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Research Article

On Complex Oscillation Property of Solutions for Higher-Order Periodic Differential Equations

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We investigate properties of the zeros of solutions for higher-order periodic differential equations, and prove that under certain hypotheses, the convergence exponent of zeros of the product of two linearly independent solutions is infinite.

1. Introduction and results

Consider the zeros of solutions of linear differential equations with periodic coefficients, for the second-order equation

$$f'' + A(z)f = 0, \tag{1.1}$$

where A is entire and nonconstant with period ω; a number of results have been obtained in [1, 2]. For the higher-order differential equation

$$f^{(k)} + A_{k-2}f^{(k-2)} + \cdots + A_0 f = 0. \tag{1.2}$$

Bank and Langley proved the following theorems in [3].

Theorem 1.1. *Let $k \geq 2$ be an integer, $A_0, \dots, A_{k-2}$ be entire periodic functions with period $2\pi i$, such that A_0 is transcendental in e^z with*

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, A_0)}{r} = c < \frac{1}{2}, \tag{1.3}$$

and for each j with $1 \le j \le k-2$, the coefficient A_j either is rational in e^z or satisfies

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r,A_j)}{r} < c. \tag{1.4}$$

Then (1.2) cannot have linearly independent solutions f_1, f_2 satisfying

$$\log^+ N\left(r, \frac{1}{f_1 f_2}\right) = O(r). \tag{1.5}$$

Theorem 1.2. *Suppose that $k \ge 2$ and $A_0, \dots, A_{k-2}$ are entire functions of period $2\pi i$, and that f is a nontrivial solution of a differential equation (1.2). Suppose further that f satisfies*

$$\log^+ N\left(r, \frac{1}{f}\right) = o(r), \tag{1.6}$$

A_0 is nonconstant and rational in e^z, and if $k \ge 3$ then $A_1, \dots, A_{k-2}$ are constants. Then there exists an integer q with $1 \le q \le k$, such that $f(z)$ and $f(z + q2\pi i)$ are linearly dependent.

The same conclusion holds if A_0 is transcendental in e^z and f satisfies

$$\log^+ N\left(r, \frac{1}{f}\right) = O(r), \tag{1.7}$$

and if $k \ge 3$, then as $r \to +\infty$ through a set L_1 of infinite linear measure, we have

$$T(r, A_j) = o(T(r, A_0)) \quad (j = 1, \dots, k-2). \tag{1.8}$$

In this paper, we will assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's value distribution theory of meromorphic functions (e.g., see [4, 5]). In addition, we will use $\sigma(f)$ and $\mu(f)$ to denote, respectively, the order and the lower order of meromorphic function $f(z)$, $\lambda(f)$ to denote the convergence exponent of zeros of $f(z)$.

Let $A(z)$ be an entire function. We define

$$\sigma_e(A) = \overline{\lim_{r\to\infty}} \frac{\log T(r,A)}{r} \tag{1.9}$$

to be the e-type order of $A(z)$. Clearly,

$$\sigma_e(A) = \overline{\lim_{r\to\infty}} \frac{\log\log M(r,A)}{r}. \tag{1.10}$$

The main aim of this paper is to improve the result of Theorem 1.1. In the following theorem (Theorem 1.3), we weaken the conditions (1.3) and (1.4) of Theorem 1.1. In particular, in Corollary 1.4, the condition $\sigma(G_0) < 1/2$, $\sigma(g_j) < \max\{\sigma(G_0), \sigma(g_0)\}$, is weaker than that of Theorem 1.1, by Remark 2.3, we see that this condition in Corollary 1.4 shows that $\sigma_e(A_0)$ may be arbitrary, that is, in Corollary 1.4, the restriction "$c < 1/2$" of Theorem 1.1 is redundant. Thus, Theorem 1.3 and Corollary 1.4 improve essentially the result of Theorem 1.1.

The other aim of this paper is to consider what condition will guarantee that every solution $f \not\equiv 0$ of (1.2) satisfies $\lambda(f) = \infty$. In Theorem 1.6 and Corollaries 1.7 and 1.8, we prove that under certain hypotheses, every solution $f(\not\equiv 0)$ of (1.2) satisfies (1.14), so $\lambda(f) = \infty$.

Theorem 1.3. *Let $k \geq 2$ and $A_j(z) = B_j(e^z) = B_j(\zeta)$, $\zeta = e^z$, $B_j(\zeta) = G_j(\zeta) + g_j(1/\zeta)$, $j = 0,1,\dots,k-2$, where $G_j(t)$ and $g_j(t)$ are entire functions. Suppose the following:*

(i) *$G_0(t)$ is transcendental and $\sigma(G_0) < \infty$ if $\sigma(G_0) > 0$, then G_0 also satisfies that for any τ satisfying $0 < \tau < \sigma(G_0)$, there exists a subset $H \subset (1,+\infty)$ with infinite logarithmic measure, such that when $|t| = r \in H$,*

$$\log|G_0(t)| > r^{\tau}; \tag{1.11}$$

(ii) *for $j > 0, G_j(t)$ either is a polynomial or $\sigma(G_j) < \sigma(G_0)$;*

(iii) *for $j > 0, g_j(t)$ either is a polynomial or $\sigma(g_j) < \max\{\sigma(G_0),\ \sigma(g_0)\}$, where $g_0(t)$ is arbitrary entire function.*

Then (1.2) cannot have linearly independent solutions f_1, f_2 satisfying (1.5).

The same conclusion remains valid if $G_j(t)$ and $g_j(t)$ $(j = 0,\dots,k-2)$ are transposed in the hypotheses (i)–(iii) above.

Corollary 1.4. *Let $k \geq 2$ and $A_j(z) = B_j(e^z) = B_j(\zeta)$, $\zeta = e^z$, $B_j(\zeta) = G_j(\zeta) + g_j(1/\zeta)$, $j = 0,1,\dots,k-2$, where $G_j(t)$ and $g_j(t)$ are entire functions. Suppose the following:*

(i)* *$G_0(t)$ is transcendental with $\sigma(G_0) < 1/2$;*

(ii)* *for $j > 0, G_j(t)$ either is a polynomial or $\sigma(G_j) < \sigma(G_0)$;*

(iii)* *for $j > 0, g_j(t)$ either is a polynomial or $\sigma(g_j) < \max\{\sigma(G_0),\ \sigma(g_0)\}$.*

Then (1.2) cannot have linearly independent solutions f_1, f_2 satisfying (1.5).

The same conclusion remains valid if $G_j(t)$ and $g_j(t)$ $(j = 0,\dots,k-2)$ are transposed in the hypotheses (i)–(iii)* above.*

We introduce the concept of gap power series before we state Corollary 1.5. An entire function f is said to be a gap power series if $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, where $\{\lambda_n\}$ is a increasing sequence of positive integers, f is said to have a *Fabry gap* if

$$\overline{\lim_{n\to\infty}} \frac{n}{\lambda_n} = 0. \tag{1.12}$$

Corollary 1.5. *Assume that the hypotheses of Corollary 1.4 are satisfied but the statements (i)* and (ii)* are replaced, respectively, by the following:*

(i)** *$G_0(t)$ is an entire function with Fabry gap with $\sigma(G_0) < +\infty$;*

(ii)** *for $j > 0$, either $G_j(t)$ is a polynomial or $\sigma(G_j) < \mu(G_0)$.*

Then the conclusion of Corollary 1.4 remains valid.

Theorem 1.6. *Let $k \geq 2$ and $A_j(z) = B_j(e^z) = B_j(\zeta)$, $\zeta = e^z$, $B_j(\zeta) = G_j(\zeta) + g_j(1/\zeta)$, $j = 0,1,\dots,k-2$, where $G_j(t)$ and $g_j(t)$ are entire functions. Suppose the following:*

(1) *$g_0(t)$ is transcendental and $\sigma(g_0) < \infty$, if $\sigma(g_0) > 0$, then for any τ satisfying $0 < \tau < \sigma(g_0)$, there exists a subset $H \subset (1,+\infty)$ with infinite logarithmic measure, such that when $|t| = r \in H$,*

$$\log|g_0(t)| > r^{\tau}; \tag{1.13}$$

(2) *for* $j > 0$, *either* $g_j(t)$ *is a polynomial or* $\sigma(g_j) < \sigma(g_0)$;

(3) *for* $j \geq 0, G_j(t)$ *is polynomial of degree* p_j *such that* $0 \leq p_s < \min\{k-s, p_0\}$ $(s = 1,\dots,k-2)$ *and* p_0 *is not divisible by* k.

Then every nontrivial solution f *of (1.2) must have* $\lambda(f) = \infty$, *and in fact, the stronger conclusion*

$$\log^{+} N\left(r, \frac{1}{f}\right) \neq o(r) \quad (r \longrightarrow \infty) \tag{1.14}$$

holds.

The same conclusion remains valid if $G_j(t)$ *and* $g_j(t)$ $(j = 0,\dots,k-2)$ *are transposed in the hypotheses (1)–(3) above.*

Corollary 1.7. *Let* $k \geq 2$ *and* $A_j(z) = B_j(e^z) = B_j(\zeta)$, $\zeta = e^z$, $B_j(\zeta) = G_j(\zeta) + g_j(1/\zeta)$, $j = 0,1,\dots,k-2$, *where* $G_j(t)$ *and* $g_j(t)$ *are entire functions. Suppose the following:*

$(1)^*$ $g_0(t)$ *is transcendental and* $\sigma(g_0) < 1/2$;

$(2)^*$ *for* $j > 0$, *either* $g_j(t)$ *is a polynomial or* $\sigma(g_j) < \sigma(g_0)$;

$(3)^*$ *for* $j \geq 0$, $G_j(t)$ *is a polynomial of degree* p_j *such that* $0 \leq p_s < \min\{k-s, p_0\}$ $(s = 1,\dots,k-2)$ *and* p_0 *is not divisible by* k.

Then every nontrivial solution f *of (1.2) must have* $\lambda(f) = \infty$, *and in fact, the stronger conclusion (1.14) holds.*

The same conclusion remains valid if $G_j(t)$ *and* $g_j(t)$ $(j = 0,\dots,k-2)$ *are transposed in the hypotheses* $(1)^*$–$(3)^*$ *above.*

Corollary 1.8. *Assume that the hypotheses of Corollary 1.7 are satisfied but the statements* $(1)^*$ *and* $(2)^*$ *are replaced, respectively, by the following:*

$(1)^{**}$ $G_0(t)$ *is an entire function with Fabry gap with* $\sigma(G_0) < +\infty$;

$(2)^{**}$ *for* $j > 0$, *either* $G_j(t)$ *is a polynomial or* $\sigma(G_j) < \mu(G_0)$.

Then the conclusion of Corollary 1.7 remains valid.

2. Lemmas for the proof of Theorem 1.3

Lemma 2.1 (see [6]). *Let* f *be a transcendental meromorphic function with* $\sigma(f) = \sigma < \infty$. *Let* $H = \{(k_1, j_1), (k_2, j_2), \dots, (k_q, j_q)\}$ *be a finite set of distinct pairs of integers that satisfy* $k_i > j_i \geq 0$ *for* $i = 1,\dots,q$. *Also let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E_1 \subset (1,\infty)$ *with finite logarithmic measure such that for all* z *satisfying* $|z| \notin [0,1] \cup E$ *and for all* $(k,j) \in H$ *one has*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.1}$$

Remark 2.2. Let $g(\zeta)$ be a function analytic in $R_0 < |\zeta| < \infty$. By [7, page 15], $g(\zeta)$ can be represented as

$$g(\zeta) = \zeta^m \psi(\zeta) F(\zeta), \tag{2.2}$$

where $\psi(\zeta)$ is analytic and does not vanish in $R_0 < |\zeta| \leq \infty$ and $\psi(\infty) = 1$, F is an entire function and

$$F(\zeta) = u(\zeta)e^{h(\zeta)}, \tag{2.3}$$

where the function $u(\zeta)$ is a Weierstrass product formed by the zeros of $g(\zeta)$ in $R_0 < |\zeta| < \infty$, $h(\zeta)$ is an entire function. If $u(\zeta)$ is of finite order of growth, set $W(\zeta) = \psi(\zeta)u(\zeta)$, since as $\zeta \to \infty$, $\psi^{(j)}(\zeta)/\psi(\zeta) = o(1)$, by Lemma 2.1, it is easy to see that there exists a subset $E_1 \subset (0, \infty)$ having finite logarithmic measure and a constant $M_1(> 0)$, such that for all ζ satisfying $|\zeta| \notin E_1$,

$$\left| \frac{W^{(j)}(\zeta)}{W(\zeta)} \right| \leq |\zeta|^{M_1}. \tag{2.4}$$

Remark 2.3. By [8, page 276], we know that if $A(z)$ is an entire function and $A(z) = B(e^z) = B(\zeta) = G(\zeta) + g(1/\zeta)$, where $G(t)$ and $g(t)$ are entire functions, then

$$\sigma_e(A) = \max\{\sigma(G), \sigma(g)\}. \tag{2.5}$$

Lemma 2.4 (see [3]). *Let $A(z)$ be a nonconstant entire function with period $2\pi i$. Then*

$$c = \underline{\lim}_{r\to\infty} \frac{T(r,A)}{r} > 0. \tag{2.6}$$

If c is finite, then $A(z)$ is rational in e^z.

We easily prove the following lemma.

Lemma 2.5. *Let $A_j(z)$ $(j = 1,2)$ be entire functions with $A_j(z) = B_j(e^z) = B_j(t)$, $t = e^z$. If $B_1(t)$ is transcendental (i.e., Laurent's expansion of $B_1(t)$ is of infinitely many terms) and $B_2(t)$ is rational, then*

$$T(r, A_2) = o\{T(r, A_1)\}. \tag{2.7}$$

Lemma 2.6. *Suppose that A_j, B_j, G_j, g_j $(j = 0,\dots,k-2)$ satisfy the hypotheses of Theorem 1.3. If $f(z)(\not\equiv 0)$ is a solution of (1.2) and satisfies (1.7), then in $1 < |\xi| < \infty$, $f(z)$ can be represented as*

$$f(z) = \xi^d \psi(\xi)u(\xi)e^{h(\xi)}, \tag{2.8}$$

where $\xi = e^{z/q}$, q is an integer and satisfies $1 \leq q \leq k$, d is some constant, $\psi(\xi)$ is analytic and does not vanish in $1 < |\xi| \leq \infty$ and $\psi(\infty) = 1$, both $u(\xi)$ and $h(\xi)$ are entire functions of finite order.

If $G_j(t)$ and $g_j(t)$ $(j = 0,\dots,k-2)$ are transposed in (i)–(iii), then the same conclusion still holds with $\xi = e^{-z/q}$.

Proof. By Remark 2.3 we see that

$$\sigma_e(A_j) = \max\{\sigma(G_j),\ \sigma(g_j)\}. \tag{2.9}$$

By the hypotheses (i)–(iii) of Theorem 1.3, we easily see that if $\sigma_e(A_0) > 0$, then there exists a set $H \subset (0,\infty)$ of infinite linear measure, such that

$$T(r,A_j) = o\{T(r,A_0)\}, \quad r \in H; \tag{2.10}$$

if $\sigma_e(A_0) = 0$, then by Lemma 2.5,

$$T(r,A_j) = o\{T(r,A_0)\} \quad (j = 1,\dots,k-2). \tag{2.11}$$

Now suppose that $f(\not\equiv 0)$ is solution of (1.2) and satisfies (1.7). By (2.10), (2.11), and Theorem 1.2, we see that there exists an integer $q : 1 \le q \le k$ such that $f(z)$ and $f(z+q2\pi i)$ are linearly dependent. By [9, page 382], we see that $f(z)$ can be represented as

$$f(z) = e^{d_1 z} G(e^{z/q}), \tag{2.12}$$

where $G(\xi)$ is analytic in $0 < |\xi| < \infty$, $\xi = e^{z/q}$. By Remark 2.2, we see that in $1 < |\xi| < \infty$, $G(\xi)$ may be represented as

$$G(\xi) = \xi^m \psi(\xi) u(\xi) e^{h(\xi)}, \tag{2.13}$$

where m is an integer, $\psi(\xi)$ is analytic and does not vanish in $1 < |\xi| \le \infty$ and $\psi(\infty) = 1$, $u(\xi)$ is a Weierstrass product formed by the zeros of $G(\xi)$ in $1 < |\xi| < \infty$, $h(\xi)$ is an entire function, hence (2.8) holds.

Firstly, we prove that $u(\xi)$ is of finite order of growth. By the transformation $\xi = e^{z/q}$ and (1.7), the counting function $N_1(\rho,1/G)$ of $G(\xi)$ in $1 < |\xi| < \infty$ satisfies $\log^+ N_1(\rho,1/G) = O(\log\rho)$. So that $u(\xi)$ is an entire function of finite order.

Secondly, we prove that $h(\xi)$ is of finite order of growth. Set $W(\xi) = \psi(\xi)u(\xi)$, then

$$f(z) = \xi^d W(\xi) e^{h(\xi)}. \tag{2.14}$$

Substituting (2.14) into (1.2), we obtain

$$(h')^k + P_{k-1}(h') = 0, \tag{2.15}$$

where $P_{k-1}(h')$ is a differential polynomial in h' of total degree $k-1$, its coefficients are polynomials in $W^{(s)}/W$ $(s = 1,\dots,k)$, $1\xi^m$ $(m = 1,\dots,k-1)$, $A_j(z)$ $(j = 0,\dots,k-2)$.

By Remark 2.2, we see that there exists a subset $E_1 \subset (0,\infty)$ with finite logarithmic measure and a constant M_1, such that for all ξ satisfying $|\xi| \notin E_1$, and for $s = 1,\dots,k$, $m = 1,\dots,k-1$,

$$\left| \frac{1}{\xi^m} \frac{W^{(s)}(\xi)}{W(\xi)} \right| \le |\xi|^{M_1}. \tag{2.16}$$

By (2.15) and (2.16), we obtain

$$m(\rho,h') \le M\{m(\rho,G_0(\xi^q)) + \log m(\rho,h') + \log\rho\}, \tag{2.17}$$

where $\rho \notin E_2$, $E_2 \subset [0,\infty)$ is a set of finite linear measure, $M(>0)$ is a constant. Since $G_0(t)$ is of finite order, by (2.17), we see that $h(\xi)$ is of finite order.

If $G_j(t)$ and $g_j(t)$ $(j = 0,\dots,k-2)$ are transposed in (i)–(iii), we can still deduce the same conclusion by setting $\zeta = 1/\eta$, $G_j^*(\eta) = g_j(\eta) = g_j(1/\zeta)$, $g_j^*(\eta) = G_j(1/\eta) = G_j(\zeta)$ $(j = 0,\dots,k-2)$, and noting that $G_j^*(\eta)$ and $g_j^*(\eta)$ satisfy (i), (ii), and (iii), respectively, $A_j(z) = B_j(\zeta) = B_j(1/\eta) = G_j^*(\eta) + g_j^*(1/\eta)$. In the previous argument, $G_j(t)$ and $g_j(t)$ are replaced, respectively, by $G_j^*(t)$ and $g_j^*(t)$. Thus, Lemma 2.6 is proved. □

Remark 2.7 (see [10, 11]). Let $h(z)$ be a transcendental entire function with order $\sigma(h) = \sigma < 1/2$. Then there exists a subset $H \subset (1,\infty)$ having infinite logarithmic measure, such that if $\sigma = 0$, then

$$\frac{\min\{\log|h(z)| : |z| = r\}}{\log r} \longrightarrow \infty \quad (|z| = r \in H,\ r \longrightarrow \infty); \tag{2.18}$$

if $\sigma > 0$, then for any α $(0 < \alpha < \sigma)$,

$$\log|h(z)| > r^{\alpha} \quad (|z| = r \in H,\ r \longrightarrow \infty). \tag{2.19}$$

3. Proof of Theorem 1.3

Suppose that (1.2) has two linearly independent solutions $f_1(z)$ and $f_2(z)$ that satisfy (1.5), then both f_1, f_2 satisfy (1.7). We deduce immediately from Lemma 2.6 that both $f_1(z)$ and $f_2(z)$ have representations in the form (2.15). In particular, we can choose an integer $q : 1 \le q \le k^2$, according to (2.14) the representations can be written as

$$f_1(z) = \xi^{d_1} W_1(\xi) e^{h_1(\xi)}, \qquad f_2(z) = \xi^{d_2} W_2(\xi) e^{h_2(\xi)}, \tag{3.1}$$

where d_j $(j = 1,2)$ are two constants, $\xi = e^{z/q}$, $W_j(\xi) = \psi_j(\xi) u_j(\xi)$ $(j = 1,2)$, $\psi_j(\xi)$ is analytic in $1 < |\xi| \le \infty$, and $\psi_j(\xi) \ne 0$, $\psi_j(\infty) \ne 0$, $u_j(\xi)$, and $h_j(\xi)$ are all entire functions of finite order. By Remark 2.2, there exists a subset $E_1 \subset (0,\infty)$ having finite logarithmic measure and a constant M $(0 < M < \infty$, M is not necessarily the same at each occurrence), such that for all ξ satisfying $|\xi| \notin E_1$, and for $s = 1,\dots,k$, $m = 1,\dots,k$,

$$\left|\xi^s \frac{W_1^{(m)}(\xi)}{W_1(\xi)}\right| + \left|\xi^s \frac{W_2^{(m)}(\xi)}{W_2(\xi)}\right| + \left|\xi^s \frac{h_1^{(m)}(\xi)}{h_1'(\xi)}\right| + \left|\xi^s \frac{h_2^{(m)}(\xi)}{h_2'(\xi)}\right| \le |\xi|^M. \tag{3.2}$$

If $\sigma(G_0) = 0$, then by Remark 2.7 we see that there exists a subset $H \subset (1,\infty)$ having infinite logarithmic measure, such that

$$\frac{\min\{\log|G_0(t)| : |t| = r\}}{\log r} \longrightarrow \infty \quad (r \in H,\ r \longrightarrow \infty), \tag{3.3}$$

and $G_j(t)$ $(j = 1,\dots,k-2)$ are polynomials on t, hence there is a constant M that satisfies

$$|G_j(t)| \le r^M \quad (|t| = r \longrightarrow \infty). \tag{3.4}$$

If $\sigma(G_0) > 0$, then by the hypothesis (i), we see that there exists a subset $H \subset (1,\infty)$ having infinite logarithmic measure (for convenience, we still assume that the subset with infinite logarithmic measure in the hypothesis (i) is H), and $\delta, \tau > 0$, such that for $j > 0$,

$$\begin{gathered} \sigma(G_j) < \delta < \tau < \sigma(G_0), \\ \log|G_j(t)| < r^{\delta} < r^{\tau} < \log|G_0(t)|, \quad (|t| = r \in H). \end{gathered} \tag{3.5}$$

Thus, we can find a sequence $\{\rho_n\}$, $\rho_1 < \rho_2 < \cdots, \rho_n \to \infty$, such that for ξ lying on $|\xi| = \rho_n$, we have, respectively, that as $\rho_n \to \infty$,

$$|B_j(\xi^q)| \le \rho_n^M \quad (j = 1,\dots,k-2), \tag{3.6}$$

$$\frac{\log|B_0(\xi^q)|}{\log\rho_n} \longrightarrow \infty \quad (\sigma(G_0) = 0), \tag{3.7}$$

$$\log|B_j(\xi^q)| < \rho_n^{q\delta} < \rho_n^{q\tau} < \log|B_0(\xi^q)|, \quad (j = 1,\dots,k-2,\ \sigma(G_0) > 0). \tag{3.8}$$

For convenience, when $\sigma(G_0) = 0$, we let $\delta = 0$. Thus, by (3.7) and (3.8) we have for $j = 1,\dots,k-2$ that

$$B_j(\xi^q) = \{\rho_n^M \exp(\rho_n^{q\delta})\} \quad (|\xi| = \rho_n). \tag{3.9}$$

We now estimate h_1' on $|\xi| = \rho_n$. Substituting f_1 in (3.1) into (1.2), we deduce that

$$(h_1')^k + P_{k-1}(\xi)(h_1')^{k-1} + \sum_{j=0}^{k-2} P_j(\xi)(h_1')^j + \frac{q^k}{\xi^k}\left[g_0\left(\frac{1}{\xi^q}\right) + G_0(\xi^q)\right] = 0, \tag{3.10}$$

where $P_{k-1}(\xi)$ is only polynomial in $W_1^{(m)}/(\xi)W_1(\xi)$, $h_1^{(m)}(\xi)/h_1'(\xi)$, $1/\xi^s$ $(1 \le s \le k-1,\ 1 \le m \le k)$ with constant coefficients; $P_j(\xi)$ $(j = 0,\dots,k-2)$ are polynomials in $W_1^{(m)}(\xi)/W_1(\xi)$, $h_1^{(m)}(\xi)/h_1'(\xi)$, $1/\xi^s$ $(1 \le s \le k-1,\ 1 \le m \le k)$, and $B_1(\xi^q),\dots,B_{k-2}(\xi^q)$ with constant coefficients. Set

$$D(\xi) = \frac{q^k}{\xi^k}\left[g_0\left(\frac{1}{\xi^q}\right) + G_0(\xi^q)\right] = \frac{q^k}{\xi^k}B_0(\xi^q). \tag{3.11}$$

On the circle $S_n = \{\xi : |\xi| = \rho_n,\ 0 < \arg\xi < 2\pi\}$, we define a single valued branch of $D(\xi)^{1/k}$. By (3.10), we have

$$\left(\frac{h_1'}{D^{1/k}}\right)^k + \left(\frac{P_{k-1}}{D^{1/k}}\right)\left(\frac{h_1'}{D^{1/k}}\right)^{k-1} + \sum_{k=0}^{k-2}\left(\frac{P_j}{D^{(k-j)/k}}\right)\left(\frac{h_1'}{D^{1/k}}\right)^j + 1 = 0. \tag{3.12}$$

By (3.7)–(3.9) and (3.12), we can deduce, on S_n,

$$|h_1'(\xi) - c_n D^{1/k}(\xi)| \le \rho_n^M \quad (|\xi| = \rho_n,\ c_n^k = -1). \tag{3.13}$$

Substituting $f_2(z)$ in (3.1) into (1.2), using a similar argument as above, for h_2', we can get the same estimation, that is, h_2' satisfies (3.13), so that by (3.13) we can deduce that for every sufficiently large n there exist M and a_n such that $a_n^k = 1$, and, on $|\xi| = \rho_n$,

$$|h_2'(\xi) - a_n h_1'(\xi)| \leq \rho_n^M. \tag{3.14}$$

Since kth root of unity has only k roots, we see that there must exist infinite many n_j such that these a_{n_j} in (3.14) are all the same, say $a_{n_j} = a$. By (3.14), we see that $h_2'(\xi) - ah_1'(\xi)$ must be a polynomial and so is $h_2(\xi) - ah_1(\xi)$. Set $h_2(\xi) - ah_1(\xi) = P$. The polynomial P and e^P may be incorporated into the factors W_1 and W_2, so that, without loss of generality, we may further assume that $h_2(\xi) \equiv ah_1(\xi)$.

Now prove $a = 1$. Since $f_j'/f_j = (1/q)(d_j + \xi(W_j'/W_j) + \xi h_j')$ $(j = 1,2)$ and $a^k = 1$, we see that for sufficiently large n, on $|\xi| = \rho_n$,

$$a\left[2k\frac{W_1'}{W_1} + k(k-1)\frac{h_1''}{h_1'}\right] = 2k\frac{W_2'}{W_2} + k(k-1)\frac{h_1''}{h_1'} + o\left(\frac{1}{\rho_n^2}\right). \tag{3.15}$$

Set

$$F_1 = W_1^{2k}(h_1')^{k(k-1)}, \qquad F_2 = W_2^{2k}(h_1')^{k(k-1)}, \tag{3.16}$$

then F_1 and F_2 are the analytic functions in $\{\xi : 1 < |\xi| < \infty\}$. Without loss of generality, we may assume that the entire function h_1' has infinite many zeros, otherwise, we may take a non-Picard exceptional value c of h_1', and replace $h_1(\xi)$ by $h_1(\xi) - c\xi$. $e^{c\xi}$ is incorporated into W_1. Here above deduction remains unchanged, yet $h_1' - c$ is of infinite many zeros. Denote by $n_1(\rho_n, 1/F_1)$ and $n_1(\rho_n, 1/F_2)$, respectively, zeros of F_1 and F_2 in annulus $\rho_1 < |\xi| < \rho_n$. Since

$$n_1\left(\rho_n, \frac{1}{F_j}\right) = \frac{1}{2\pi i}\int_{s_n + s_1^-}\frac{F_j'}{F_j}d\xi = \frac{1}{2\pi i}\int_{s_n+s_1^-}\left[2k\frac{W_j'}{W_j} + k(k-1)\frac{h_1''}{h_1'}\right]d\xi, \tag{3.17}$$

by (3.15), we get

$$an_1\left(\rho_n, \frac{1}{F_1}\right) = n_1\left(\rho_n, \frac{1}{F_2}\right) + O(1), \tag{3.18}$$

combining this with $a^k = 1$, we get $a = 1$.

Lastly, we easily prove that f_1 and f_2 are linearly dependent.

We remark that the above proof remains valid if we interchange the roles of G_j and g_j as at the end of the proof of Lemma 2.6. The proof of Theorem 1.3 is completed.

4. Lemma for the proof of Theorem 1.6

LEMMA 4.1. *Let $k \geq 2$ and $A_j(z) = B_j(e^z) = B_j(\zeta)$, $\zeta = e^z$, $B_j(\zeta) = G_j(\zeta) + g_j(1/\zeta)$, $j = 0,1,\dots,k-2$, where $G_j(t)$ and $g_j(t)$ are entire functions. Suppose the following:*

(i) *$g_0(t)$ is transcendental and $\sigma(g_0) < \infty$;*

(ii) *for $j > 0$, either $g_j(t)$ is a polynomial or $\sigma(g_j) < \sigma(g_0)$;*

(iii) *for $j \geq 0, G_j(t)$ is polynomial.*

If $f(z)(\not\equiv 0)$ is a solution of (1.2) and satisfies (1.7), then in $1 < |\xi| < \infty$, $f(z)$ can be represented as

$$f(z) = \xi^{d_0}\psi_0(\xi)u_0(\xi)e^{h_0(\xi)}, \tag{4.1}$$

where $\xi = e^{z/q}$, q is an integer and satisfies $1 \leq q \leq k$, d_0 is some constant, $\psi_0(\xi)$ is analytic and does not vanish in $1 < |\xi| \leq \infty$, and $\psi_0(\infty) = 1$, both $u_0(\xi)$ and $h_0(\xi)$ are entire functions, and $h_0(\xi)$ and $u_0(\xi)$ also satisfy the following:

(a) *$h_0(\xi)$ is a polynomial;*

(b) *if the condition (1.7) is replaced by (1.6), then $u_0(\xi)$ is a polynomial.*

If $G_j(t)$ and $g_j(t)$ $(j = 0,\dots,k-2)$ are transposed in (i)–(iii), then the same conclusion still holds with $\xi = e^{-z/q}$.

We give the following two remarks in order to prove Lemma 4.1.

Remark 4.2. Under the hypotheses of Lemma 4.1, in (2.8) of Lemma 2.6, $\xi = e^{-z/q}$. But in Lemma 4.1, we do not proceed transformation $\zeta = 1/\eta$, so, in (4.1), $\xi = e^{z/q}$.

Remark 4.3. Wiman-Valiron theory and its applications to differential equations (see [1, pages 5-6] or [12, pages 71-72]).

Consider the linear differential equation

$$a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f = 0, \tag{4.2}$$

if $a_0,\dots,a_k$ are polynomials, $f(z)$ is an entire transcendental function, then we have the following:

(a) $M(r,f)$ satisfies the relation

$$\log M(r,f) = c_1 r^\sigma + o(r^\sigma) \quad \text{as } r \longrightarrow \infty \tag{4.3}$$

for some positive real constant c_1;

(b) σ is a positive rational number.

The same conclusion holds to the differential equation of the form (4.2) whose coefficients are analytic in a neighborhood of $z = \infty$ and have at most a pole at $z = \infty$.

Proof of Lemma 4.1. Using a method similar to the proof of Lemma 2.6, combining Remarks 4.2 and 4.3, we can prove Lemma 4.1. □

5. Proof of Theorem 1.6

Suppose that f is a nontrivial solution of (1.2) and satisfies (1.6). Then Theorem 1.3 implies that $f(z)$ and $f(z+2\pi i)$ must be linearly dependent. On the other hand, by Lemma 4.1, $f(z)$ has the representation in $1 < |\xi| < \infty$,

$$f(z) = \zeta^d \psi(\zeta)u(\zeta)e^{h(\zeta)} = \zeta^d W(\zeta)e^{h(\zeta)}, \tag{5.1}$$

where $\zeta = e^z$, d is some constant, $\psi(\zeta)$ is analytic and does not vanish in $1 < |\zeta| \leq \infty$, and $\psi(\infty) = 1$, both $u(\zeta)$ and $h(\zeta)$ are entire functions and have at most a pole at $\zeta = \infty$, as

$\zeta \to \infty$, set

$$W(\zeta) = \alpha\zeta^{s}(1+o(1)), \quad h(\zeta) = \beta\zeta^{\nu}(1+o(1)), \quad \alpha\beta \neq 0. \tag{5.2}$$

Substituting (5.1) into (1.2), we get

$$(\zeta h')^{k} + Q_{k-1}(\zeta)(\zeta h')^{k-1} + \sum_{j=0}^{k-2} Q_j(\zeta)(\zeta h')^{j} = 0, \tag{5.3}$$

where $Q_{k-1}(\zeta)$ is a polynomial in $\zeta^m(W^{(m)}/W)$, $\zeta^{m-1}(h^{(m)}/h')$, $1/\zeta^m$ with constant coefficients; $Q_j(\zeta)$ $(j = 0,\dots,k-2)$ are polynomials in $B_j, B_{j+1},\dots,B_{k-2}$ and $\zeta^m(W^{(m)}/W)$, $\zeta^{m-1}(h^{(m)}/h')$, $1/\zeta^m$ with constant coefficients, every $Q_j(\zeta)$ is linear in $B_j, B_{j+1},\dots,B_{k-2}$ and the coefficient of B_j is 1. Substituting (5.2) into (5.3), we get, as $\zeta \to \infty$,

$$\begin{aligned}(\beta\nu)^{k}\zeta^{k\nu}(1+o(1)) &+ O(|\zeta|^{(k-1)\nu}) \\ &+ \sum_{j=0}^{k-2}\Big[b_j\zeta^{p_j+j\nu}(1+o(1)) + C_{j+1}^{j}b_{j+1}\zeta^{p_{j+1}+j\nu}(1+o(1)) \\ &\qquad + \cdots + C_{k-2}^{j}b_{k-2}\zeta^{p_{k-2}+j\nu}(1+o(1))\Big](\beta\nu)^{j} = 0.\end{aligned} \tag{5.4}$$

If $\nu \geq 1$, from the hypotheses $p_j < \min\{k-j,\ p_0\}$ $(j = 1,\dots,k-2)$, we see that for $j \geq 1$, when $j \leq m \leq k-2$, $p_m + j\nu < k - m + j\nu \leq (k-j)\nu + j\nu = k\nu$; for $j = 0$, when $1 \leq n \leq k-2$, $p_n < p_0$. Thus, when $\nu \geq 1$, (5.4) can be written as

$$(\beta\nu)^{k}\zeta^{k\nu}(1+o(1)) + b_0\zeta^{p_0}(1+o(1)) = 0. \tag{5.5}$$

But p_0 is not divisible by k, hence (5.5) is a contradiction.

If $\nu = 0$, then $h(\zeta)$ and $e^{h(\zeta)}$ are constants, by Lemma 4.1 and (5.2) we see that $f(z) = \zeta^{d}W(\zeta)e^{h(\zeta)}$ has at most a pole at $\zeta = \infty$, but the coefficients $B_{k-2}(\zeta),\dots,B_0(\zeta)$ of (1.2) are analytic in a neighborhood of $\zeta = \infty$ and have at most a pole at $\zeta = \infty$, and the order of pole of $B_0(\zeta)$ at $\zeta = \infty$ is the highest, this is impossible.

Thus, (1.2) cannot admit a solution that satisfies (1.6), hence every solution $f \not\equiv 0$ of (1.2) satisfies (1.14). This completes the proof of Theorem 1.6.

6. Proofs of corollaries

Proof of Corollary 1.4. By Remark 2.7, we see that the hypotheses of Theorem 1.3 are satisfied. This completes the proof. □

To prove Corollary 1.5, we need the following lemma that can be deduced from [13, Theorem 4].

Lemma 6.1. *Let $A(z)$ be an entire function with Fabry gap, and*

$$\log M(r,A) < r^{\lambda} \tag{6.1}$$

for some sufficiently large $r > 0$, where $\lambda > 0$ is a fixed constant. Let $\eta_1,\eta_2 \in (0,1)$ be two constants, then there exists a set $E \subset (0,\infty)$, such that the logarithmic measure of $E \cap [1,r]$

is at least $(1-\eta_1)\log r + O(1)$*; as* $r \to +\infty$ *through values satisfying (6.1) and for* $r \in E$*, one has*

$$\log L(r,A) > (1-\eta_2)\log M(r,A), \tag{6.2}$$

where $L(r,A) = \min_{|z|=r}\{|A(z)|\}$, $M(r,A) = \max_{|z|=r}\{|A(z)|\}$.

Proof of Corollary 1.5. Let $\mu(G_0) = \tau_0$. According to the definition, we have, for infinitely many r with $r \to \infty$,

$$\log M(r, G_0) < r^{\tau_0+1}, \tag{6.3}$$

thus (6.1) holds. We deduce from Lemma 6.1 that there exists a set $H_0 \subset (1,+\infty)$ with infinite logarithmic measure, and when $|t| = r \in H_0$, we have

$$\log|G_0(t)| > (1-\eta_2)\log M(r,G_0). \tag{6.4}$$

When $\mu(G_0) > 0$, for $j > 0$ we choose $\sigma(G_j) < \tau < \mu(G_0)$. We deduce from the definition that there exists $r_0 > 0$, such that for all $r > r_0$, $\log M(r,G_0) > (1/(1-\eta_2))r^{\tau}$. Thus, when $r \in H = H_0 \cap (r_0,+\infty)$, condition (i) of Theorem 1.3 holds. Clearly, H has infinite logarithmic measure. Thus, Corollary 1.5 follows from Theorem 1.3. □

Proofs of Corollaries 1.7 and 1.8. Using a similar argument as in proof of Corollaries 1.4 and 1.5, respectively, we see that the conditions of Corollaries 1.7 and 1.8 satisfy the hypotheses of Theorem 1.6, respectively. Thus, by Theorem 1.6, we see that Corollaries 1.7 and 1.8 hold. □

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