the asymptotic behavior of solutions to (1.1).

## 2. Main results

In this section, we study the asymptotic behavior of solutions to the evolution equation (1.1) under appropriate assumptions on the operator  $A$  and the functions  $p(t)$  and  $r(t)$ , similar to those assumed by Véron [8, 9], implying the existence of solutions to (1.1). Throughout the paper, we assume that (1.2) holds and  $A^{-1}(0) \neq \emptyset$ .

First we prove two lemmas.

**LEMMA 2.1.** *Assume that  $u(t)$  is a solution to (1.1). Then for each  $p \in A^{-1}(0)$ ,  $|u(t) - p|$  is either nonincreasing, or eventually increasing.*

*Proof.* Let  $p \in A^{-1}(0)$ . By monotonicity of  $A$  and (1.1), we have

$$(p(t)u''(t) + r(t)u'(t), u(t) - p) \geq 0 \quad \text{a.e. on } (0, +\infty). \tag{2.1}$$

It follows that

$$p(t) \frac{d^2}{dt^2} |u(t) - p|^2 + r(t) \frac{d}{dt} |u(t) - p|^2 \geq 0. \quad (2.2)$$

Dividing both sides of the above inequality by  $p(t)$  and multiplying by  $e^{\int_0^t (r(s)/p(s)) ds}$ , we obtain

$$\frac{d}{dt} \left( e^{\int_0^t (r(s)/p(s)) ds} \frac{d}{dt} |u(t) - p|^2 \right) \geq 0. \quad (2.3)$$

We consider two cases.

If  $(d/dt)|u(t) - p|^2 \leq 0$  for each  $t > 0$ , then  $|u(t) - p|^2$  is nonincreasing. Otherwise, there exists  $t_0 > 0$  such that  $(d/dt)|u(t) - p|^2|_{t=t_0} > 0$ . Integrating (2.3), we get for each  $t \geq t_0$  that

$$e^{\int_0^t (r(s)/p(s)) ds} \frac{d}{dt} |u(t) - p|^2 \geq 2e^{\int_0^{t_0} (r(s)/p(s)) ds} (u'(t_0), u(t_0) - p) > 0. \quad (2.4)$$

Hence,  $(d/dt)|u(t) - p|^2 > 0$  for each  $t > t_0$ . This means that  $|u(t) - p|$  is eventually increasing.  $\square$

Note that in the proof of Lemma 2.1, we did not use the boundedness of  $u$ .

LEMMA 2.2. *Suppose that  $u(t)$  is a solution to (1.1). Then for each  $p \in A^{-1}(0)$ ,  $\lim_{t \rightarrow +\infty} |u(t) - p|^2$  exists and  $\liminf_{t \rightarrow +\infty} (d/dt)|u(t) - p|^2 \leq 0$ . In addition, if either (1.3) is satisfied or  $A$  is strongly monotone, then  $|u(t) - p|^2$  is nonincreasing.*

*Proof.* The existence of  $\lim_{t \rightarrow +\infty} |u(t) - p|^2$  follows from Lemma 2.1.

By contradiction, assume that  $\liminf_{t \rightarrow +\infty} (d/dt)|u(t) - p|^2 > 0$ . Then there exist  $t_0 > 0$  and  $\lambda > 0$ , such that for each  $t \geq t_0$ ,

$$\frac{d}{dt} |u(t) - p|^2 \geq \lambda. \quad (2.5)$$

Integrating from  $t = t_0$  to  $t = T$ , we get

$$|u(T) - p|^2 - |u(t_0) - p|^2 \geq \lambda T - \lambda t_0. \quad (2.6)$$

Letting  $T \rightarrow +\infty$ , we deduce that  $u$  is not bounded, a contradiction. If in addition (1.3) is satisfied, assume that  $|u(t) - p|$  is eventually increasing. Then there exists  $t_0 > 0$  such that  $(u'(t_0), u(t_0) - p) > 0$ . Dividing both sides of (2.4) by  $e^{\int_0^t (r(s)/p(s)) ds}$  and integrating from  $t = t_0$  to  $t = T$ , we get

$$|u(T) - p|^2 - |u(t_0) - p|^2 \geq 2e^{\int_0^{t_0} (r(s)/p(s)) ds} (u'(t_0), u(t_0) - p) \int_{t_0}^T e^{-\int_0^t (r(s)/p(s)) ds} dt. \quad (2.7)$$

Letting  $T \rightarrow +\infty$ , we obtain a contradiction to assumption (1.3). This implies that  $|u(t) - p|$  is nonincreasing.

Finally, assume that  $A$  is strongly monotone, and let  $p \in A^{-1}(0)$ . Then we have

$$(p(t)u''(t) + r(t)u'(t), u(t) - p) \geq \beta |u(t) - p|^2. \tag{2.8}$$

This implies that

$$p(t) \frac{d^2}{dt^2} |u(t) - p|^2 + r(t) \frac{d}{dt} |u(t) - p|^2 \geq 2\beta |u(t) - p|^2. \tag{2.9}$$

Suppose to the contrary that  $|u(t) - p|$  is increasing for  $t \geq T_0 > 0$ . Let  $K$  (resp.,  $M$ ) be an upper bound for  $p(t)$  (resp.,  $|r(t)|$ ). Integrating both sides of this inequality from  $t = T_0$  to  $t = T$ , we get

$$\begin{aligned} & 2\beta \int_{T_0}^T |u(t) - p|^2 dt \\ & \leq K \left( \frac{d}{dT} |u(T) - p|^2 - 2(u'(T_0), u(T_0) - p) + \int_{T_0}^T \frac{r(t)}{p(t)} \frac{d}{dt} |u(t) - p|^2 dt \right) \\ & \leq K \left( \frac{d}{dT} |u(T) - p|^2 - 2(u'(T_0), u(T_0) - p) + \frac{M}{\alpha} |u(T) - p|^2 - \frac{M}{\alpha} |u(T_0) - p|^2 \right). \end{aligned} \tag{2.10}$$

Since  $|u(t) - p|$  is increasing for  $t \geq T_0 > 0$ , we have

$$\begin{aligned} & 2\beta |u(T_0) - p|^2 (T - T_0) \\ & \leq K \left( \frac{d}{dT} |u(T) - p|^2 - 2(u'(T_0), u(T_0) - p) \right. \\ & \quad \left. + \frac{M}{\alpha} |u(T) - p|^2 - \frac{M}{\alpha} |u(T_0) - p|^2 \right). \end{aligned} \tag{2.11}$$

Taking  $\liminf$  as  $T \rightarrow +\infty$  of both sides in the above inequality, by the first part of this lemma we deduce that  $u(t)$  is unbounded, a contradiction.  $\square$

In the following, we prove a mean ergodic theorem when  $A$  is the subdifferential of a proper, convex, and lower semicontinuous function.

**THEOREM 2.3.** *Suppose that  $u(t)$  is a solution to (1.1) and  $A = \partial\varphi$ , where  $\varphi : H \rightarrow ]-\infty, +\infty]$  is a proper, convex, and lower semicontinuous function. If (1.3) is satisfied, then  $\sigma_T := (1/T) \int_0^T u(t) dt \rightarrow p \in A^{-1}(0)$ , as  $T \rightarrow +\infty$ .*

*Proof.* By the subdifferential inequality and (1.1), we get for each  $p \in A^{-1}(0)$  that

$$\begin{aligned} \varphi(u(t)) - \varphi(p) & \leq (p(t)u''(t) + r(t)u'(t), u(t) - p) \\ & \leq \frac{p(t)}{2} \frac{d^2}{dt^2} |u(t) - p|^2 + \frac{r(t)}{2} \frac{d}{dt} |u(t) - p|^2 \\ & = \frac{p(t)}{2} e^{-\int_0^t (r(s)/p(s)) ds} \frac{d}{dt} \left( e^{\int_0^t (r(s)/p(s)) ds} \frac{d}{dt} |u(t) - p|^2 \right). \end{aligned} \tag{2.12}$$

Let  $K$  be an upper bound for  $p(t)/2$ . Integrating the above inequality from  $t = 0$  to  $t = T$ , and using integration by parts, we get

$$\begin{aligned} & \int_0^T (\varphi(u(t)) - \varphi(p)) dt \\ & \leq K \left( \frac{d}{dT} |u(T) - p|^2 - 2(u'(0), u(0) - p) + \int_0^T \frac{r(t)}{p(t)} \frac{d}{dt} |u(t) - p|^2 dt \right) \quad (2.13) \\ & \leq K \left( -2(u'(0), u(0) - p) + \int_0^T \frac{r(t)}{p(t)} \frac{d}{dt} |u(t) - p|^2 dt \right) \end{aligned}$$

(the second inequality holds by Lemma 2.2). Let  $R$  be an upper bound for  $|r(t)|$ , which exists by assumption (1.2). Since  $|u(t) - p|$  is nonincreasing (by Lemma 2.2), we get from (2.13) that

$$\begin{aligned} & \limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T (\varphi(u(t)) - \varphi(p)) dt \\ & \leq \limsup_{T \rightarrow +\infty} \frac{K}{T} \int_0^T \frac{r(t)}{p(t)} \frac{d}{dt} |u(t) - p|^2 dt \quad (2.14) \\ & \leq \frac{-KR}{\alpha} \limsup_{T \rightarrow +\infty} \frac{1}{T} [ |u(T) - p|^2 - |u(0) - p|^2 ] = 0. \end{aligned}$$

Since  $p \in A^{-1}(0)$  and  $A = \partial\varphi$ ,  $p$  is a minimum point of  $\varphi$ . Convexity of  $\varphi$  implies that

$$0 \leq \varphi(\sigma_T) - \varphi(p) \leq \frac{1}{T} \int_0^T \varphi(u(t)) dt - \varphi(p). \quad (2.15)$$

Taking the lim sup as  $T \rightarrow +\infty$  in the above inequality, we get by (2.14)

$$\limsup_{T \rightarrow +\infty} \varphi(\sigma_T) \leq \varphi(p). \quad (2.16)$$

Assume that  $\sigma_{T_n} \rightarrow q$  for some sequence  $\{T_n\}$  converging to  $+\infty$  as  $n \rightarrow +\infty$ . Since  $\varphi$  is lower semicontinuous, we have

$$\liminf_{n \rightarrow +\infty} \varphi(\sigma_{T_n}) \geq \varphi(q). \quad (2.17)$$

Therefore,

$$\varphi(p) \geq \limsup_{T \rightarrow +\infty} \varphi(\sigma_T) \geq \liminf_{n \rightarrow +\infty} \varphi(\sigma_{T_n}) \geq \varphi(q). \quad (2.18)$$

Hence,  $q \in A^{-1}(0)$  and by Lemma 2.2  $\lim_{t \rightarrow +\infty} |u(t) - q|^2$  exists. Now if  $p$  is another weak cluster point of  $\sigma_T$ , then  $\lim_{t \rightarrow +\infty} (|u(t) - p|^2 - |u(t) - q|^2)$  exists. It follows that  $\lim_{t \rightarrow +\infty} (u(t), p - q)$  exists, hence  $\lim_{T \rightarrow +\infty} (\sigma_T, p - q)$  exists. This implies that  $p = q$ , and therefore  $\sigma_T \rightarrow p \in A^{-1}(0)$ , as  $T \rightarrow +\infty$ .  $\square$

**THEOREM 2.4.** *Let  $u$  be a solution to (1.1). If (1.3) is satisfied and there exist  $t_0 > 0$  and a positive constant  $M$ , such that  $r(t) \geq -Mt^{-2}$  for  $t \geq t_0$ , then*

$$\lim_{T \rightarrow +\infty} \left| u(T) - \frac{1}{T} \int_0^T u(t) dt \right| = 0. \tag{2.19}$$

*Proof.* From (2.1), we have

$$|u'(t)|^2 \leq \frac{1}{2} \frac{d^2}{dt^2} |u(t) - p|^2 + \frac{1}{2} \frac{r(t)}{p(t)} \frac{d}{dt} |u(t) - p|^2. \tag{2.20}$$

Multiplying both sides of the above inequality by  $t^2$ , integrating from  $t = 0$  to  $t = T$ , and dividing by  $T$ , since  $|u(t) - p|^2$  is nonincreasing, we get after integration by parts that

$$\frac{1}{T} \int_0^T t^2 |u'(t)|^2 dt \leq -|u(T) - p|^2 + \frac{1}{T} \int_0^T |u(t) - p|^2 dt + \frac{1}{2T} \int_0^T \frac{t^2 r(t)}{p(t)} \frac{d}{dt} |u(t) - p|^2 dt. \tag{2.21}$$

Since  $|u(t) - p|^2$  is nonincreasing (by Lemma 2.2),  $r(t) \geq -Mt^{-2}$  for  $t \geq t_0$ , and  $p(t)$  is bounded from below and by  $\alpha$ , we get

$$\begin{aligned} \limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T t^2 |u'(t)|^2 dt &\leq \limsup_{T \rightarrow +\infty} \frac{1}{2T} \int_0^T \frac{t^2 r(t)}{p(t)} \frac{d}{dt} |u(t) - p|^2 dt \\ &\leq \frac{-M}{2\alpha} \limsup_{T \rightarrow +\infty} \frac{1}{T} [ |u(T) - p|^2 - |u(t_0) - p|^2 ] = 0. \end{aligned} \tag{2.22}$$

Integrating by parts and using the Cauchy-Schwartz inequality, we have

$$\begin{aligned} \left| u(t) - \frac{1}{t} \int_0^t u(s) ds \right|^2 &= \left| \frac{1}{t} \int_0^t s u'(s) ds \right|^2 \leq \left( \frac{1}{t} \int_0^t s |u'(s)| ds \right)^2 \\ &\leq \frac{1}{t^2} \left( \int_0^t ds \right) \left( \int_0^t s^2 |u'(s)|^2 ds \right) = \frac{1}{t} \int_0^t s^2 |u'(s)|^2 ds. \end{aligned} \tag{2.23}$$

Thus by (2.22),

$$\limsup_{t \rightarrow +\infty} \left| u(t) - \frac{1}{t} \int_0^t u(s) ds \right|^2 \leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t s^2 |u'(s)|^2 ds = 0. \tag{2.24}$$

□

As a corollary to Theorem 2.4, we have the following weak convergence theorem.

**THEOREM 2.5.** *Suppose that the assumptions in Theorems 2.3 and 2.4 are satisfied. Then  $u(t) \rightharpoonup p \in A^{-1}(0)$  as  $t \rightarrow +\infty$ .*

In our next theorem, we prove the strong convergence of  $u$  by assuming  $A$  to be strongly monotone.

**THEOREM 2.6.** *Assume that the operator  $A$  is strongly monotone, and let  $u$  be a solution to (1.1). Then  $u(t)$  converges strongly to  $p \in A^{-1}(0)$  as  $t \rightarrow +\infty$ .*

*Proof.* By the strong monotonicity of  $A$ , and for  $p \in A^{-1}(0)$  (in this case  $A^{-1}(0)$  is a singleton), we have

$$(p(t)u''(t) + r(t)u'(t), u(t) - p) \geq \beta |u(t) - p|^2. \quad (2.25)$$

Let  $K$  be an upper bound for  $p(t)$ . Integrating this inequality from  $t = 0$  to  $t = T$  and using Lemma 2.2, we obtain

$$2\beta \int_0^T |u(t) - p|^2 dt \leq K \left( \frac{d}{dT} |u(T) - p|^2 - 2(u'(0), u(0) - p) + \int_0^T \frac{r(t)}{p(t)} \frac{d}{dt} |u(t) - p|^2 dt \right). \quad (2.26)$$

Let  $R$  be an upper bound for  $|r(t)|$ , which exists by assumption (1.2). Dividing both sides of this inequality by  $T$  and using Lemma 2.2, we get

$$\begin{aligned} 2\beta \lim_{T \rightarrow +\infty} |u(T) - p|^2 &= \limsup_{T \rightarrow +\infty} \frac{\beta}{T} \int_0^T |u(t) - p|^2 dt \\ &\leq \limsup_{T \rightarrow +\infty} \frac{K}{T} \int_0^T \frac{r(t)}{p(t)} \frac{d}{dt} |u(t) - p|^2 dt \\ &\leq \frac{-KR}{\alpha} \limsup_{T \rightarrow +\infty} \frac{1}{T} [ |u(T) - p|^2 - |u(0) - p|^2 ] = 0. \end{aligned} \quad (2.27)$$

This completes the proof of the theorem. □

Now, we apply our results to an example presented by Véron [8] and Apreutesei [10].

*Example 2.7.* Let  $H = L^2(\Omega)$  where  $\Omega \subseteq \mathbb{R}^n$  is a bounded domain with smooth boundary  $\Gamma$ . Let  $j : \mathbb{R} \rightarrow (-\infty, +\infty]$  be proper, convex, and lower semicontinuous and  $\beta = \partial j$ . We assume for simplicity that  $0 \in \beta(0)$ . Define

$$Au = -\Delta u = - \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} \quad (2.28)$$

with

$$D(A) = \left\{ u \in H^2(\Omega), \frac{-\partial u}{\partial \eta}(x) \in \beta(u(x)) \text{ a.e. on } \Gamma \right\}, \quad (2.29)$$

where  $((\partial u / \partial \eta)(x))$  is the outward normal derivative to  $\Gamma$  at  $x \in \Gamma$ . We know that  $A = \partial \phi$ , where  $\phi : L^2(\Omega) \rightarrow (-\infty, +\infty]$  is the Brézis functional:

$$\phi(u) = \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma} \beta(u(x)) d\sigma & \text{if } u \in H^1(\Omega), \beta(u) \in L^1(\Gamma), \\ +\infty & \text{otherwise.} \end{cases} \quad (2.30)$$

Consider the following equation:

$$\begin{aligned}
 p(t) \frac{\partial^2 u}{\partial t^2}(t, x) + r(t) \frac{\partial u}{\partial t}(t, x) + \sum_i \frac{\partial^2 u}{\partial x_i^2}(t, x) &= 0 \quad \text{a.e. on } \mathbb{R}^+ \times \Omega, \\
 -\frac{\partial u}{\partial \eta}(t, x) &\in \beta u(t, x) \quad \text{a.e. on } \mathbb{R}^+ \times \Gamma, \\
 u(0, x) &= u_0(x) \quad \text{a.e. on } \Omega.
 \end{aligned}
 \tag{2.31}$$

Assume that  $p(t)$  and  $r(t)$  are real functions satisfying (1.2) and (1.3). Then Theorem 2.3 implies the weak mean ergodic convergence of  $u(t, \cdot)$ . In addition, if  $r(t) \geq -Mt^{-2}$  eventually, Corollary 2.5 implies the weak convergence of the solution to the above equation.

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