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Research Article

Semigroup Approach to Semilinear Partial Functional Differential Equations with Infinite Delay

Hassane Bouzahir

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We describe a semigroup of abstract semilinear functional differential equations with infinite delay by the use of the Crandall Liggett theorem. We suppose that the linear part is not necessarily densely defined but satisfies the resolvent estimates of the Hille-Yosida theorem. We clarify the properties of the phase space ensuring equivalence between the equation under investigation and the nonlinear semigroup.

1. Introduction

Most of the existing results about functional differential equations with finite delay have been recently under verification in the case of infinite delay. Our objective in this paper is to study the solution semigroup generated by the following partial functional differential equation with infinite delay:

$$\begin{gathered} \frac{d}{dt}x(t) = A_T x(t) + F(x_t), \quad t \geq 0, \\ x_0 = \phi \in \mathcal{B}, \end{gathered} \tag{1.1}$$

where A_T is a nondensely defined linear operator on a Banach space $(E, |\cdot|)$. The phase space $\mathcal{B}$ can be the space C_γ, γ being a positive real constant, of all continuous functions $\phi : (-\infty, 0] \to E$ such that $\lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta)$ exists in E, endowed with the norm $\|\phi\|_\gamma := \sup_{\theta \leq 0} e^{\gamma\theta}|\phi(\theta)|$, $\phi \in C_\gamma$. For every $t \geq 0$, the function $x_t \in \mathcal{B}$ is defined by

$$x_t(\theta) = x(t+\theta), \quad \text{for } \theta \in (-\infty, 0]. \tag{1.2}$$

We assume the following.

(H1) $F:\mathcal{B}\to E$ is globally Lipschitz continuous; that is, there exists a positive constant L such that $|F(\psi_1)-F(\psi_2)|\le L\|\psi_1-\psi_2\|_{\mathcal{B}}$ for all $\psi_1,\psi_2\in\mathcal{B}$.

A typical example that can be transformed into (1.1) is the following:

$$\begin{gathered}\frac{\partial}{\partial t}w(t,\xi)=a\frac{\partial^2}{\partial\xi^2}w(t,\xi)+bw(t,\xi)+c\int_{-\infty}^{0}G(\theta)w(t+\theta,\xi)d\theta\\ +f(w(t-\tau,\xi)),\quad t\ge 0,\ 0\le\xi\le\pi,\\ w(t,0)=w(t,\pi)=0,\quad t\ge 0,\\ w(\theta,\xi)=w_0(\theta,\xi),\quad -\infty<\theta\le 0,\ 0\le\xi\le\pi,\end{gathered}\tag{1.3}$$

where a, b, c, and τ are positive constants, $f:\mathbb{R}\to\mathbb{R}$ is a continuous function, G is a positive integrable function on $(-\infty,0]$, and $w_0:(-\infty,0]\times[0,\pi]\to\mathbb{R}$ is an appropriate continuous function.

Effectively, in [1], an abstract treatment of (1.3) as (1.1) leads to a characterization of exponential asymptotic stability near an equilibrium of (1.3) provided that the associated linearized semigroup is exponentially stable.

For many quantitative studies of any problem of type (1.1) in a concrete space of functions mapping $(-\infty,0]$ into E, one should choose a space that verifies at least the fundamental axioms first introduced in [2]. That is, $(\mathcal{B},\|\cdot\|_{\mathcal{B}})$ is a (semi)normed abstract linear space of functions mapping $(-\infty,0]$ into E, which satisfies the following.

(A) There is a positive constant H and functions $K(\cdot),M(\cdot):\mathbb{R}^+\to\mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma\in\mathbb{R}$, $a>0$, if $x:(-\infty,\sigma+a]\to E$, $x_\sigma\in\mathcal{B}$, and $x(\cdot)$ is continuous on $[\sigma,\sigma+a]$, then for every t in $[\sigma,\sigma+a]$ the following conditions hold:

(i) $x_t\in\mathcal{B}$;

(ii) $|x(t)|\le H\|x_t\|_{\mathcal{B}}$, which is equivalent to

(ii)′ for each $\varphi\in\mathcal{B}$, $|\varphi(0)|\le H\|\varphi\|_{\mathcal{B}}$;

(iii) $\|x_t\|_{\mathcal{B}}\le K(t-\sigma)\sup_{\sigma\le s\le t}|x(s)|+M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$.

(A1) For the function $x(\cdot)$ in (A), $t\mapsto x_t$ is a $\mathcal{B}$-valued continuous function for t in $[\sigma,\sigma+a]$.

(B) The space $\mathcal{B}$ or the space of equivalence classes $\widehat{\mathcal{B}}:=\mathcal{B}/\|\cdot\|_{\mathcal{B}}=\{\widehat{\varphi}:\varphi\in\mathcal{B}\}$ is complete.

However, to obtain interesting qualitative results, a concrete choice should be made on a space that verifies additional properties which are essential to investigate the equation. A class of employed spaces is called uniform fading memory spaces. They verify that the function $K(\cdot)$ is constant, $\lim_{t\to+\infty}M(t)=0$, and the following extra property.

(C) If $\{\phi_n\}_{n\ge 0}$ is a Cauchy sequence in $\mathcal{B}$ with respect to the (semi)norm and if ϕ_n converges compactly to ϕ on $(-\infty,0]$, then ϕ is in $\mathcal{B}$ and $\|\phi_n-\phi\|_{\mathcal{B}}\to 0$ as $n\to+\infty$.

There are many examples of concrete spaces that verify the above properties. In [3], it was proved, for instance, that if $\gamma>0$, the above-defined space C_γ is a uniform fading memory space. Another example is given by

$$C_g^0 := \left\{\phi \in C((-\infty,0];E) : \lim_{\theta\to-\infty} \frac{|\phi(\theta)|}{g(\theta)} = 0\right\}, \tag{1.4}$$

equipped with the norm

$$\|\phi\|_g := \sup_{-\infty<\theta\le 0} \frac{|\phi(\theta)|}{g(\theta)}, \tag{1.5}$$

where $g : (-\infty,0] \to [1,+\infty)$ is a continuous function such that (g1): g is nonincreasing and $g(0) = 1$, and (g2): the function $G : [0,+\infty) \to [0,+\infty)$ defined by $G(t) := \sup_{-\infty<\theta\le -t}(g(t+\theta)/g(\theta))$ tends to 0 as t tends to ∞.

In general, set for any positive continuous function g on $(-\infty,0]$,

$$\begin{aligned} C_g &:= \left\{\phi \in C((-\infty,0];E) : \frac{|\phi(\theta)|}{g(\theta)} \text{ is bounded}\right\}, \\ LC_g &:= \left\{\phi \in C_g : \lim_{\theta\to-\infty}\left(\frac{|\phi(\theta)|}{g(\theta)}\right) \text{ exists in } E\right\}, \\ UC_g &:= \left\{\phi \in C_g : \frac{|\phi(\theta)|}{g(\theta)} \text{ is uniformly continuous on } (-\infty,0]\right\}, \end{aligned} \tag{1.6}$$

such that (g3): G is locally bounded for $t \ge 0$, (g4): $g(\theta)$ tends to ∞ as θ tends to $-\infty$, and (g2) are satisfied. Then LC_g is a uniform fading memory space; the additional condition (g5): $\log g(\theta)$ is uniformly continuous on $(-\infty,0]$, ensuring that UC_g is a uniform fading memory space. Precisely, $K(t) = \sup_{-t\le\theta\le 0}(1/g(\theta))$ and $M(t) = G(t)$. Note that for the space C_γ, as defined above, $g(\theta) = e^{-\gamma\theta}$.

On the contrary, despite its consideration in some recent separate publications concerned with abstract stability investigations, unfortunately, the space $C_0 := \{\phi \in C((-\infty,0];E) : \lim_{\theta\to-\infty}\phi(\theta) = 0\}$ is not a uniform fading memory space. In [4] for instance, some restrictive results about asymptotic behavior of solutions were obtained in the linear positive case on C_0. The followed method uses evolution semigroups, extrapolation spaces, and critical spectrum on Banach lattices spaces. For the basic discussion about the general phase space $\mathcal{B}$, we refer the reader to [3, especially Chapter 1] and [5, pages 401–406].

Although many authors have avoided repetitions by working on the abstract space $\mathcal{B}$, the delicacy of some investigations restricts their work on a class of concrete spaces that verify many properties such as C_γ with $\gamma > 0$. In [1, 6–8], we have considered (1.1) with A_T being nondensely defined and satisfying the Hille-Yosida condition. Precisely, since $D(A_T)$ is not densely defined, we have addressed the problems of existence, uniqueness, regularity, existence of global attractor, existence of periodic solutions, and local stability by means of the integrated semigroups theory. In this article, we use the Crandall Liggett approach. We show the relation between a nonlinear semigroup and integral solutions to (1.1).

Notice that the most general results about functional differential equations with infinite delay are obtained notably in [9–15] and in [16] also in the situation where A_T depends on t. Our results extend earlier ones which require the delay to be finite and A_T to have a dense domain in E.

In this paper, we proceed as follows. In Section 2, we recall some basic results on existence, uniqueness, and properties of integral solutions to (1.1). Then, in Section 3, we establish properties of the solution operator in nonlinear case. Next, we rely upon the well-known Crandall and Liggett theorem in order to compute the nonlinear solution semigroup by an exponential formula. Finally, we give the link between the semigroup given by the Crandall and Liggett theorem and the integral solution to (1.1).

2. Basic results

Throughout, we assume that A_T satisfies the Hille-Yosida condition:

(H2) there exist two constants $\beta \geq 1$ and $\omega_0 \in \mathbb{R}$ with $(\omega_0, +\infty) \subset \rho(A_T)$ and $\sup\{\|(\lambda - \omega_0)^n R(\lambda, A_T)^n\| : \lambda > \omega_0,\ n \in \mathbb{N}\} \leq \beta$,

where $\rho(A_T)$ is the resolvent set of A_T and $R(\lambda, A_T) = (\lambda I - A_T)^{-1}$.

Definition 2.1. A function $x : (-\infty, a] \to E$, $a > 0$, is an integral solution of (1.1) in $(-\infty, a]$ if the following conditions hold:

(i) x is continuous on $[0, a]$;

(ii) $\int_0^t x(s)ds \in D(A_T)$, for $t \in [0, a]$;

(iii)

$$x(t) = \begin{cases} \phi(0) + A_T \displaystyle\int_0^t x(s)ds + \int_0^t F(s, x_s)ds, & 0 \leq t \leq a, \\ \phi(t), & -\infty < t \leq 0. \end{cases} \tag{2.1}$$

It follows from (ii) of the above definition that for an integral solution x, one has $x(t) \in \overline{D(A_T)}$ for all $t \geq 0$. In particular, $\phi(0) \in \overline{D(A_T)}$.

Define the part A_0 of A_T in $\overline{D(A_T)}$ by

$$\begin{gathered} D(A_0) = \{x \in D(A_T) : A_T x \in \overline{D(A_T)}\}, \\ A_0 x = A_T x, \quad \text{for } x \in D(A_0). \end{gathered} \tag{2.2}$$

Recall (cf. [17]) that A_0 generates a C_0-semigroup $(T_0(t))_{t \geq 0}$ on $\overline{D(A_T)}$.

It is known (see [1, 6]) that under (H1) and (H2), for $\phi \in \mathcal{B}$ such that $\phi(0) \in \overline{D(A_T)}$, (1.1) admits a unique integral solution $x(\cdot, \phi)$ given by the following formula:

$$x(t, \phi) = \begin{cases} T_0(t)\phi(0) + \lim\limits_{\lambda \to +\infty} \displaystyle\int_0^t T_0(t - s) B_\lambda F(x_s(\cdot, \phi))ds, & \text{for } t \geq 0, \\ \phi(t), & \text{for } t \in (-\infty, 0], \end{cases} \tag{2.3}$$

where $B_\lambda = \lambda R(\lambda, A_T)$.

Set

$$\mathcal{X} := \{\varphi \in \mathcal{B} : \varphi(0) \in \overline{D(A_T)}\}. \tag{2.4}$$

Define $\mathcal{U}(t)$ on $\mathcal{X}$ for $t \geq 0$ by

$$\mathcal{U}(t)\varphi = x_t(\cdot,\varphi), \tag{2.5}$$

where $x(\cdot,\phi)$ is the integral solution of (1.1). The point of departure for our results is [1] where we have proved that $(\mathcal{U}(t))_{t\geq 0}$ is a strongly continuous semigroup satisfying the following properties:

(i) $(\mathcal{U}(t))_{t\geq 0}$ satisfies, for $t \geq 0$ and $\theta \in (-\infty,0]$, the following translation property

$$(\mathcal{U}(t)\varphi)(\theta) = \begin{cases} (\mathcal{U}(t+\theta)\varphi)(0), & \text{if } t+\theta \geq 0, \\ \varphi(t+\theta), & \text{if } t+\theta \leq 0, \end{cases} \tag{2.6}$$

(ii) there exist two positive locally bounded functions $m(\cdot), n(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $\varphi_1, \varphi_2 \in X$, and $t \geq 0$,

$$\|\mathcal{U}(t)\varphi_1 - \mathcal{U}(t)\varphi_2\|_{\mathcal{B}} \leq m(t)e^{n(t)}\|\varphi_1 - \varphi_2\|_{\mathcal{B}}. \tag{2.7}$$

Moreover, if F is a bounded linear operator and $\mathcal{B}$ is a subspace of $C((-\infty,0];E)$ satisfying axioms (A1), (A2), (B) and the following axiom which was introduced in [13]:

(D) for a sequence $(\varphi_n)_{n\geq 0}$ in $\mathcal{B}$, if $\|\varphi_n\|_{\mathcal{B}} \to 0$, then $|\varphi_n(s)| \to 0$ for each $s \in (-\infty,0]$,

then $A_{\mathcal{U}} : D(A_{\mathcal{U}}) \subseteq \mathcal{X} \to \mathcal{X}$ such that $A_{\mathcal{U}}\varphi = \varphi'$ for any $\varphi \in D(A_{\mathcal{U}})$, where

$$\begin{aligned} D(A_{\mathcal{U}}) = \{&\varphi \in \mathcal{X} : \varphi \text{ is continuously differentiable, } \varphi(0) \in D(A_T), \\ &\varphi' \in \mathcal{X},\ \varphi'(0) = A_T\varphi(0) + F(\varphi)\}, \end{aligned} \tag{2.8}$$

is the infinitesimal generator of $(\mathcal{U}(t))_{t\geq 0}$.

3. Main results

Our first main result can be considered as an extension of the above result to the case where F is nonlinear. The concrete choice of $\mathcal{B}$ (LC_g^0, LC_g, or UC_g) seems to be best adapted to obtain our results. Here, we suppose sufficient conditions on g. The proof combines the ideas of [1, 18] or [19].

Proposition 3.1. *Let $\mathcal{B} = LC_g^0$, $\mathcal{B} = LC_g$ (with (g1)), or $\mathcal{B} = UC_g$ (with (g5)). Then Conditions (H1) and (H2) imply that $A_{\mathcal{U}}$ is the infinitesimal generator of $(\mathcal{U}(t))_{t\geq 0}$.*

Proof. Let $\varphi \in \mathscr{X}$ be continuously differentiable such that

$$\varphi' \in \mathscr{X},\ \varphi(0) \in D(A_T), \qquad \varphi'(0) = A_T\varphi(0) + F(\varphi). \tag{3.1}$$

Let $x(\cdot,\varphi) : (-\infty,+\infty) \to E$ be the unique integral solution of (1.1). We have to show that $\lim_{t\to 0^+}(1/t)(\mathscr{U}(t)\varphi - \varphi)$ exists in $\mathscr{X}$ and is equal to φ'. By definition of $x_t(\cdot,\varphi)$ and $\mathscr{U}$,

$$x(t,\varphi) = \begin{cases} (\mathscr{U}(t)\varphi)(0), & \text{if } t \geq 0, \\ \varphi(t), & \text{if } t \in (-\infty,0], \end{cases} \tag{3.2}$$

then

$$\frac{1}{t}(\mathscr{U}(t)\varphi - \varphi)(\theta) = \begin{cases} \dfrac{1}{t}(x(t+\theta,\varphi) - \varphi(\theta))(0), & t+\theta > 0, \\ \dfrac{1}{t}(\varphi(t+\theta) - \varphi(\theta)), & t+\theta \in (-\infty,0]. \end{cases} \tag{3.3}$$

If $t+\theta \leq 0$, $(1/t)(\mathscr{U}(t)\varphi - \varphi)(\theta)$ tends to $D^+\varphi(\theta)$ as $t \to 0^+$, where $D^+\varphi(\theta)$ is the right derivative of φ in θ.

If $t+\theta > 0$, we have

$$\begin{aligned} &\frac{1}{t}(\mathscr{U}(t)\varphi - \varphi)(\theta) - \varphi'(\theta) \\ &\quad = \frac{1}{t}\left[T_0(t+\theta)\varphi(0) + \lim_{\lambda\to\infty}\int_0^{t+\theta} T_0(t+\theta-s)B_\lambda F(x_s)\,ds - \varphi(\theta)\right] - \varphi'(\theta). \end{aligned} \tag{3.4}$$

Let $S(t)$, $t \geq 0$, be the integrated semigroup associated with $T_0(t)$, $t \geq 0$. We obtain from the last equality

$$\begin{aligned} &\frac{1}{t}(\mathscr{U}(t)\varphi - \varphi)(\theta) - \varphi'(\theta) \\ &\quad = \frac{1}{t}\Bigg[\varphi(0) + S(t+\theta)A_T\varphi(0) + \lim_{\lambda\to\infty}\int_0^{t+\theta} T_0(t+\theta-s)B_\lambda(F(x_s) - F(\varphi))\,ds \\ &\qquad + \lim_{\lambda\to\infty}\int_0^{t+\theta} T_0(t+\theta-s)B_\lambda F(\varphi)ds - \varphi(\theta)\Bigg] - \varphi'(\theta). \end{aligned} \tag{3.5}$$

Since $T_0(t)\varphi(0) = \varphi(0) + A_T S(t)\varphi(0)$ and

$$\begin{aligned} \lim_{\lambda\to\infty}\int_0^{t+\theta} T_0(t+\theta-s)B_\lambda F(\varphi)ds &= \lim_{\lambda\to\infty} S(t+\theta)B_\lambda F(\varphi) \\ &= \lim_{\lambda\to\infty} B_\lambda S(t+\theta)F(\varphi) \\ &= S(t+\theta)F(\varphi), \end{aligned} \tag{3.6}$$

we deduce that

$$\begin{aligned}
&\frac{1}{t}(\mathcal{U}(t)\varphi-\varphi)(\theta)-\varphi'(\theta)\\
&\quad=\frac{1}{t}\left[\varphi(0)+S(t+\theta)\varphi'(0)+\lim_{\lambda\to\infty}\int_0^{t+\theta}T_0(t+\theta-s)B_\lambda(F(x_s)-F(\varphi))ds-\varphi(\theta)\right]\\
&\qquad-\varphi'(\theta)\\
&\quad=\frac{1}{t}\left[S(t+\theta)\varphi'(0)-\int_0^{t+\theta}\varphi'(0)ds+\lim_{\lambda\to\infty}\int_0^{t+\theta}T_0(t+\theta-s)B_\lambda(F(x_s)-F(\varphi))ds\right]\\
&\qquad+\frac{1}{t}\varphi(0)+\frac{t+\theta}{t}\varphi'(0)-\frac{1}{t}\varphi(\theta)-\varphi'(\theta)\\
&\quad=\frac{1}{t}\left[\int_0^{t+\theta}(T_0(s)\varphi'(0)-\varphi'(0))ds+\lim_{\lambda\to\infty}\int_0^{t+\theta}T_0(t+\theta-s)B_\lambda(F(x_s)-F(\varphi))ds\right]\\
&\qquad+\frac{1}{t}\int_\theta^0\varphi'(s)ds-\varphi'(\theta)+\varphi'(0)-\frac{1}{t}\int_\theta^0\varphi'(0)ds\\
&\quad=\frac{1}{t}\left[\int_0^{t+\theta}(T_0(s)\varphi'(0)-\varphi'(0))ds+\lim_{\lambda\to\infty}\int_0^{t+\theta}T_0(t+\theta-s)B_\lambda(F(x_s)-F(\varphi))ds\right]\\
&\qquad+\frac{1}{t}\int_\theta^0(\varphi'(s)-\varphi'(0))ds+\varphi'(0)-\varphi'(\theta).
\end{aligned}\tag{3.7}$$

Hence

$$\begin{aligned}
\left|\frac{1}{t}(\mathcal{U}(t)\varphi-\varphi)(\theta)-\varphi'(\theta)\right|&\le\frac{1}{t}\int_0^{t+\theta}|T_0(s)\varphi'(0)-\varphi'(0)|ds\\
&\quad+\frac{1}{t}\lim_{\lambda\to\infty}\left|\int_0^{t+\theta}T_0(t+\theta-s)B_\lambda(F(x_s)-F(\varphi))\right|ds\\
&\quad+\frac{1}{t}\int_\theta^0|\varphi'(s)-\varphi'(0)|ds+|\varphi'(0)-\varphi'(\theta)|.
\end{aligned}\tag{3.8}$$

Let $\varepsilon>0$ and choose $\alpha>0$ small enough such that if $0<t<\alpha$, $-\infty<\theta\le 0$, and $t+\theta>0$, then

$$\begin{gathered}
\frac{1}{t}\int_0^{t+\theta}|T_0(s)\varphi'(0)-\varphi'(0)|ds<\frac{\varepsilon}{4},\\
\frac{1}{t}\lim_{\lambda\to\infty}\left|\int_0^{t+\theta}T_0(t+\theta-s)B_\lambda(F(x_s)-F(\varphi))ds\right|<\frac{\varepsilon}{4},\\
\frac{1}{t}\int_\theta^0|\varphi'(s)-\varphi'(0)|ds+|\varphi'(0)-\varphi'(\theta)|<\frac{\varepsilon}{4},
\end{gathered}\tag{3.9}$$

and if $0<t<\alpha$, $-\infty<\theta\le 0$, and $t+\theta\le 0$, then

$$\begin{aligned}
\frac{1}{g(\theta)}\left|\frac{1}{t}(\mathcal{U}(t)\varphi-\varphi)(\theta)-\varphi'(\theta)\right|&\le\left|\frac{1}{t}(\mathcal{U}(t)\varphi-\varphi)(\theta)-\varphi'(\theta)\right|\\
&=\left|\frac{1}{t}(\varphi(t+\theta)-\varphi(\theta))-\varphi'(\theta)\right|<\frac{\varepsilon}{4}.
\end{aligned}\tag{3.10}$$

Consequently, if $0 < t < \alpha$, then

$$\frac{1}{g(\theta)}\left|\frac{1}{t}(\mathcal{U}(t)\varphi-\varphi)(\theta)-\varphi'(\theta)\right| < \left|\frac{1}{t}(\mathcal{U}(t)\varphi-\varphi)(\theta)-\varphi'(\theta)\right| < \varepsilon. \tag{3.11}$$

Hence

$$\left\|\frac{1}{t}(\mathcal{U}(t)\varphi-\varphi)-\varphi'\right\|_{\mathcal{B}} < \varepsilon. \tag{3.12}$$

This proves that $\lim_{t\to 0^+}(1/t)(\mathcal{U}(t)\varphi-\varphi)$ exists and is equal to φ'. Then, $\varphi \in D(A_{\mathcal{U}})$.

Conversely, let $\varphi \in \mathcal{X}$ such that

$$\lim_{t\to 0^+}\frac{1}{t}(\mathcal{U}(t)\varphi-\varphi) = \lim_{t\to 0^+}\frac{1}{t}(x_t(\cdot,\varphi)-\varphi) = \psi = A_{\mathcal{U}}\varphi \quad \text{exists in } \mathcal{X}. \tag{3.13}$$

We can easily see that axiom (D) is verified by C_γ, LC_g, and UC_g which implies that

$$\lim_{t\to 0^+}\frac{1}{t}(x_t(\theta,\varphi)-\varphi(\theta)) \quad \text{exists for all } \theta \le 0 \text{ and is equal to } \psi(\theta). \tag{3.14}$$

Then, for $\theta \in (-\infty,0)$, we have

$$\psi(\theta) = \lim_{t\to 0^+}\frac{1}{t}(\varphi(t+\theta)-\varphi(\theta)) = D^+\varphi(\theta); \tag{3.15}$$

that is, $D^+\varphi = \psi$ in $(-\infty,0)$. Since ψ is continuous, $D^+\varphi$ is also continuous in $(-\infty,0)$. Let us recall the following result.

Lemma 3.2 [20]. *Let φ be continuous and differentiable on the right on $[a,b)$. If $D^+\varphi$ is continuous on $[a,b)$, then φ is continuously differentiable on $[a,b)$.*

From the above lemma, we deduce that the function φ is continuously differentiable in $(-\infty,0)$ and $\varphi' = \psi$. On the other hand, for $\theta = 0$, one has $\lim_{\theta\to 0^-}\varphi'(\theta)$ exists and equals $\psi(0)$. From this we infer that the function φ is continuously differentiable in $(-\infty,0]$ and $\varphi' = \psi \in \mathcal{X}$. We also deduce that $t \mapsto \mathcal{U}(t)\varphi$ is continuously differentiable. On the other hand, we have

$$x(t) = \varphi(0) + A_T\left(\int_0^t x(s)ds\right) + \int_0^t F(x_s)ds. \tag{3.16}$$

This implies that $\lim_{t\to 0^+} A_T[((1/t)\int_0^t x(s)ds) + (1/t)\int_0^t F(x_s)ds]$ exists and hence $\lim_{t\to 0^+} A_T((1/t)\int_0^t x(s)ds)$ exists. From the closedness of A_T and the fact that $(1/t)\int_0^t x(s)ds \in D(A_T)$ for $t > 0$, we deduce that $\lim_{t\to 0^+}(1/t)\int_0^t x(s)ds$ exists in $D(A_T)$ and is equal to $\varphi(0)$. Consequently, $\varphi(0) \in D(A_T)$ and $\varphi'(0) = A_T\varphi(0) + F(\varphi)$. This completes the proof of the proposition. □

The next result may be considered as an extension of a similar one in [19]. Our goal is to establish the Crandall and Liggett exponential formula

$$\lim_{n\to+\infty}\left(I-\frac{t}{n}A_{\mathcal{U}}\right)^{-n}\varphi = \mathcal{U}(t)\varphi, \quad \forall \varphi \in \mathcal{X}_\gamma,\ t \ge 0. \tag{3.17}$$

We restrict our choice to $\mathscr{X}_\gamma := \{\phi \in C_\gamma : \phi(0) \in \overline{D(A_T)}\}$ with $\gamma > 0$. Recall that this specified space is a uniform fading memory one.

Proposition 3.3. *Let $\mathscr{B} = C_\gamma$ with $\gamma > 0$. Suppose that (H1) and (H2) are satisfied. Then, the operator $A_{\mathscr{U}}$ given by Proposition 3.1 satisfies the following Crandall Liggett conditions.*

(a) $\operatorname{Im}(I - \lambda A_{\mathscr{U}}) = \mathscr{X}_\gamma$ *for all* $\lambda \in (0, 1/(L+\omega_0))$.

(b) *For all* $\psi_1, \psi_2 \in \mathscr{X}_\gamma$ *and* $\lambda \in (0, 1/(L+\omega_0))$,

$$||(I-\lambda A_{\mathscr{U}})^{-1}\psi_1 - (I-\lambda A_{\mathscr{U}})^{-1}\psi_2||_\gamma \le \frac{1}{1-\lambda(L+\omega_0)}||\psi_1 - \psi_2||_\gamma. \tag{3.18}$$

(c) $D(A_{\mathscr{U}})$ *is dense in* $\mathscr{X}_\gamma$.

Proof. (a) It is well known that one can suppose without loss of generality that $\omega_0 > -L$ and $\|T_0(t)\| \le e^{\omega_0 t}$. To prove (a), it is clear from the definition of $A_{\mathscr{U}}$ that $(I - \lambda A_{\mathscr{U}})(D(A_{\mathscr{U}})) \subseteq \mathscr{X}_\gamma$ for $\lambda > 0$. On the other hand, for $\psi \in \mathscr{X}_\gamma$ and $\lambda > 0$, let us solve the following equation:

$$(I - \lambda A_{\mathscr{U}})\varphi = \psi, \quad \varphi \in D(A_{\mathscr{U}}). \tag{3.19}$$

Recall that with $\gamma > 0$ and $\lambda > 0$, the function $W(1/\lambda)\psi(0) : \theta \mapsto e^{(1/\lambda)\theta}\psi(0)$, $\theta \le 0$, belongs to $\mathscr{X}_\gamma$. Also, the fact that $\mathscr{C}_\gamma$ with $\gamma > 0$ is a uniform fading memory space implies that the function $M_\lambda\psi : \theta \mapsto (1/\lambda)\int_\theta^0 e^{(1/\lambda)(\theta - s)}\psi(s)ds$, $\theta \le 0$, belongs to $\mathscr{X}_\gamma$ (see [21, 22]). Moreover, we can see that the solution of (3.19) is

$$\varphi(\theta) = \left(W\left(\frac{1}{\lambda}\right)\varphi(0)\right)(\theta) + (M_\lambda\psi)(\theta) = e^{(1/\lambda)\theta}\varphi(0) + \frac{1}{\lambda}\int_\theta^0 e^{(1/\lambda)(\theta-s)}\psi(s)ds. \tag{3.20}$$

Next, we suppose that $0 < \lambda\omega_0 < 1$. From (3.19) evaluated at 0 and the definition of $A_{\mathscr{U}}$ we get

$$\varphi(0) = (I - \lambda A_T)^{-1}(\psi(0) + \lambda F(\varphi)). \tag{3.21}$$

Introduce the following mapping $G_\psi^\lambda : E \to E$ defined by

$$G_\psi^\lambda(x) = (I - \lambda A_T)^{-1}\left(\psi(0) + \lambda F\left(W\left(\frac{1}{\lambda}\right)x + M_\lambda\psi\right)\right) \quad \forall x \in E. \tag{3.22}$$

For $x, y \in E$, we have

$$\begin{aligned}
|G_\psi^\lambda(x) - G_\psi^\lambda(y)| &\le \left|R\left(\frac{1}{\lambda}, A_T\right)\left[F\left(W\left(\frac{1}{\lambda}\right)x + M_\lambda\psi\right) - F\left(W\left(\frac{1}{\lambda}\right)y + M_\lambda\psi\right)\right]\right| \\
&\le \frac{\lambda L}{1-\lambda\omega_0}\left\|W\left(\frac{1}{\lambda}\right)x - W\left(\frac{1}{\lambda}\right)y\right\|_\gamma \\
&\le \frac{\lambda L}{1-\lambda\omega_0}\sup_{\theta\le 0} e^{\gamma\theta}\,|e^{(1/\lambda)\theta}(x-y)| \\
&\le \frac{\lambda L}{1-\lambda\omega_0}|x-y|.
\end{aligned} \tag{3.23}$$

Next, we suppose that $\lambda \in (0, 1/(L+\omega_0))$. Then, G_ψ^λ is a strict contraction and it has a unique fixed point x in E. Knowing that $(I-\lambda A_T)^{-1}(E) \subseteq D(A_T)$, we deduce that this fixed point belongs to $D(A_T)$. Consequently, $\operatorname{Im}(I-\lambda A_{\mathcal{U}}) = \mathcal{X}_\gamma$ for all $\lambda \in (0, 1/(L+\omega_0))$.

(b) Let $\lambda \in (0, 1/(L+\omega_0))$ be fixed. Set $\mathcal{J}_\lambda := (I-\lambda A_{\mathcal{U}})^{-1}$ which is well defined from $\mathcal{X}_\gamma$ to $D(A_{\mathcal{U}})$. We prove that $\mathcal{J}_\lambda$ is Lipschitz continuous with Lipschitz constant less than $(1-\lambda\omega_0)/(1-\lambda(L+\omega_0))$. In fact, let $\lambda > 0$ with $\lambda \in (0, 1/(L+\omega_0))$ and $\mathcal{J}_\lambda\psi_1 := \varphi_1$ $\mathcal{J}_\lambda\psi_2 := \varphi_2$ for $\psi_1, \psi_2 \in \mathcal{X}_\gamma$. Given $\varepsilon > 0$, by definition, there exists $\theta \in (-\infty, 0]$ such that

$$e^{\gamma\theta}\,|\,\varphi_1(\theta) - \varphi_2(\theta)\,| > \|\varphi_1 - \varphi_2\|_\gamma - \varepsilon. \tag{3.24}$$

Using (3.21) and (H2), we get

$$\begin{aligned}
e^{\gamma\theta}\,|\,\varphi_1(\theta)-\varphi_2(\theta)\,| &\le e^{\gamma\theta}\left\{\left|e^{(1/\lambda)\theta}R\left(\frac{1}{\lambda},A_T\right)\left[\frac{1}{\lambda}(\psi_1(0)-\psi_2(0))+(F(\varphi_1)-F(\varphi_2))\right]\right|\right. \\
&\qquad \left.+\left|\frac{1}{\lambda}\int_\theta^0 e^{(1/\lambda)(\theta-s)}(\psi_1(s)-\psi_2(s))\,ds\right|\right\} \\
&\le \left\{e^{(1/\lambda)\theta}\frac{1}{1-\lambda\omega_0}\left(\|\psi_1-\psi_2\|_\gamma+\lambda L\|\varphi_1-\varphi_2\|_\gamma\right)\right. \\
&\qquad \left.+\left|\frac{1}{\lambda}e^{(1/\lambda)\theta}\int_\theta^0 e^{(\gamma-1/\lambda)s}\,ds\right|\,\|\psi_1-\psi_2\|_\gamma\right\} \\
&\le \left(\frac{e^{(1/\lambda)\theta}}{1-\lambda\omega_0}+\left(1-e^{(1/\lambda)\theta}\right)\right)\|\psi_1-\psi_2\|_\gamma+\frac{\lambda L e^{(1/\lambda)\theta}}{1-\lambda\omega_0}\|\varphi_1-\varphi_2\|_\gamma
\end{aligned} \tag{3.25}$$

which implies that

$$\begin{aligned}
\frac{1-\lambda\omega_0-\lambda L e^{(1/\lambda)\theta}}{1-\lambda\omega_0}\|\varphi_1-\varphi_2\|_\gamma &\le \varepsilon+\left(\frac{e^{(1/\lambda)\theta}}{1-\lambda\omega_0}+\left(1-e^{(1/\lambda)\theta}\right)\right)\|\psi_1-\psi_2\|_\gamma, \\
\|\varphi_1-\varphi_2\|_\gamma &\le \frac{1-\lambda\omega_0}{1-\lambda\omega_0-\lambda L}\left(\frac{e^{(1/\lambda)\theta}+1-\lambda\omega_0-e^{(1/\lambda)\theta}+\lambda\omega_0 e^{(1/\lambda)\theta}}{1-\lambda\omega_0}\right)\|\psi_1-\psi_2\|_\gamma \\
&\le \frac{1}{1-\lambda(\omega_0+L)}\|\psi_1-\psi_2\|_\gamma.
\end{aligned} \tag{3.26}$$

To prove (c), using similar arguments as in [23, the proof of Proposition 3.5], we can verify that for all $\lambda > 0$ with $\lambda \in (0, 1/(L+\omega_0))$ and $\psi \in \mathcal{X}_\gamma$,

$$\begin{aligned}
\|\psi-\mathcal{J}_\lambda\psi\|_\gamma &\le \frac{\lambda L}{1-\lambda\omega}\|\psi\|_\gamma+\frac{\lambda}{1-\lambda\omega}\,|\,F(0)\,| \\
&\quad +\|\psi-\psi(0)-M_\lambda(\psi-\psi(0))\|_\gamma+\left|\,(I-\lambda A_T)^{-1}\psi(0)-\psi(0)\,\right|.
\end{aligned} \tag{3.27}$$

Since by its definition $\mathcal{J}_\lambda\psi$ belongs to $D(A_{\mathcal{U}})$, assertion (c) follows from the fact that $\lim_{\lambda\to 0+}|(I-\lambda A_T)^{-1}x - x| = 0$ for all $x \in \overline{D(A_T)}$ (see [24]) and the following result.

Lemma 3.4. *Set $\mathcal{X}_\gamma^0 := \{\phi \in \mathcal{X}_\gamma$ such that $\phi(0) = 0\}$. Then for all $\lambda > 0$ and $\psi \in \mathcal{X}_\gamma^0$ the function $M_\lambda\psi$ tends to ψ in $\mathcal{X}_\gamma^0$ as λ tends to zero.*

Proof. Set $\mathscr{X}_\gamma^0 := \{\phi \in \mathscr{X}_\gamma$ such that $\phi(0) = 0\}$. We can see that the operator $B_0 : D(B_0) \subseteq \mathscr{X}_\gamma^0 \to \mathscr{X}_\gamma^0$ such that $B_0\phi = \phi'$ for any $\phi \in D(B_0)$ where

$$D(B_0) = \{\phi \in \mathscr{X}_\gamma^0 : \phi \text{ is continuously differentiable and } \phi' \in \mathscr{X}_\gamma^0\} \tag{3.28}$$

is the infinitesimal generator of a C_0 semigroup on $\mathscr{X}_\gamma^0$. Moreover,

$$M_\lambda \psi = (I - \lambda B_0)^{-1}\psi \quad \text{for any } \lambda > 0,\ \psi \in \mathscr{X}_\gamma^0. \tag{3.29}$$

This implies that the assertion of the lemma is a direct consequence of a basic result that can be found, for instance, in [24, page 248]. □

Corollary 3.5. *Let $\mathscr{X} = \mathscr{X}_\gamma$ with $\gamma > 0$. Suppose that (H1) and (H2) are satisfied. Then, for all $\varphi \in \mathscr{X}_\gamma$ and $t \geq 0$,*

$$\lim_{n \to +\infty} \left(I - \frac{t}{n} A_{\mathscr{U}}\right)^{-n} \varphi = \mathscr{U}(t)\varphi. \tag{3.30}$$

The proof of the above result is based on the following special case of the well-known Crandall and Liggett theorem (see [25]).

Theorem 3.6 [25]. *Let $(Y, \|\cdot\|)$ be a Banach space and B a nonlinear operator with dense domain $D(B)$ in Y. Suppose that there exists a positive real constant ω such that*

(a) $\operatorname{Im}(I - \lambda B) = Y$ *for all* $\lambda \in (0, 1/\omega)$,

(b) *for all* $y_1, y_2 \in Y$ *and* $\lambda \in (0, 1/\omega)$,

$$\|(I - \lambda B)^{-1} y_1 - (I - \lambda B)^{-1} y_2\| \leq \frac{1}{1 - \lambda\omega} \|y_1 - y_2\|. \tag{3.31}$$

Then for all $y \in Y$, the limit

$$W_0(t)y := \lim_{n \to +\infty} \left(I - \frac{t}{n} B\right)^{-n} y \tag{3.32}$$

exists in Y. Moreover, the family of operators $(W_0(t))_{t \geq 0}$ satisfies

(i) $W_0(0) = I$,

(ii) $W_0(t_1 + t_2) = W_0(t_1)W_0(t_2)$ *for all* $t_1, t_2 \geq 0$,

(iii) $\|W_0(t)y_1 - W_0(t)y_2\| \leq e^{\omega t}\|y_1 - y_2\|$ *for all* $y_1, y_2 \in Y$ *and* $t \geq 0$.

Let us now give the link between the semigroup given by the Crandall and Liggett theorem and the integral solution to (1.1). The proof will be omitted because it is very similar to the case of finite delay (see [18] or [19]).

Proposition 3.7. *Let $\mathscr{X} = \mathscr{X}_\gamma$ with $\gamma > 0$. Suppose that (H1) and (H2) are satisfied and the operator $A : D(A) \subseteq \mathscr{X}_\gamma \to \mathscr{X}_\gamma$ such that $A\varphi = \varphi'$ for any $\varphi \in D(A)$, where*

$$\begin{aligned} D(A) = \{&\varphi \in \mathscr{X}_\gamma : \varphi \text{ is continuously differentiable, } \varphi(0) \in D(A_T), \\ &\varphi' \in \mathscr{X}_\gamma,\ \varphi'(0) = A_T\varphi(0) + F(\varphi)\}, \end{aligned} \tag{3.33}$$

satisfies the hypotheses of Crandall and Liggett theorem in $\mathscr{X}_\gamma$. Let $(U(t))_{t\ge 0}$ be the nonlinear semigroup given by

$$U(t)\varphi = \lim_{n\to+\infty}\left(I-\frac{t}{n}A\right)^{-n}\varphi \quad \forall\varphi\in\mathscr{X}_\gamma,\ t\ge 0. \tag{3.34}$$

Then for all $\varphi\in\mathscr{X}_\gamma$, the function $y := y(\cdot,\varphi) : (-\infty,+\infty)\to\mathbb{R}$ defined by

$$y(t,\varphi)=\begin{cases}(U(t)\varphi)(0), & t\ge 0,\\ \varphi(t), & t\le 0,\end{cases} \tag{3.35}$$

is an integral solution of (1.1).

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Hassane Bouzahir: National Engineering School of Applied Sciences (ENSA), University Ibn Zohr, P.O. Box 1136, Agadir 80000, Morocco
Email address: bouzahir@ensa-agadir.ac.ma