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Research Article

Inclusion Properties for Certain Subclasses of Analytic Functions Associated with the Dziok-Srivastava Operator

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The purpose of the present paper is to introduce several new classes of analytic functions defined by using the Choi-Saigo-Srivastava operator associated with the Dziok-Srivastava operator and to investigate various inclusion properties of these classes. Some interesting applications involving classes of integral operators are also considered.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. For $0 \le \eta, \beta < 1$, we denote by $\mathcal{S}^*(\eta)$, $\mathcal{K}(\eta)$, and $\mathcal{C}(\eta,\beta)$ the subclasses of $\mathcal{A}$ consisting of all analytic functions which are, respectively, starlike of order η, convex of order η, close-to-convex of order η, and type β in $\mathbb{U}$. For various other interesting developments involving functions in the class $\mathcal{A}$, the reader may be referred (for example) to the work of Srivastava and Owa [1].

Let $\mathcal{N}$ be the class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $\text{Re}\{\phi(z)\} > 0$ for $z \in \mathbb{U}$.

Making use of the principle of subordination between analytic functions, we introduce the subclasses $\mathcal{S}^*(\eta;\phi)$, $\mathcal{K}(\eta;\phi)$, and $\mathcal{C}(\eta,\delta;\phi,\psi)$ of the class $\mathcal{A}$ for $0 \le \eta,\ \beta < 1$, and $\phi,\psi \in \mathcal{N}$ (cf. [2, 3]), which are defined by

$$\mathcal{S}^*(\eta;\phi) := \left\{ f \in \mathcal{A} : \frac{1}{1-\eta}\left(\frac{zf'(z)}{f(z)} - \eta\right) \prec \phi(z) \text{ in } \mathbb{U} \right\},$$
$$\mathcal{K}(\eta;\phi) := \left\{ f \in \mathcal{A} : \frac{1}{1-\eta}\left(1 + \frac{zf''(z)}{f'(z)} - \eta\right) \prec \phi(z) \text{ in } \mathbb{U} \right\}, \tag{1.2}$$
$$\mathcal{C}(\eta,\beta;\phi,\psi) := \left\{ f \in \mathcal{A} : \exists g \in \mathcal{S}^*(\eta;\phi) \text{ s.t. } \frac{1}{1-\beta}\left\{\frac{zf'(z)}{g(z)} - \beta\right\} \prec \psi(z) \text{ in } \mathbb{U} \right\}.$$

We note that the classes mentioned above are the familiar classes which have been used widely on the space of analytic and univalent functions in $\mathbb{U}$, and for special choices for the functions ϕ and ψ involved in these definitions, we can obtain the well-known subclasses of $\mathcal{A}$. For examples, we have

$$\mathcal{S}^*\left(\eta;\frac{1+z}{1-z}\right) = \mathcal{S}^*(\eta), \qquad \mathcal{K}\left(\eta;\frac{1+z}{1-z}\right) = \mathcal{K}(\eta),$$
$$\mathcal{C}\left(\eta,\beta;\frac{1+z}{1-z},\frac{1+z}{1-z}\right) = \mathcal{C}(\eta,\beta). \tag{1.3}$$

Also let the Hadamard product (or convolution) $f * g$ of two analytic functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k \tag{1.4}$$

be given (as usual) by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k. \tag{1.5}$$

Making use of the Hadamard product (or convolution) given by (1.5), we now define the Dziok-Srivastava operator

$$H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s) : \mathcal{A} \longrightarrow \mathcal{A}, \tag{1.6}$$

which was introduced and studied in a series of recent papers by Dziok and Srivastava ([4–6]; see also [7, 8]). Indeed, for complex parameters

$$\alpha_1,\ldots,\alpha_q, \quad \beta_1,\ldots,\beta_s (\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-; \mathbb{Z}_0^- = 0,-1,-2,\ldots; j = 1,\ldots,s), \tag{1.7}$$

the generalized hypergeometric function ${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ is given by

$${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!}$$
$$(q \le s+1;\ q,s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ z \in \mathbb{U}), \tag{1.8}$$

where $(\nu)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(\nu)_k := \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1 & \text{if } k=0,\ \nu \in \mathbb{C}\backslash\{0\}, \\ \nu(\nu+1)\cdots(\nu+k-1) & \text{if } k \in \mathbb{N},\ \nu \in \mathbb{C}. \end{cases} \tag{1.9}$$

Corresponding to a function $\mathcal{F}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$, defined by

$$\mathcal{F}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) := z_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z), \tag{1.10}$$

Dziok and Srivastava [5] considered a linear operator defined by the following Hadamard product (or convolution):

$$H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z) := \mathcal{F}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) * f(z). \tag{1.11}$$

We note that the linear operator $H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)$ includes various other linear operators which were introduced and studied by Carlson and Shaffer [9], Hohlov [10], Ruscheweyh [11], and so on [12, 13].

Corresponding to the function $\mathcal{F}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$, defined by (1.10), we introduce a function $\mathcal{F}_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ given by

$$\mathcal{F}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) * \mathcal{F}_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) = \frac{z}{(1-z)^\lambda} \quad (\lambda > 0). \tag{1.12}$$

Analogous to $H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)$, we now define the linear operator $H_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)$ on $\mathcal{A}$ as follows:

$$\begin{gathered} H_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z) = \mathcal{F}_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) * f(z) \\ (\alpha_i,\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ i=1,\ldots,q;\ j=1,\ldots,s;\ \lambda>0;\ z\in\mathbb{U};\ f\in\mathcal{A}). \end{gathered} \tag{1.13}$$

For convenience, we write

$$H_{\lambda,q,s}(\alpha_1) := H_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s). \tag{1.14}$$

It is easily verified from the definition (1.13) that

$$z(H_{\lambda,q,s}(\alpha_1+1)f(z))' = \alpha_1 H_{\lambda,q,s}(\alpha_1)f(z) - (\alpha_1-1)H_{\lambda,q,s}(\alpha_1+1)f(z), \tag{1.15}$$

$$z(H_{\lambda,q,s}(\alpha_1)f(z))' = \lambda H_{\lambda+1,q,s}(\alpha_1)f(z) - (\lambda-1)H_{\lambda,q,s}(\alpha_1)f(z). \tag{1.16}$$

In particular, the operator $H_\lambda(\gamma+1,1;1)(\lambda>0;\gamma>-1)$ was introduced by Choi et al. [2], who investigated (among other things) several inclusion properties involving various subclasses of analytic and univalent functions. For $\gamma = n(n\in\mathbb{N}\cup 0;\mathbb{N}=\{1,2,\ldots\})$ and $\lambda=2$, we also note that the Choi-Sago-Srivastava operator $H_{\lambda,2,1}(\gamma+1,1;1)f$ is the Noor integral operator of nth order of f studied by Liu [14] and K. I. Noor and M. A. Noor [15, 16].

Next, by using the operator $H_{\lambda,q,s}(\alpha_1)$, we introduce the following classes of analytic functions for $\phi,\psi \in \mathcal{N}$, and $0 \le \eta,\ \beta < 1$:

$$\begin{aligned}
\mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi) &:= \{f \in \mathcal{A} : H_{\lambda,q,s}(\alpha_1)f \in \mathcal{S}^*(\eta;\phi)\},\\
\mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;\phi) &:= \{f \in \mathcal{A} : H_{\lambda,q,s}(\alpha_1)f \in \mathcal{K}(\eta;\phi)\},\\
\mathcal{C}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi) &:= \{f \in \mathcal{A} : H_{\lambda,q,s}(\alpha_1)f \in \mathcal{C}(\eta,\beta;\phi,\psi)\}.
\end{aligned} \tag{1.17}$$

We also note that

$$f(z) \in \mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;\phi) \Longleftrightarrow zf'(z) \in \mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi). \tag{1.18}$$

In particular, we set

$$\begin{aligned}
\mathcal{S}_{\lambda,\alpha_1}\left(q,s;\eta;\frac{1+Az}{1+Bz}\right) &=: \mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;A,B) \quad (-1 \le B < A \le 1),\\
\mathcal{K}_{\lambda,\alpha_1}\left(q,s;\eta;\frac{1+Az}{1+Bz}\right) &=: \mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;A,B) \quad (-1 \le B < A \le 1).
\end{aligned} \tag{1.19}$$

In this paper, we investgate several inclusion properties of the classes $\mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)$, $\mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;\phi)$, and $\mathcal{C}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$ associated with the operator $H_{\lambda,q,s}(\alpha_1)$. Some applications involving integral operators are also considered.

2. Inclusion Properties Involving the Operator $H_{\lambda,q,s}(\alpha_1)$

The following results will be required in our investigation.

Lemma 2.1 [17]. *Let ϕ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and* $\operatorname{Re}\{\kappa\phi(z) + \nu\} > 0$ $(\kappa,\nu \in \mathbb{C})$. *If p is analytic in $\mathbb{U}$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\kappa p(z) + \nu} \prec \phi(z) \quad (z \in \mathbb{U}) \tag{2.1}$$

implies

$$p(z) \prec \phi(z) \quad (z \in \mathbb{U}). \tag{2.2}$$

Lemma 2.2 [18]. *Let ϕ be convex univalent in $\mathbb{U}$ and let ω be analytic in $\mathbb{U}$ with* $\operatorname{Re}\{\omega(z)\} \ge 0$. *If p is analytic in $\mathbb{U}$ and $p(0) = \phi(0)$, then*

$$p(z) + \omega(z)zp'(z) \prec \phi(z) \quad (z \in \mathbb{U}) \tag{2.3}$$

implies

$$p(z) \prec \phi(z) \quad (z \in \mathbb{U}). \tag{2.4}$$

Theorem 2.3. *Let $\alpha_1, \lambda > 1$ and $\phi \in \mathcal{N}$. Then,*

$$\mathcal{S}_{\lambda+1,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{S}_{\lambda,\alpha_1+1}(q,s;\eta;\phi). \tag{2.5}$$

Proof. First of all, we will show that

$$\mathcal{S}_{\lambda+1,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi). \tag{2.6}$$

Let $f \in \mathcal{S}_{\lambda+1,\alpha_1}(q,s;\eta;\phi)$ and set

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)f(z)} - \eta\right), \tag{2.7}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Using (1.16) and (2.7), we have

$$\frac{1}{1-\eta}\left(\frac{z(H_{\lambda+1,q,s}(\alpha_1)f(z))'}{H_{\lambda+1,q,s}(\alpha_1)f(z)} - \eta\right) = p(z) + \frac{zp'(z)}{(1-\eta)p(z)+\lambda-1+\eta} \quad (z \in \mathbb{U}). \tag{2.8}$$

Since $\lambda > 1$ and $\phi \in \mathcal{N}$, we see that

$$\operatorname{Re}\{(1-\eta)\phi(z)+\lambda-1+\eta\} > 0 \quad (z \in \mathbb{U}). \tag{2.9}$$

Applying Lemma 2.1 to (2.8), it follows that $p \prec \phi$, that is, $f \in \mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)$.

To prove the second part, let $f \in \mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)$ and put

$$s(z) = \frac{1}{1-\eta}\left(\frac{z(H_{\lambda,q,s}(\alpha_1+1)f(z))'}{H_{\lambda,q,s}(\alpha_1+1)f(z)} - \eta\right), \tag{2.10}$$

where s is analytic function with $s(0) = 1$. Then, by using the arguments similar to those detailed above with (1.15), it follows that $s \prec \phi$ in $\mathbb{U}$, which implies that $f \in \mathcal{S}_{\lambda,\alpha_1+1}(q,s;\eta;\phi)$. Therefore, we complete the proof of Theorem 2.3. □

THEOREM 2.4. *Let $\alpha_1, \lambda > 1$ and $\phi \in \mathcal{N}$. Then,*

$$\mathcal{K}_{\lambda+1,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{K}_{\lambda,\alpha_1+1}(q,s;\eta;\phi). \tag{2.11}$$

Proof. Applying (1.18) and Theorem 2.3, we observe that

$$\begin{aligned}
f(z) \in \mathcal{K}_{\lambda+1,\alpha_1}(q,s;\eta;\phi) &\Longleftrightarrow H_{\lambda+1,q,s}(\alpha_1)f(z) \in \mathcal{K}(\eta;\phi) \\
&\Longleftrightarrow H_{\lambda+1,q,s}(\alpha_1)(zf'(z)) \in \mathcal{S}(\eta;\phi) \\
&\Longleftrightarrow zf'(z) \in \mathcal{S}_{\lambda+1,\alpha_1}(q,s;\eta;\phi) \\
&\Longrightarrow zf'(z) \in \mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi) \\
&\Longleftrightarrow z(H_{\lambda,q,s}(\alpha_1)f(z))' \in \mathcal{S}(\eta;\phi) \\
&\Longleftrightarrow f(z) \in \mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;\phi), \\
f(z) \in \mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;\phi) &\Longleftrightarrow zf'(z) \in \mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi) \\
&\Longrightarrow zf'(z) \in \mathcal{S}_{\lambda,\alpha_1+1}(q,s;\eta;\phi) \\
&\Longleftrightarrow f(z) \in \mathcal{K}_{\lambda,\alpha_1+1}(q,s;\eta;\phi),
\end{aligned} \tag{2.12}$$

which evidently proves Theorem 2.4. □

Taking

$$\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1;\ z \in \mathbb{U}) \tag{2.13}$$

in Theorems 2.3 and 2.4, we have the following.

Corollary 2.5. *Let $\alpha_1, \lambda > 1$. Then,*

$$\begin{gathered}\mathcal{S}_{\lambda+1,\alpha_1}(q,s;\eta;A,B) \subset \mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;A,B) \subset \mathcal{S}_{\lambda,\alpha_1+1}(q,s;\eta;A,B),\\ \mathcal{K}_{\lambda+1,\alpha_1}(q,s;\eta;A,B) \subset \mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;A,B) \subset \mathcal{K}_{\lambda,\alpha_1+1}(q,s;\eta;A,B).\end{gathered} \tag{2.14}$$

Next, by using Lemma 2.2, we obtain the following inclusion relation for the class $\mathcal{C}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$.

Theorem 2.6. *Let $\alpha_1, \lambda > 1$ and $\phi, \psi \in \mathcal{N}$. Then,*

$$\mathcal{C}_{\lambda+1,\alpha_1}(q,s;\eta,\beta;\phi,\psi) \subset \mathcal{C}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi) \subset \mathcal{C}_{\lambda,\alpha_1+1}(q,s;\eta,\beta;\phi,\psi). \tag{2.15}$$

Proof. We begin by proving that

$$\mathcal{C}_{\lambda+1,\alpha_1}(q,s;\eta,\beta;\phi,\psi) \subset \mathcal{C}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi). \tag{2.16}$$

Let $f \in \mathcal{C}_{\lambda+1,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$. Then, from the definition of $\mathcal{C}_{\lambda+1,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$, there exists a function $r \in \mathcal{S}^*(\eta;\phi)$ such that

$$\frac{1}{1-\beta}\left(\frac{z(H_{\lambda+1,q,s}(\alpha_1)f(z))}{r(z)} - \beta\right) \prec \psi(z) \quad (z \in \mathbb{U}). \tag{2.17}$$

Choose the function g such that $H_{\lambda+1,q,s}(\alpha_1)g(z) = r(z)$. Then, $g \in \mathcal{S}_{\lambda+1,\alpha_1}(q,s;\eta;\phi)$ and

$$\frac{1}{1-\beta}\left(\frac{z(H_{\lambda+1,q,s}(\alpha_1)f(z))'}{H_{\lambda+1,q,s}(\alpha_1)g(z)} - \beta\right) \prec \psi(z) \quad (z \in \mathbb{U}). \tag{2.18}$$

Now let

$$p(z) = \frac{1}{1-\beta}\left(\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)g(z)} - \beta\right), \tag{2.19}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Using (1.16), we have

$$\begin{aligned}(1-\beta)zp'(z)H_{\lambda,q,s}(\alpha_1)g(z) + ((1-\beta)p(z)+\beta)z(H_{\lambda,q,s}(\alpha_1)g(z))'\\ = \lambda z(H_{\lambda+1,q,s}(\alpha_1)f(z))' - (\lambda-1)z(H_{\lambda,q,s}(\alpha_1)f(z))'.\end{aligned} \tag{2.20}$$

Since $g \in \mathcal{S}_{\lambda+1,\alpha_1}(q,s;\eta;\phi)$, by Theorem 2.3, we know that $g \in \mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)$. Let

$$q(z) = \frac{1}{1-\eta}\left(\frac{z(H_{\lambda,q,s}(\alpha_1)g(z))'}{H_{\lambda,q,s}(\alpha_1)g(z)} - \eta\right). \tag{2.21}$$

Then, using (1.16) once again, we have

$$\lambda \frac{H_{\lambda+1,q,s}(\alpha_1)g(z)}{H_{\lambda,q,s}(\alpha_1)g(z)} = (1-\eta)q(z) + \lambda - 1 + \eta. \tag{2.22}$$

From (2.20) and (2.22), we obtain

$$\frac{1}{1-\beta}\left(\frac{z(H_{\lambda+1,q,s}(\alpha_1)f(z))'}{H_{\lambda+1,q,s}(\alpha_1)g(z)} - \beta\right) = p(z) + \frac{zp'(z)}{(1-\eta)q(z)+\lambda-1+\eta}. \tag{2.23}$$

Since $\lambda > 1$ and $q \prec \phi$ in $\mathbb{U}$,

$$\operatorname{Re}\{(1-\eta)q(z)+\lambda-1+\eta\} > 0 \quad (z \in \mathbb{U}). \tag{2.24}$$

Hence, applying Lemma 2.2, we can show that $p \prec \psi$, so that $f \in \mathscr{C}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$.

For the second part, by using the arguments similar to those detailed above with (1.15), we obtain

$$\mathscr{C}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi) \subset \mathscr{C}_{\lambda,\alpha_1+1}(q,s;\eta,\beta;\phi,\psi). \tag{2.25}$$

Therefore, we complete the proof of Theorem 2.6. □

3. Inclusion Properties Involving the Integral Operator F_c

In this section, we consider the generalized Libera integral operator F_c [13] (cf. [2, 12]) defined by

$$F_c(f) := F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (f \in \mathscr{A};\ c > -1). \tag{3.1}$$

We first prove the following.

Theorem 3.1. *If $f \in \mathscr{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)$, then $F_c(f) \in \mathscr{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)$ $(c \geq 0)$.*

Proof. Let $f \in \mathscr{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)$ and set

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(H_{\lambda,q,s}(\alpha_1)F_c(f)(z))'}{H_{\lambda,q,s}(\alpha_1)F_c(f)(z)} - \eta\right), \tag{3.2}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. From (3.1), we have

$$z(H_{\lambda,q,s}(\alpha_1)F_c(f)(z))' = (c+1)H_{\lambda,q,s}(\alpha_1)f(z) - cH_{\lambda,q,s}(\alpha_1)F_c(f)(z). \tag{3.3}$$

Then, by using (3.2) and (3.3), we obtain

$$(c+1)\frac{H_{\lambda,q,s}(\alpha_1)f(z)}{H_{\lambda,q,s}(\alpha_1)F_c(f)(z)} = (1-\eta)p(z) + c + \eta. \tag{3.4}$$

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by z, we have

$$p(z)+\frac{zp'(z)}{(1-\eta)p(z)+c+\eta}=\frac{1}{1-\eta}\left(\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)f(z)}-\eta\right)\quad(z\in\mathbb{U}).\tag{3.5}$$

Hence, by virtue of Lemma 2.1, we conclude that $p\prec\phi$ in $\mathbb{U}$, which implies that $F_c(f)\in\mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)$. □

Next, we derive an inclusion property involving F_c, which is given by the following.

Theorem 3.2. *If $f\in\mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;\phi)$, then $F_c(f)\in\mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;\phi)$ $(c\geq 0)$.*

Proof. By applying Theorem 3.1, it follows that

$$\begin{aligned}f(z)\in\mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;\phi)&\Longleftrightarrow zf'(z)\in\mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)\\&\Longrightarrow F_c(zf'(z))\in\mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)\\&\Longleftrightarrow z(F_c(f)(z))'\in\mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)\\&\Longleftrightarrow F_c(f)(z)\in\mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;\phi),\end{aligned}\tag{3.6}$$

which proves Theorem 3.2. □

From Theorems 3.1 and 3.2, we have the following.

Corollary 3.3. *If f belongs to the class $\mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;A,B)$ (or $\mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;A,B)$), then $F_c(f)$ belongs to the class $\mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;A,B)$ (or $\mathcal{K}_{\lambda,\alpha_1}(q,s;\eta;A,B)$) $(c\geq 0)$.*

Finally, we prove.

Theorem 3.4. *If $f\in\mathcal{C}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$, then $F_c(f)\in\mathcal{C}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$ $(c\geq 0)$.*

Proof. Let $f\in\mathcal{C}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$. Then, in view of the definition of the class $\mathcal{C}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$, there exists a function $g\in\mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)$ such that

$$\frac{1}{1-\beta}\left(\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)g(z)}-\beta\right)\prec\psi(z)\quad(z\in\mathbb{U}).\tag{3.7}$$

Thus, we set

$$p(z)=\frac{1}{1-\beta}\left(\frac{z(H_{\lambda,q,s}(\alpha_1)F_c(f)(z))'}{H_{\lambda,q,s}(\alpha_1)F_c(g)(z)}-\beta\right),\tag{3.8}$$

where p is analytic in $\mathbb{U}$ with $p(0)=1$. Since $g\in\mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)$, we see from Theorem 3.1 that $F_c(g)\in\mathcal{S}_{\lambda,\alpha_1}(q,s;\eta;\phi)$. Using (3.3), we have

$$((1-\beta)p(z)+\beta)H_{\lambda,q,s}(\alpha_1)F_c(g)(z)+cH_{\lambda,q,s}(\alpha_1)F_c(f)(z)=(c+1)H_{\lambda,q,s}(\alpha_1)f(z).\tag{3.9}$$

Then, by a simple calculation, we get

$$(c+1)\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)F_c(g)(z)} = ((1-\beta)p(z)+\beta)((1-\eta)q(z)+c+\eta)+(1-\beta)zp'(z), \tag{3.10}$$

where

$$q(z) = \frac{1}{1-\eta}\left(\frac{z(H_{\lambda,q,s}(\alpha_1)F_c(g)(z))'}{H_{\lambda,q,s}(\alpha_1)F_c(g)(z)} - \eta\right). \tag{3.11}$$

Hence, we have

$$\frac{1}{1-\beta}\left(\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)g(z)} - \beta\right) = p(z) + \frac{zp'(z)}{(1-\eta)q(z)+c+\eta}. \tag{3.12}$$

The remaining part of the proof in Theorem 3.4 is similar to that of Theorem 2.6 and so we omit it. □

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References

[1] S. Owa and H. M. Srivastava, "Analytic solution of a class of Briot-Bouquet differential equations," in *Current Topics in Analytic Function Theory*, H. M. Srivastava and S. Owa, Eds., pp. 252–259, World Scientific, River Edge, NJ, USA, 1992.

[2] J. H. Choi, M. Saigo, and H. M. Srivastava, "Some inclusion properties of a certain family of integral operators," *Journal of Mathematical Analysis and Applications*, vol. 276, no. 1, pp. 432–445, 2002.

[3] W. Ma and D. Minda, "An internal geometric characterization of strongly starlike functions," *Annales Universitatis Mariae Curie-Skłodowska, Sectio A*, vol. 45, pp. 89–97, 1991.

[4] J. Dziok and H. M. Srivastava, "Classes of analytic functions associated with the generalized hypergeometric function," *Applied Mathematics and Computation*, vol. 103, no. 1, pp. 1–13, 1999.

[5] J. Dziok and H. M. Srivastava, "Some subclasses of analytic functions with fixed argument of coefficients associated with the generalized hypergeometric function," *Advanced Studies in Contemporary Mathematics*, vol. 5, no. 2, pp. 115–125, 2002.

[6] J. Dziok and H. M. Srivastava, "Certain subclasses of analytic functions associated with the generalized hypergeometric function," *Integral Transforms and Special Functions*, vol. 14, no. 1, pp. 7–18, 2003.

[7] J.-L. Liu and H. M. Srivastava, "Certain properties of the Dziok-Srivastava operator," *Applied Mathematics and Computation*, vol. 159, no. 2, pp. 485–493, 2004.

[8] J.-L. Liu and H. M. Srivastava, "Classes of meromorphically multivalent functions associated with the generalized hypergeometric function," *Mathematical and Computer Modelling*, vol. 39, no. 1, pp. 21–34, 2004.

[9] B. C. Carlson and D. B. Shaffer, "Starlike and prestarlike hypergeometric functions," *SIAM Journal on Mathematical Analysis*, vol. 15, no. 4, pp. 737–745, 1984.

[10] Yu. E. Hohlov, "Operators and operations in the class of univalent functions," *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, vol. 10, pp. 83–89, 1978.

[11] S. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society*, vol. 49, pp. 109–115, 1975.
[12] R. M. Goel and N. S. Sohi, "A new criterion for p-valent functions," *Proceedings of the American Mathematical Society*, vol. 78, no. 3, pp. 353–357, 1980.
[13] S. Owa and H. M. Srivastava, "Some applications of the generalized Libera integral operator," *Proceedings of the Japan Academy, Series A*, vol. 62, no. 4, pp. 125–128, 1986.
[14] J.-L. Liu, "The Noor integral and strongly starlike functions," *Journal of Mathematical Analysis and Applications*, vol. 261, no. 2, pp. 441–447, 2001.
[15] K. I. Noor, "On new classes of integral operators," *Journal of Natural Geometry*, vol. 16, no. 1-2, pp. 71–80, 1999.
[16] K. I. Noor and M. A. Noor, "On integral operators," *Journal of Mathematical Analysis and Applications*, vol. 238, no. 2, pp. 341–352, 1999.
[17] P. Eenigenburg, S. S. Miller, P. T. Mocanu, and M. O. Reade, "On a Briot-Bouquet differential subordination," in *General Inequalities, 3 (Oberwolfach, 1981)*, vol. 64 of *Internationale Schriftenreihe zur Numerischen Mathematik*, pp. 339–348, Birkhäuser, Basel, Switzerland, 1983.
[18] S. S. Miller and P. T. Mocanu, "Differential subordinations and univalent functions," *The Michigan Mathematical Journal*, vol. 28, no. 2, pp. 157–172, 1981.

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