# Research Article <br> Wiener-Hopf Equations Technique for General Variational Inequalities Involving Relaxed Monotone Mappings and Nonexpansive Mappings 

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Received 1 July 2007; Accepted 3 October 2007
Recommended by Simeon Reich

We show that the general variational inequalities are equivalent to the general WienerHopf equations and use this alterative equivalence to suggest and analyze a new iterative method for finding the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the general variational inequality involving multivalued relaxed monotone operators. Our results improve and extend recent ones announced by many others.

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## 1. Introduction

Variational inequalities introduced by Stampacchia [1] in the early sixties have witnessed explosive growth in theoretical advances, algorithmic development, and applications across all disciplines of pure and applied sciences (see [1,2] and the references therein). It combines novel theoretical and algorithmic advances with new domain of applications. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis, and numerical analysis. In recent years, variational inequality theory has been extended and generalized in several directions, using new and powerful methods, to study a wide class of unrelated problems in a unified and general framework. In 1988, Noor [3] introduced and studied a new class of variational inequalities involving two operators, which is known as general variational inequality. We remark that the general variational inequalities are also called Noor variational inequalities. It turned out that oddorder, nonsymmetric obstacle, free, unilateral, nonlinear equilibrium, and moving boundary problems arising in various branches of pure and applied sciences can be studied via Noor variational inequalities (see [3-5]). On the other hand, in 1997, Verma considered the solvability of a new class of variational inequalities involving multivalued
relaxed monotone operators (see [6]). Relaxed monotone operators have applications to constrained hemivariational inequalities. Since in the study of constrained problems in reflexive Banach spaces $E$ the set of all admissible elements is nonconvex but starshaped, corresponding variational formulations are no longer variational inequalities. Using hemivariational inequalities, one can prove the existence of solutions to the following type of nonconvex constrained problems $(P)$ : find $u$ in $C$ such that

$$
\begin{equation*}
\langle A u-g, v\rangle \geq 0, \quad \forall v \in T_{C}(u) \tag{1.1}
\end{equation*}
$$

where the admissible set $C \subset E$ is a star-shaped set with respect to a certain ball $B_{E}\left(u_{0}, \rho\right)$, and $T_{C}(u)$ denotes Clarke's tangent cone of $C$ at $u$ in $C$. It is easily seen that when $C$ is convex, (1.1) reduces to the variational inequality of finding $u$ in $C$ such that

$$
\begin{equation*}
\langle A u-g, v-u\rangle \geq 0, \quad \forall v \in C \tag{1.2}
\end{equation*}
$$

Example 1.1 [7]. Let $A: E \rightarrow E^{*}$ be a maximal monotone operator from a reflexive Banach space $E$ into $E^{*}$ with strong monotonicity and let $C \subset E$ be star-shaped with respect to a ball $B_{E}\left(u_{0}, \rho\right)$. Suppose that $A u_{0}-g \neq 0$ and that distance function $d_{C}$ satisfies the condition of relaxed monotonicity

$$
\begin{equation*}
\left\langle u^{*}-v^{*}, u-v\right\rangle \geq-c\|u-v\|^{2}, \quad \forall u, v \in E, \tag{1.3}
\end{equation*}
$$

and for any $u^{*} \in \partial d_{C}(u)$ and $v^{*} \in \partial d_{C}(v)$ with $c$ satisfying $0<c<4 a^{2} \rho /\left\|A u_{0}-g\right\|^{2}$, where a is the constant for strong monotonicity of $A$. Here, $\partial d_{C}$ is a relaxed monotone operator. Then, the problem $(P)$ has at least one solution.

As a result of interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various numerical methods including projection technique and its variant forms, auxiliary principle and Wiener-Hopf equations for solving variational inequalities and related optimization problems. In this paper, using essentially the projection technique, we show that the general variational inequalities are equivalent to the general Wiener-Hopf equations, whose origin can be traced back to Shi [8]. It has been shown [4, 8-10] that the Wiener-Hopf equations are more flexible and general than the projection methods. Noor $[4,9]$ has used the Wiener-Hopf equations technique to study the sensitivity analysis and the dynamical systems as well as to suggest and analyze several iterative methods for solving variational inequalities.

Related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to consider a unified approach to these two different problems.

Motivated and inspired by the research going on in this direction, we first introduce a new class of the general Wiener-Hopf equations involving. Using the projection technique, we show that the general Wiener-Hopf equations are equivalent to the general variational inequalities. We use this alterative equivalence from the numerical and approximation viewpoints to suggest and analyze an new iterative scheme for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of the general variational inequalities.

## 2. Preliminaries

Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$, whose inner product and norm are denoted by $\langle\cdot, \cdot\rangle$ and $\|\cdot\|$, respectively. Let $T, g: H \rightarrow H$ be two nonlinear operators, $A: H \rightarrow 2^{H}$ a multivalued relaxed monotone operator, and $S_{1}, S_{2}$ two nonexpansive self-mappings of $K$. Let $P_{K}$ be the projection of $H$ into the convex set $K$.

We now consider the problem of finding $u \in H: g(u) \in K$ such that

$$
\begin{equation*}
\langle T u+w, g(v)-g(u)\rangle \geq 0, \quad \forall v \in H: g(v) \in K, w \in A u . \tag{2.1}
\end{equation*}
$$

Note what follows.
(1) If $g \equiv I$, the identity operator, then problem (2.1) is equivalent to finding $u \in K$ such that

$$
\begin{equation*}
\langle T u+w, v-u\rangle \geq 0, \quad \forall v \in K, w \in A u, \tag{2.2}
\end{equation*}
$$

which is considered as the Verma general variational inequality introduced and studied by Verma [6] in 1997. Next, we will denote the set of solutions of the general variational inequality (2.2) by $\operatorname{GVI}(K, T, A)$.
(2) If $w \equiv 0$, then problem (2.1) reduces to finding $u \in H: g(u) \in K$ such that

$$
\begin{equation*}
\langle T u, g(v)-g(u)\rangle \geq 0, \quad \forall v \in H: g(v) \in K, \tag{2.3}
\end{equation*}
$$

which is known as the general variational inequality introduced and studied by Noor [3] in 1988.
(3) If $w \equiv 0$ and $g \equiv I$, the identity operator, then problem (2.1) collapses to finding $u \in K$ such that

$$
\begin{equation*}
\langle T u, v-u\rangle \geq 0, \quad \forall v \in K, \tag{2.4}
\end{equation*}
$$

which is known as the variational inequality problem, originally introduced and studied by Stampacchia [1] in 1964. Next, we will denote the set of solutions of the variational inequality (2.4) by $V I(K, T)$.
Related to the variational inequalities, we have the problems of solving the WienerHopf equations. To be more precise, Let $Q_{K}=I-S P_{K}$, where $P_{K}$ is the projection of $H$ onto the closed convex set $K, I$ is the identity operator, and $S$ is a nonexpansive selfmapping of $K$. If $g^{-1}$ exists, then we consider the problem of finding $z \in H$ such that

$$
\begin{equation*}
T g^{-1} S P_{K} z+w+\rho^{-1} Q_{K} z=0, \quad \forall w \in A g^{-1} S P_{K} z \tag{2.5}
\end{equation*}
$$

where $\rho>0$ is a constant, which is called the general Wiener-Hopf equation involving nonexpansive mappings and multivalued relaxed monotone operators. Next, we denote by $G W H E(H, T, g, S, A)$ the set of solutions of the general Wiener-Hopf equation (2.5).

If $w \equiv 0$, then (2.5) reduces to

$$
\begin{equation*}
T g^{-1} S P_{K} z+\rho^{-1} Q_{K} z=0 \tag{2.6}
\end{equation*}
$$

which is called the general Wiener-Hopf equation involving nonexpansive mappings.
If $w \equiv 0$ and $S \equiv I$, the identity operator, then (2.5) is equivalent to

$$
\begin{equation*}
T g^{-1} P_{K} z+\rho^{-1} Q_{K} z=0 \tag{2.7}
\end{equation*}
$$

where $Q_{K}=I-P_{K}$. Equation (2.7) is considered as the classical general Wiener-Hopf equation (see [4]).

If $w \equiv 0$ and $S \equiv g \equiv I$, the identity operator, then (2.5) collapses to

$$
\begin{equation*}
T P_{K} z+\rho^{-1} Q_{K} z=0 \tag{2.8}
\end{equation*}
$$

which is known as the original Wiener-Hopf equation, introduced by Shi [8]. It is well known that the variational inequalities and Wiener-Hopf equations are equivalent. This equivalence has played a fundamental and basic role in developing some efficient and robust methods for solving variational inequalities and related optimization problems.

We now recall some well-known concepts and results.
Definition 2.1. A mapping $T: K \rightarrow H$ is said to be relaxed $(\gamma, r)$-coercive if there exist two constants $\gamma, r>0$ such that

$$
\begin{equation*}
\langle T x-T y, x-y\rangle \geq(-\gamma)\|T x-T y\|^{2}+r\|x-y\|^{2}, \quad \forall x, y \in K . \tag{2.9}
\end{equation*}
$$

Definition 2.2. A mapping $A: H \rightarrow 2^{H}$ is called $t$-relaxed monotone if there exists a constant $t>0$ such that

$$
\begin{equation*}
\left\langle w_{1}-w_{2}, u-v\right\rangle \geq-t\|u-v\|^{2}, \quad \forall w_{1} \in A u, w_{2} \in A v . \tag{2.10}
\end{equation*}
$$

Definition 2.3. A multivalued mapping $A: H \rightarrow 2^{H}$ is said to be $\mu$-Lipschitzian if there exists a constant $\mu>0$ such that

$$
\begin{equation*}
\left\|w_{1}-w_{2}\right\| \leq \mu\|u-v\|, \quad \forall w_{1} \in A u, w_{2} \in A v \tag{2.11}
\end{equation*}
$$

Lemma 2.4 (Reich [11]). Suppose that $\left\{\delta_{k}\right\}_{k=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality:

$$
\begin{equation*}
\delta_{k+1} \leq\left(1-\lambda_{k}\right) \delta_{k}+\sigma_{k}, \quad k \geq 0 \tag{2.12}
\end{equation*}
$$

with $\lambda_{k} \in[0,1], \sum_{k=0}^{\infty} \lambda_{k}=\infty$, and $\sigma_{k}=\circ\left(\lambda_{k}\right)$. Then, $\lim _{k \rightarrow \infty} \delta_{k}=0$.

Lemma 2.5. For a given $z \in H, u \in K$ satisfies the inequality

$$
\begin{equation*}
\langle u-z, v-u\rangle \geq 0, \quad \forall v \in K \tag{2.13}
\end{equation*}
$$

if and only if $u=P_{K} z$, where $P_{K}$ is the projection of $H$ into $K$.
It is well-known that the projection operator $P_{K}$ is nonexpansive.
Lemma 2.6. The function $u \in H: g(u) \in K$ satisfies the general variational inequality (2.1) if and only if $u \in H$ satisfies the relation

$$
\begin{equation*}
g(u)=P_{K}[g(u)-\rho(T u+w)], \quad \forall w \in A u, \tag{2.14}
\end{equation*}
$$

where $\rho>0$ is a constant and $P_{K}$ is the metric projection of $H$ onto $K$.
Proof. The proof follows from Lemma 2.5.
Remark 2.7. If $u \in G V I(K, T, g, A)$ such that $g(u) \in F\left(S_{1}\right) \subset K$, where $S_{1}$ is nonexpansive self-mapping of $K$, one can easily see that

$$
\begin{equation*}
g(u)=S_{1} g(u)=P_{K}[g(u)-\rho(T u+w)]=S_{1} P_{K}[g(u)-\rho(T u+w)], \tag{2.15}
\end{equation*}
$$

where $\rho>0$ is a constant. If further, assume, $u \in F\left(S_{2}\right)$, where $S_{2}$ is also a nonexpansive self-mapping of $K$, then we obtain

$$
\begin{equation*}
u=\left(1-a_{n}\right) u+a_{n} S_{2} u \tag{2.16}
\end{equation*}
$$

where the sequence $\left\{a_{n}\right\} \subset[0,1]$ for all $n \geq 0$. If $u \in H$ such that $g(u) \in F\left(S_{1}\right)$ is a common element of $F\left(S_{2}\right)$ and $\operatorname{GVI}(K, T, g, A)$, then combining (2.15) with (2.16), we have

$$
\begin{equation*}
u=\left(1-a_{n}\right) u+a_{n} S_{2}\left\{u-g(u)+S_{1} P_{K}[g(u)-\rho(T u+w)]\right\}, \tag{2.17}
\end{equation*}
$$

where $\rho>0$ is a constant and the sequence $\left\{a_{n}\right\} \subset[0,1]$ for all $n>0$.

## 3. Main results

In this section, we use the general Wiener-Hopf equation (2.5) to suggest and analyze a new iterative method for finding the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the general variational inequality (2.1). For this purpose, we need the following result.

Proposition 3.1. The general variational inequality (2.1) has a solution $u \in H$ such that $g(u) \in F\left(S_{1}\right)$ if and only if the general Wiener-Hopf equation (2.5) involving a nonexpansive

## 6 Journal of Inequalities and Applications

self-mapping $S_{1}$ has a solution $z \in H$, where

$$
\begin{gather*}
z=g(u)-\rho(T u+w), \quad w \in A u, \\
g(u)=S_{1} P_{K} z, \tag{3.1}
\end{gather*}
$$

where $P_{K}$ is the projection of $H$ onto $K$ and $\rho>0$ is a constant.
Proof. Pick $u \in \operatorname{GVI}(K, T, g, A)$ such that $g(u) \in F\left(S_{1}\right)$. Observe that (2.15) yields

$$
\begin{equation*}
g(u)=S_{1} P_{K}[g(u)-\rho(T u+w)], \quad \forall w \in A u . \tag{3.2}
\end{equation*}
$$

Let

$$
\begin{equation*}
z=g(u)-\rho(T u+w), \quad \forall w \in A u . \tag{3.3}
\end{equation*}
$$

Combining (3.2) with (3.3), we have

$$
\begin{gather*}
g(u)=S_{1} P_{K} z \\
z=g(u)-\rho(T u+w), \quad \forall w \in A u, \tag{3.4}
\end{gather*}
$$

which yields

$$
\begin{equation*}
z=S_{1} P_{K} z-\rho\left(T g^{-1} S_{1} P_{K} z+w\right), \quad \forall w \in A g^{-1} S_{1} P_{K} z . \tag{3.5}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
T g^{-1} S_{1} P_{K} z+w+\rho^{-1} Q_{K} z=0, \quad \forall w \in A g^{-1} S_{1} P_{K} z \tag{3.6}
\end{equation*}
$$

where $Q_{K}=I-S_{1} P_{K}$.
So, $z \in H$ is a solution of the general Wiener-Hopf equation (2.5). This completes the proof.

Remark 3.2. Observing Proposition 3.1, one can easily see the general variational inequality (2.1) and the general Wiener-Hopf equation (2.5) are equivalent. This equivalence is very useful from the numerical point of view. Using the equivalence and by an appropriate rearrangement, we suggest and analyze a new iterative algorithm for finding the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the general variational inequality.

Algorithm 3.3. The approximate solution $\left\{u_{n}\right\}$ is generated by the following iterative algorithm: $u_{0} \in K$ and

$$
\begin{gather*}
z_{n}=g\left(u_{n}\right)-\rho\left(T u_{n}+w_{n}\right), \\
u_{n+1}=\left(1-a_{n}\right) u_{n}+a_{n} S_{2}\left[u_{n}-g\left(u_{n}\right)+S_{1} P_{K} z_{n}\right], \tag{3.7}
\end{gather*}
$$

where $\left\{a_{n}\right\}$ is a sequence in $[0,1]$ for all $n \geq 0$ and $S_{1}$ and $S_{2}$ are two nonexpansive selfmappings of $K$.

If $\left\{w_{n}\right\} \equiv 0$ and $S_{1} \equiv I$, the identity operator, Algorithm 3.3 reduces to the following algorithm, which is essentially a one-step iterative method refined from Noor [12].

Algorithm 3.4. The approximate solution $\left\{u_{n}\right\}$ is generated by the following iterative algorithm: $u_{0} \in K$ and

$$
\begin{gather*}
z_{n}=g\left(u_{n}\right)-\rho T u_{n} \\
u_{n+1}=\left(1-a_{n}\right) u_{n}+a_{n} S_{2}\left[u_{n}-g\left(u_{n}\right)+P_{K} z_{n}\right] \tag{3.8}
\end{gather*}
$$

where $\left\{a_{n}\right\}$ is a sequence in $[0,1]$ for all $n \geq 0$ and $S_{2}$ is a nonexpansive self-mappings of $K$.

If $\left\{w_{n}\right\} \equiv 0$ and $g \equiv S_{1} \equiv I$, the identity operator, Algorithm 3.3 reduces to the following algorithm.

Algorithm 3.5. The approximate solution $\left\{u_{n}\right\}$ is generated by the following iterative algorithm: $u_{0} \in K$ and

$$
\begin{gather*}
z_{n}=u_{n}-\rho T u_{n}, \\
u_{n+1}=\left(1-a_{n}\right) u_{n}+a_{n} S_{2} P_{K} z_{n}, \tag{3.9}
\end{gather*}
$$

where $\left\{a_{n}\right\}$ is a sequence in $[0,1]$ for all $n \geq 0$ and $S_{2}$ is a nonexpansive self-mappings of $K$.

If the mapping $T$ is $\alpha$-inverse strongly monotone mapping, then Algorithm 3.5 can be viewed as Takahashi and Toyoda's [2].

If $\left\{a_{n}\right\}=1,\left\{w_{n}\right\} \equiv 0$, and $g=S_{1}=S_{2}=I$, the identity operator, Algorithm 3.3 reduces to the following algorithm, which was considered by Noor [4].

Algorithm 3.6. The approximate solution $\left\{u_{n}\right\}$ is generated by the following iterative algorithm: $u_{0} \in K$ and

$$
\begin{gather*}
z_{n}=u_{n}-\rho T u_{n}  \tag{3.10}\\
u_{n+1}=P_{K} z_{n}
\end{gather*}
$$

where $\left\{a_{n}\right\}$ is a sequence in $[0,1]$ for all $n \geq 0$.
If $\left\{a_{n}\right\}=1$ and $g=S_{1}=S_{2}=I$, the identity operator, Algorithm 3.3 collapses to the following algorithm, which was studied by Verma [6].

Algorithm 3.7. Given $u_{0} \in H$, the approximate solution $\left\{u_{n}\right\}$ is generated by the following iterative algorithm:

$$
\begin{equation*}
u_{n+1}=P_{K}\left[u_{n}-\rho\left(T u_{n}+w_{n}\right)\right] . \tag{3.11}
\end{equation*}
$$

Theorem 3.8. Let $K$ be a nonempty closed convex subset of a real Hilbert space H. Let $T: K \rightarrow H$ be a relaxed $\left(\gamma_{1}, r_{1}\right)$-coercive and $\mu_{1}$-Lipschitz continuous mapping, $g: K \rightarrow H$ a relaxed ( $\gamma_{2}, r_{2}$ )-coercive and $\mu_{2}$-Lipschitz continuous mapping, $A: H \rightarrow 2^{H}$ a $t$-relaxed monotone and $\mu_{3}$-Lipschitz continuous mapping, and $S_{1}, S_{2}$ two nonexpansive self-mappings of $K$ such that $F\left(S_{1}\right) \neq \varnothing, F\left(S_{2}\right) \cap \operatorname{GVI}(K, T, g, A) \neq \varnothing$, and $\operatorname{GWHE}(H, T, g, S, A) \neq \varnothing$, respectively. Let $\left\{z_{n}\right\},\left\{u_{n}\right\}$, and $\left\{g\left(u_{n}\right)\right\}$ be sequences generated by Algorithm 3.3, where $\left\{\alpha_{n}\right\}$ is a
sequence in $[0,1]$. Assume that the following conditions are satisfied:
(C1) $\theta=k_{1}+2 k_{2}<1$,
where $k_{1}=\sqrt{1+2 \rho\left(\gamma_{1} \mu_{1}^{2}-r_{1}+t\right)+\rho^{2}\left(\mu_{1}+\mu_{3}\right)^{2}}$ and $k_{2}=\sqrt{1+2 \mu_{2}^{2} \gamma_{2}-2 r_{2}+\mu_{2}^{2}}$;
(C2) $\sum_{n=0}^{\infty}, a_{n}=\infty$.
Then, the sequences $\left\{z_{n}\right\},\left\{u_{n}\right\}$, and $\left\{g\left(u_{n}\right)\right\}$ converge strongly to $z \in G W H E(H$, $\left.T, g, S_{1}, A\right), u \in F\left(S_{2}\right) \cap G V I(K, T, g, A)$, and $g(u) \in F\left(S_{1}\right)$, respectively.

Proof. Let $z \in H$ be an element of $G W H E\left(H, T, g, S_{1}, A\right)$ and $u \in F\left(S_{2}\right) \cap G V I(K, T, g, A)$ such that $g(u) \in F\left(S_{1}\right)$. From (2.17) and Proposition 3.1, we have

$$
\begin{gather*}
z=g(u)-\rho\left(T_{u}+w\right), \\
u=\left(1-a_{n}\right) u+a_{n} S_{2}\left[u-g(u)+S_{1} P_{K} z\right] . \tag{3.12}
\end{gather*}
$$

First, we estimate that $\left\|u_{n+1}-u\right\|$. From (3.7) and (3.12), we obtain

$$
\begin{align*}
& \left\|u_{n+1}-u\right\| \\
& \quad=\left\|\left(1-a_{n}\right) u_{n}+a_{n} S_{2}\left[u_{n}-g\left(u_{n}\right)+S_{1} P_{K} z_{n}\right]-u\right\| \\
& \quad \leq\left(1-a_{n}\right)\left\|u_{n}-u\right\|+a_{n}\left\|S_{2}\left[u_{n}-g\left(u_{n}\right)+S_{1} P_{K} z_{n}\right]-S_{2}\left[u-g(u)+S_{1} P_{K} z\right]\right\|  \tag{3.13}\\
& \quad \leq\left(1-a_{n}\right)\left\|u_{n}-u\right\|+a_{n}\left\|\left(u_{n}-u\right)-\left[g\left(u_{n}\right)-g(u)\right]\right\|+a_{n}\left\|z_{n}-z\right\| .
\end{align*}
$$

Next, we evaluate $\|\left(u_{n}-u\right)-\left[g\left(u_{n}\right)-g(u) \|\right.$. By the relaxed $\left(\gamma_{2}, r_{2}\right)$-coercive and $\mu_{2}{ }^{-}$ Lipschitzian definition on $g$, we have

$$
\begin{align*}
& \left\|\left(u_{n}-u\right)-\left[g\left(u_{n}\right)-g(u)\right]\right\|^{2} \\
& \quad=\left\|u_{n}-u\right\|^{2}-2\left\langle g\left(u_{n}\right)-g(u), u_{n}-u\right\rangle+\left\|g\left(u_{n}\right)-g(u)\right\|^{2} \\
& \quad \leq\left\|u_{n}-u\right\|^{2}-2\left[-\gamma_{2}\left\|g\left(u_{n}\right)-g(u)\right\|^{2}+r_{2}\left\|u_{n}-u\right\|^{2}\right]+\mu_{2}^{2}\left\|u_{n}-u\right\|^{2}  \tag{3.14}\\
& \quad \leq\left(1+2 \mu_{2}^{2} \gamma_{2}-2 r_{2}+\mu_{2}^{2}\right)\left\|u_{n}-u\right\|^{2}=k_{2}^{2}\left\|u_{n}-u\right\|^{2},
\end{align*}
$$

where $k_{2}=\sqrt{1+2 \mu_{2}^{2} \gamma_{2}-2 r_{2}+\mu_{2}^{2}}$. Next, we evaluate $\left\|z_{n}-z\right\|$. In a similar way, using the relaxed ( $\gamma_{1}, r_{1}$ )-coercive and $\mu_{1}$-Lipschitzian definition on Tand the $t$-relaxed monotone and $\mu_{3}$-Lipschitzian definition on $A$, we have

$$
\begin{align*}
&\left\|\left(u_{n}-u\right)-\rho\left[\left(T u_{n}+w_{n}\right)-\left(T_{u}+w\right)\right]\right\|^{2} \\
&=\left\|u_{n}-u\right\|^{2}-2 \rho\left\langle T u_{n}+w_{n}-(T u+w), u_{n}-u\right\rangle+\rho^{2}\left\|\left(T u_{n}+w_{n}\right)-(T u+w)\right\|^{2} \\
& \leq\left\|u_{n}-u\right\|^{2}-2 \rho\left(\left\langle T u_{n}-T u, u_{n}-u\right\rangle+\left\langle w_{n}-w, u_{n}-u\right\rangle\right) \\
& \quad+\rho^{2}\left(\left\|T u_{n}-T u\right\|+\left\|w_{n}-w\right\|\right)^{2} \\
& \leq {\left[1+2 \rho\left(\gamma_{1} \mu_{1}^{2}-r_{1}+t\right)+\rho^{2}\left(\mu_{1}+\mu_{3}\right)^{2}\right]\left\|u_{n}-u\right\|^{2}=k_{1}^{2}\left\|u_{n}-u\right\|^{2}, } \tag{3.15}
\end{align*}
$$

where $k_{1}=\sqrt{1+2 \rho\left(\gamma_{1} \mu_{1}^{2}-r_{1}+t\right)+\rho^{2}\left(\mu_{1}+\mu_{3}\right)^{2}}$. From (3.7) and (3.12), we have

$$
\begin{align*}
\left\|z_{n}-z\right\| & =\left\|g\left(u_{n}\right)-g(u)-\rho\left[\left(T u_{n}+w_{n}\right)-(T u+w)\right]\right\| \\
& \leq\left\|u_{n}-u-\left[g\left(u_{n}\right)-g(u)\right]\right\|+\left\|u_{n}-u-\rho\left[\left(T u_{n}+w_{n}\right)-(T u+w)\right]\right\| . \tag{3.16}
\end{align*}
$$

Now, substituting (3.14) and (3.15) into (3.16), we have

$$
\begin{equation*}
\left\|z_{n}-z\right\| \leq\left(k_{1}+k_{2}\right)\left\|u_{n}-u\right\| . \tag{3.17}
\end{equation*}
$$

Substituting (3.14) and (3.17) into (3.13), we have

$$
\begin{align*}
\left\|u_{n+1}-u\right\| & \leq\left[1-\left(1-k_{1}-2 k_{2}\right) a_{n}\right]\left\|u_{n}-u\right\| \\
& =\left[1-(1-\theta) a_{n}\right]\left\|u_{n}-u\right\|, \tag{3.18}
\end{align*}
$$

where $\theta=k_{1}+2 k_{2}<1$. Thus, from (C1), (C2) and Lemma 2.4, we have $\lim _{n \rightarrow \infty} \| u_{n}-$ $u \|=0$. Also from (3.17), we have $\lim _{n \rightarrow \infty}\left\|z_{n}-z\right\|=0$. On the other hand, we have

$$
\begin{equation*}
\left\|g\left(u_{n}\right)-g(u)\right\| \leq \mu_{2}\left\|u_{n}-u\right\| . \tag{3.19}
\end{equation*}
$$

It follows that $\lim _{n \rightarrow \infty}\left\|g\left(u_{n}\right)-g(u)\right\|=0$. This completes the proof.
Remark 3.9. In this paper, we show that the general variational inequalities involving three nonlinear operators are equivalent to a new class of general Wiener-Hopf equations. The iterative methods suggested and analyzed in this paper are very convenient and are reasonably easy to use for the computation. It is interesting to use the technique in this paper to develop other new iterative methods for solving the general variational inequalities in different directions.

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