Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 67430, 4 pages
doi:10.1155/2007/67430

Research Article

On Shafer-Fink-Type Inequality

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Received 5 January 2007; Accepted 14 April 2007

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A new simple proof of Shafer-Fink-type inequality proposed by Malešević is given.

1. Introduction

R. E. Shafer (see Mitrinović [1, page 247]) gives us a result as follows.

Theorem 1.1. *Let* $x > 0$. *Then*

$$\arcsin x > \frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} > \frac{3x}{2+\sqrt{1-x^2}}. \tag{1.1}$$

The theorem is generalized by Fink [2] as follows.

Theorem 1.2. *Let* $0 \le x \le 1$. *Then*

$$\frac{3x}{2+\sqrt{1-x^2}} \le \arcsin x \le \frac{\pi x}{2+\sqrt{1-x^2}}. \tag{1.2}$$

Furthermore, 3 and π *are the best constants in (1.2).*

From the theorems above, it is possible to improve the upper bound of inverse sine and deduce the following property (see [3, 4]).

Theorem 1.3. *Let* $0 \le x \le 1$. *Then*

$$\begin{aligned} \frac{3x}{2+\sqrt{1-x^2}} &\le \frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \le \arcsin x \\ &\le \frac{\pi(\sqrt{2}+1/2)(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \le \frac{\pi x}{2+\sqrt{1-x^2}}. \end{aligned} \tag{1.3}$$

Furthermore, 3 and π, 6 and $\pi(\sqrt{2}+1/2)$ are the best constants in (1.3).

Malešević [5, 6] obtained the following theorem by using λ-method and computer separately.

Theorem 1.4. *For all $x \in [0,1]$, the following inequality is valid:*

$$\arcsin x \leq \frac{(\pi(2-\sqrt{2})/(\pi-2\sqrt{2}))(\sqrt{1+x}-\sqrt{1-x})}{\sqrt{2}(\pi-4)/(\pi-2\sqrt{2})+\sqrt{1+x}+\sqrt{1-x}}. \tag{1.4}$$

Recently, Malešević [7] obtains the inequality (1.4) by using further method on computer.

In this paper, we show a new simple proof of inequality (1.4), and obtain the following further result.

Theorem 1.5. *Let $0 \leq x \leq 1$. Then*

$$\frac{6(\sqrt{1+x}-\sqrt{1-x})}{4+\sqrt{1+x}+\sqrt{1-x}} \leq \arcsin x \leq \frac{(\pi(2-\sqrt{2})/(\pi-2\sqrt{2}))(\sqrt{1+x}-\sqrt{1-x})}{\sqrt{2}(\pi-4)/(\pi-2\sqrt{2})+\sqrt{1+x}+\sqrt{1-x}}. \tag{1.5}$$

Furthermore, 4 and $\sqrt{2}(4-\pi)/(\pi-2\sqrt{2})$ are the best constants in (1.5).

2. One lemma: L'Hospital's rule for monotonicity

Lemma 2.1 [8–10]. *Let $f,g:[a,b]\to\mathbb{R}$ be two continuous functions which are differentiable on (a,b). Further, let $g'\neq 0$ on (a,b). If f'/g' is increasing (or decreasing) on (a,b), then the functions*

$$\frac{f(x)-f(b)}{g(x)-g(b)}, \qquad \frac{f(x)-f(a)}{g(x)-g(a)} \tag{2.1}$$

are also increasing (or decreasing) on (a,b).

3. A concise proof of Theorem 1.5

In view of the fact that $(\alpha+2)(\sqrt{1+x}-\sqrt{1-x})/(\alpha+\sqrt{1+x}+\sqrt{1-x}) = \arcsin x = (\beta+2)(\sqrt{1+x}-\sqrt{1-x})/(\beta+\sqrt{1+x}+\sqrt{1-x})$ for $x=0$, the existence of Theorem 1.5 is ensured when the following result is proved.

Corollary 3.1. *Let $0 < x \leq 1$. Then the double inequality*

$$\frac{(\alpha+2)(\sqrt{1+x}-\sqrt{1-x})}{\alpha+\sqrt{1+x}+\sqrt{1-x}} \leq \arcsin x \leq \frac{(\beta+2)(\sqrt{1+x}-\sqrt{1-x})}{\beta+\sqrt{1+x}+\sqrt{1-x}} \tag{3.1}$$

holds if and only if $\alpha \geq 4$ and $\beta \leq \sqrt{2}(4-\pi)/(\pi-2\sqrt{2})$.

Proof of Corollary 3.1. Let

$$G(x) = \frac{2(\sqrt{1+x} - \sqrt{1-x}) - (\sqrt{1+x} + \sqrt{1-x}) \arcsin x}{\arcsin x - (\sqrt{1+x} - \sqrt{1-x})}, \quad x \in (0,1], \tag{3.2}$$

and $\sqrt{1+x} = \sqrt{2}\cos\theta$, $\sqrt{1-x} = \sqrt{2}\sin\theta$, in which case we have $\theta \in [0, \pi/4)$, $x = \cos 2\theta$, and

$$G(x) =: I(\theta) = \frac{4\cos(\theta + \pi/4) - 2(\pi/2 - 2\theta)\sin(\theta + \pi/4)}{(\pi/2) - 2\theta - 2\cos(\theta + \pi/4)}. \tag{3.3}$$

Let $\theta + \pi/4 = \pi/2 - t$, then $t \in (0, \pi/4]$ and

$$G(x) = I(\theta) =: J(t) = 2\frac{\sin t - t\cos t}{t - \sin t} = 2H(t), \tag{3.4}$$

where $H(t) = (\sin t - t\cos t)/(t - \sin t) =: f_1(t)/g_1(t)$, and $f_1(t) = \sin t - t\cos t$, $g_1(t) = t - \sin t$, $f_1(0) = 0$, $g_1(0) = 0$.

Now, processing the monotonicity of the function $H(t)$ on $(0, \pi/4]$, we have

$$\frac{f_1'(t)}{g_1'(t)} = \frac{t\sin t}{1 - \cos t} =: \frac{f_2(t)}{g_2(t)}, \tag{3.5}$$

where $f_2(t) = t\sin t$, $g_1(t) = 1 - \cos t$, and $f_2(0) = 0$, $g_2(0) = 0$. Since $f_2'(t)/g_2'(t) = 1 + t/\tan t$ is decreasing on $(0, \pi/4]$, we find that $H(t)$ is decreasing on $(0, \pi/4]$ by using Lemma 2.1 repeatedly.

So we obtain that $G(x)$ is decreasing on $(0,1]$. Furthermore, $G(0^+) = 4$ and $G(1) = \sqrt{2}(4-\pi)/(\pi - 2\sqrt{2})$. Thus, 4 and $\sqrt{2}(4-\pi)/(\pi - 2\sqrt{2})$ are the best constants in (1.5). □

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