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Research Article

Nonexistence of Positive Solution for Quasilinear Elliptic Problems in the Half-Space

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Liouville-type results in $\mathbb{R}^N$ or in the half-space $\mathbb{R}^N_+$ might be important to obtain a priori estimates for positive solutions of associated problems in bounded domains via some procedure of blow up. In this work, we obtain a nonexistence result for the positive solution of $u^p \leq -\Delta_m u \leq Cu^p$, in the half-space.

1. Introduction

Consider the following problem:

$$-\Delta_m u \geq u^p \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $1 < m < N$ and $m-1 < p < N(m-1)/(N-m)$. Mitidieri and Pohozaev proved in [1], among other results, that problem (1.1) has no positive solution.

On the other hand, as far as we know, there is not a similar result in the half-space $\mathbb{R}^N_+ = \{x = (x_1,\dots,x_N) \in \mathbb{R}^N : x_N > 0\}$.

This kind of results may be used to prove existence results for associated problems in bounded domains: $-\Delta_m u = f(x,u)$ in Ω; $u = 0$ on $\partial\Omega$. This is particularly useful if the problem under consideration is nonvariational (see, e.g., [2–4] and the references therein). Usually these a priori estimates are obtained by using a blow up technique. Suppose by contradiction that there exists a sequence $(u_n)_n$ of solutions of the associated problem, with u_n unbounded (in the L^∞ norm). Let x_n be a point at which u_n attain their maxima. With suitable assumptions on the function f, the blow up methods provide a nontrivial solution of the problem

$$-\Delta_m u \geq u^p, \tag{1.2}$$

in $\mathbb{R}^N$ or in the half-space.

To avoid the case of the half-space, it is assumed in [3] that Ω is convex, f does not depend on x, and $1 < m \leq 2$. These assumptions together with the moving plane method allow to obtain a positive solution of $-\Delta_m u \geq u^p$ in $\mathbb{R}^N$, which is a contradiction with the Liouville result in [1].

In [4], a variant of the blow up technique is proposed, but it is centered on a certain point y_0 instead of on the points x_n. In order to do that, the values of the solutions in different points of Ω are compared through some Harnack-type inequalities (see [4–7]). Using this procedure, the limit problem obtained with the blow up method is defined in all $\mathbb{R}^N$, obtaining again a contradiction with [1].

Nevertheless, it is not used that the limit function also satisfies $-\Delta_m u \leq Cu^p$. In this work, we employ local integral inequalities together with Harnack-type inequalities to prove that these additional assumptions imply the nonexistence of a positive solution of $-\Delta_m u \geq u^p$ in the half-space (Theorem 3.1).

In Section 2, we state a local integral estimate and a Harnack-type inequality. In Section 3, we prove our nonexistence result in $\mathbb{R}^N_+$.

2. Preliminaries

We state two results which will be useful in the next section. The first one is a known local integral estimate (see [4, 6, 8]). Here and in the sequel, by $B(x_0;R)$ we will mean a ball of radius R and center x_0.

Lemma 2.1. *Let u be a positive weak C^1 solution of the inequality*

$$-\Delta_m u \geq u^p, \tag{2.1}$$

in a domain $\Omega \subset \mathbb{R}^N$, where $p > m - 1$. Let $R > 0$ and $x_0 \in 2$ be such that $B(x_0;2R) \subset \Omega$. Then, for any $r \in 2(0,p)$, there exists a positive constant $c = c(N,m,p,)$ such that

$$\int_{B(x_0;R)} u^r \leq cR^{N-mr/(p+1-m)}. \tag{2.2}$$

We will also use the following weak Harnack inequality due to Trudinger [7].

Lemma 2.2. *Let u be a nonnegative weak solution of $-\Delta u \geq 0$ in Ω. Take $\gamma \in (0, N(m-1)/(N-m))$ and $x_0 \in \Omega$ $R > 0$ such that $B(\cdot;2R) \subset \Omega$. Then there exists $C = C(N,m,\gamma)$ such that*

$$\inf_{B(\cdot;R)} u \geq CR^{-N/\gamma}\|u\|_{L^\gamma(B(x_0;2R))}. \tag{2.3}$$

3. Nonexistence in $\mathbb{R}^N_+$

As already mentioned in the introduction, nonexistence results in $\mathbb{R}^N$ or in the half-space might be important to obtain the existence of solutions via some procedure of blow up. Nevertheless, Liouville theorems are often more difficult to obtain in the second case than in the first one.

Consider the following problem:

$$u^p \leq -\Delta_m u \leq Cu^p \quad \text{in } \mathbb{R}^N_+, \tag{3.1}$$

where $C \geq 1$. We have the following result.

Theorem 3.1. *Assume that $m-1 < p < N(m-1)/(N-m)$. Then, there is no positive solution to (3.1) in $C^1(\mathbb{R}^N_+)$.*

Proof. Assume by contradiction that u is a positive solution of (3.1). Take $x_0 \in \mathbb{R}^N_+$ such that $u(x_0) > 0$ and put $\delta = d(x_0, \partial\mathbb{R}^N_+)$. By translation, we may assume that $x_0 = (0,\dots,\delta)$. By continuity of the function u, there are $\widetilde{\delta} \in (0,\delta)$ and $k > 0$ such that

$$u(x) > k > 0 \tag{3.2}$$

for all x in $B(x_0;\widetilde{\delta})$.

Take $\beta > 0$, the functions $v_\beta(x) = \beta u(\beta^{(p+1-m)/m}x)$ also verify (3.1) and

$$v_\beta(x) > k\beta \tag{3.3}$$

for all x in $B(\beta^{-(p+1-m)/m}x_0;\widetilde{\delta}\beta^{-(p+1-m)/m})$.

Now, take $x \in B(\beta^{-(p+1-m)/m}x_0;\widetilde{\delta}\beta^{-(p+1-m)/m})$ and $\beta > 1$, we get

$$\begin{aligned} |x - x_0| &\leq |x - \beta^{-(p+1-m)/m}x_0| + |\beta^{-(p+1-m)/m}x_0 - x_0| \\ &< \widetilde{\delta}\beta^{-(p+1-m)/m} + (1 - \beta^{-(p+1-m)/m})|x_0| < \delta. \end{aligned} \tag{3.4}$$

Thus, $B(\beta^{-(p+1-m)/m}x_0;\widetilde{\delta}\beta^{-(p+1-m)/m}) \subset B(x_0;\delta)$.

In order to apply Lemma 2.2, we note that any function v_β is nonnegative and verifies the inequality $-\Delta_m v_\beta \geq 0$. We choose γ such that $(p+1-m)N/m < \gamma < N(m-1)/(N-m)$, and then by Lemma 2.2 we get

$$\begin{aligned} \min_{B(x_0;\delta/2)} v_\beta &\geq c\delta^{-N/\gamma}\left(\int_{B(x_0;\delta)} v_\beta^\gamma\right)^{1/\gamma} \\ &\geq c\delta^{-N/\gamma}\left(\int_{B(\beta^{-(p+1-m)/m}x_0;\widetilde{\delta}\beta^{-(p+1-m)/m})} v_\beta^\gamma\right)^{1/\gamma} \\ &\geq ck\beta^{(-(p+1-m)N/m+\gamma)/\gamma} \end{aligned} \tag{3.5}$$

for any $\beta > 1$. To conclude the proof, by Lemma 2.1 we have for $r \in (0, p)$,

$$c\delta^N k^r \beta^{(-(p+1-m)N/m+\gamma)r/\gamma} \leq \int_{B(x_0;\delta/2)} v_\beta^r \leq c_1 \delta^{N-mr/(p+1-m)}, \tag{3.6}$$

which is a contradiction for $\beta \to \infty$. □

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