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Research Article

Steffensen's Integral Inequality on Time Scales

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We establish generalizations of Steffensen's integral inequality on time scales via the diamond-α dynamic integral, which is defined as a linear combination of the delta and nabla integrals.

1. Introduction

Steffensen [1] stated that if f and g are integrable functions on (a,b) with f nonincreasing and $0 \le g \le 1$, then

$$\int_{b-\lambda}^{b} f(t)\,dt \le \int_a^b f(t)g(t)\,dt \le \int_a^{a+\lambda} f(t)\,dt, \tag{1.1}$$

where $\lambda = \int_a^b g(t)dt$. This inequality is usually called Steffensen's inequality in the literature. A comprehensive survey on Steffensen's inequality can be found in [2].

Recently, Anderson [3] has given the time scale version of Steffensen's integral inequality, using nabla integral as follows: let $a,b \in \mathbb{T}_\kappa^\kappa$ and let $f,g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be nabla integrable functions, with f of one sign and decreasing and $0 \le g \le 1$ on $[a,b]_{\mathbb{T}}$. Assume $\ell, \gamma \in [a,b]_{\mathbb{T}}$ such that

$$\begin{aligned} b-\ell \le \int_a^b g(t)\nabla t \le \gamma - a \quad & \text{if } f \ge 0, t \in [a,b]_{\mathbb{T}}, \\ \gamma - a \le \int_a^b g(t)\nabla t \le b-\ell \quad & \text{if } f \le 0 t \in [a,b]_{\mathbb{T}}. \end{aligned} \tag{1.2}$$

Then

$$\int_{\ell}^{b} f(t)\nabla t \le \int_{a}^{b} f(t)g(t)\nabla t \le \int_{a}^{\gamma} f(t)\nabla t. \tag{1.3}$$

In the theorem above which can be found in [3] as Theorem 3.1, we could replace the nabla integrals with delta integrals under the same hypotheses and get a completely analogous result.

Wu [4] has given some generalizations of Steffensen's integral inequality which can be written as the following inequality: let f, g, and h be integrable functions defined on $[a,b]$ with f nonincreasing. Also let

$$0 \le g(t) \le h(t) \quad (t \in [a,b]). \tag{1.4}$$

Then

$$\int_{b-\lambda}^{b} f(t)h(t)\,dt \le \int_{a}^{b} f(t)g(t)\,dt \le \int_{a}^{a+\lambda} f(t)h(t)\,dt, \tag{1.5}$$

where λ is given by

$$\int_{a}^{a+\lambda} h(t)\,dt = \int_{a}^{b} g(t)\,dt = \int_{b-\lambda}^{b} h(t)\,dt. \tag{1.6}$$

The aim of this paper is to extend some generalizations of Steffensen's integral inequality to an arbitrary time scale. We obtain Steffensen's integral inequality using the diamond-α derivative on time scales. The diamond-α derivative reduces to the standard Δ derivative for $\alpha = 1$, or the standard ∇ derivative for $\alpha = 0$. We refer the reader to [5] for an account of the calculus corresponding to the diamond-α dynamic derivative. The paper is organized as follows: the next section contains basic definitions and theorems of time scales theory, which can also be found in [5–9], and of delta, nabla, and diamond-α dynamic derivatives. In Section 3, we present our results, which are generalizations of Steffensen's integral inequality on time scales.

2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers. The calculus of time scales was initiated by Stefan Hilger in his Ph.D. thesis [9] in order to create a theory that can unify discrete and continuous analysis. Let $\mathbb{T}$ be a time scale. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. Let $\sigma(t)$ and $\rho(t)$ be the forward and backward jump operators in $\mathbb{T}$, respectively. For $t \in \mathbb{T}$, we define the forward, jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \tag{2.1}$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}. \tag{2.2}$$

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, then t is called right-dense, and if $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. Let $t \in \mathbb{T}$, then two mappings $\mu, \nu : \mathbb{T} \to [0, \infty)$ satisfying

$$\mu(t) := \sigma(t) - t, \qquad \nu(t) := t - \rho(t) \tag{2.3}$$

are called the graininess functions.

We introduce the sets $\mathbb{T}^\kappa$, $\mathbb{T}_\kappa$, and $\mathbb{T}^\kappa_\kappa$ which are derived from the time scales $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^\kappa = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_\kappa = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. Finally, $\mathbb{T}^\kappa_\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$.

Let $f : \mathbb{T} \to \mathbb{R}$ be a function on time scales. Then for $t \in \mathbb{T}^\kappa$, we define $f^\Delta(t)$ to be the number, if one exists, such that for all $\varepsilon > 0$, there is a neighborhood U of t such that for all $s \in U$,

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right| \le \varepsilon |\sigma(t) - s|. \tag{2.4}$$

We say that f is delta differentiable on $\mathbb{T}^\kappa$, provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. Similarly, for $t \in \mathbb{T}_\kappa$, we define $f^\nabla(t)$ to be the number value, if one exists, such that for all $\varepsilon > 0$, there is a neighborhood V of t such that for all $s \in V$,

$$\left| f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s] \right| \le \varepsilon \left| \rho(t) - s \right|. \tag{2.5}$$

We say that f is nabla differentiable on $\mathbb{T}_\kappa$, provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, that is, $f^\sigma = f \circ \sigma$.

If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\rho : \mathbb{T} \to \mathbb{R}$ by $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbb{T}$, that is, $f^\rho = f \circ \rho$.

Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa (t \neq \min \mathbb{T})$. Then we have the following.

(i) If f is delta differentiable at t, then f is continuous at t.

(ii) If f is left continuous at t and t is right-scattered, then f is delta differentiable at t with

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}. \tag{2.6}$$

(iii) If t is right-dense, then f is delta differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s} \tag{2.7}$$

exists as a finite number. In this case,

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{2.8}$$

(iv) If f is delta differentiable at t, then

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.9}$$

Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_{\kappa} (t \neq \max \mathbb{T})$. Then we have the following.

(i) If f is nabla differentiable at t, then f is continuous at t.

(ii) If f is right continuous at t and t is left-scattered, then f is nabla differentiable at t with

$$f^{\nabla}(t) = \frac{f(t) - f^{\rho}(t)}{\nu(t)}. \tag{2.10}$$

(iii) If t is left-dense, then f is nabla differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s} \tag{2.11}$$

exists as a finite number. In this case,

$$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{2.12}$$

(iv) If f is nabla differentiable at t, then

$$f^{\rho}(t) = f(t) - \nu(t) f^{\nabla}(t). \tag{2.13}$$

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits finite at all left-dense points in $\mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous, provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits finite at all right-dense points in $\mathbb{T}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{\kappa}$. Then the delta integral of f is defined by

$$\int_a^b f(t) \Delta t = F(b) - F(a). \tag{2.14}$$

A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_{\kappa}$. Then the nabla integral of g is defined by

$$\int_a^b g(t) \nabla t = G(b) - G(a). \tag{2.15}$$

Many other information sources concerning time scales can be found in [6–8].

Now, we briefly introduce the diamond-α dynamic derivative and the diamond-α dynamic integra,l and we refer the reader to [5] for a comprehensive development of the calculus of the diamond-α dynamic derivative and the diamond-α dynamic integration.

Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}$, we define the diamond-α dynamic derivative $f^{\diamond_\alpha}(t)$ by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \quad 0 \le \alpha \le 1. \tag{2.16}$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable. The diamond-α derivative reduces to the standard Δ derivative for $\alpha = 1$, or the standard ∇ derivative for $\alpha = 0$. On the other hand, it represents a "weighted dynamic derivative" for $\alpha \in (0,1)$. Furthermore, the combined dynamic derivative offers a centralized derivative formula on any uniformly discrete time scale $\mathbb{T}$ when $\alpha = 1/2$.

Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$. Then

(i) $f + g : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$ with

$$(f+g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) + g^{\diamond_\alpha}(t); \tag{2.17}$$

(ii) for any constant c, $cf : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$ with

$$(cf)^{\diamond_\alpha}(t) = c f^{\diamond_\alpha}(t); \tag{2.18}$$

(iii) $fg : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$ with

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) g(t) + \alpha f^{\sigma}(t) g^{\Delta}(t) + (1-\alpha) f^{\rho}(t) g^{\nabla}(t). \tag{2.19}$$

Let $a, t \in \mathbb{T}$, and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(\tau) \Diamond_\alpha \tau = \alpha \int_a^t h(\tau) \Delta \tau + (1-\alpha) \int_a^t h(\tau) \nabla \tau, \quad 0 \le \alpha \le 1. \tag{2.20}$$

We may notice that since the $\Diamond_\alpha$ integral is a combined Δ and ∇ integral, we, in general, do not have

$$\left(\int_a^t h(\tau) \Diamond_\alpha \tau \right)^{\Diamond_\alpha} = h(t), \quad t \in \mathbb{T}. \tag{2.21}$$

Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$, then

(i) $\int_a^t [f(\tau) + g(\tau)] \Diamond_\alpha \tau = \int_a^t f(\tau) \Diamond_\alpha \tau + \int_a^t g(\tau) \Diamond_\alpha \tau$,

(ii) $\int_a^t c f(\tau) \Diamond_\alpha \tau = c \int_a^t f(\tau) \Diamond_\alpha \tau$,

(iii) $\int_a^t f(\tau) \Diamond_\alpha \tau = \int_a^b f(\tau) \Diamond_\alpha \tau + \int_b^t f(\tau) \Diamond_\alpha \tau$.

3. Main results

Throughout this section, we suppose that $\mathbb{T}$ is a time scale, $a < b$ are points in $\mathbb{T}$. For a q-difference equation version of the following result, including proof techniques, see [10]. We refer the reader to [10] for an account of q-calculus and its applications.

Theorem 3.1. *Let $a, b \in \mathbb{T}_\kappa^\kappa$ with $a < b$ and f, g, and $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\Diamond_\alpha$-integrable functions, with f of one sign and decreasing and $0 \le g(t) \le h(t)$ on $[a,b]_{\mathbb{T}}$. Assume $\ell, \gamma \in [a,b]_{\mathbb{T}}$*

such that

$$\begin{aligned}
&\int_{\ell}^{b} h(t)\Diamond_\alpha t \le \int_{a}^{b} g(t)\Diamond_\alpha t \le \int_{a}^{\gamma} h(t)\Diamond_\alpha t \quad \text{if } f \ge 0,\ t \in [a,b]_{\mathbb{T}},\\
&\int_{a}^{\gamma} h(t)\Diamond_\alpha t \le \int_{a}^{b} g(t)\Diamond_\alpha t \le \int_{\ell}^{b} h(t)\Diamond_\alpha t \quad \text{if } f \le 0,\ t \in [a,b]_{\mathbb{T}}.
\end{aligned} \tag{3.1}$$

Then

$$\int_{\ell}^{b} f(t)h(t)\Diamond_\alpha t \le \int_{a}^{b} f(t)g(t)\Diamond_\alpha t \le \int_{a}^{\gamma} f(t)h(t)\Diamond_\alpha t. \tag{3.2}$$

Proof. The proof given in the q-difference case [10] can be extended to general time scales. We prove only the left inequality in (3.2) in the case $f \ge 0$. The proofs of the other cases are similar. Since f is decreasing and g is nonnegative, we get

$$\begin{aligned}
\int_{a}^{b} f(t)g(t)\Diamond_\alpha t - \int_{\ell}^{b} f(t)h(t)\Diamond_\alpha t &= \int_{a}^{\ell} f(t)g(t)\Diamond_\alpha t + \int_{\ell}^{b} f(t)g(t)\Diamond_\alpha t - \int_{\ell}^{b} f(t)h(t)\Diamond_\alpha t\\
&= \int_{a}^{\ell} f(t)g(t)\Diamond_\alpha t - \int_{\ell}^{b} f(t)[h(t)-g(t)]\Diamond_\alpha t\\
&\ge \int_{a}^{\ell} f(t)g(t)\Diamond_\alpha t - f(\ell)\int_{\ell}^{b} [h(t)-g(t)]\Diamond_\alpha t\\
&= \int_{a}^{\ell} f(t)g(t)\Diamond_\alpha t - f(\ell)\int_{\ell}^{b} h(t)\Diamond_\alpha t + f(\ell)\int_{\ell}^{b} g(t)\Diamond_\alpha t\\
&\ge \int_{a}^{\ell} f(t)g(t)\Diamond_\alpha t - f(\ell)\int_{a}^{b} g(t)\Diamond_\alpha t + f(\ell)\int_{\ell}^{b} g(t)\Diamond_\alpha t\\
&= \int_{a}^{\ell} f(t)g(t)\Diamond_\alpha t - f(\ell)\left(\int_{a}^{b} g(t)\Diamond_\alpha t - \int_{\ell}^{b} g(t)\Diamond_\alpha t\right)\\
&= \int_{a}^{\ell} f(t)g(t)\Diamond_\alpha t - f(\ell)\int_{a}^{\ell} g(t)\Diamond_\alpha t\\
&= \int_{a}^{\ell} [f(t)-f(\ell)]g(t)\Diamond_\alpha t \ge 0.
\end{aligned} \tag{3.3}$$

□

Remark 3.2. When $\alpha = 0$ and setting $h(t) = 1$, inequality (3.2) reduces to inequality [3, (3.1)].

In order to obtain our other results, we need the following lemma.

LEMMA 3.3. *Let $a, b \in \mathbb{T}_\kappa^\kappa$ with $a < b$ and f, g, and $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\Diamond_\alpha$-integrable functions. Suppose also that $\ell, \gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\int_{a}^{\gamma} h(t)\Diamond_\alpha t = \int_{a}^{b} g(t)\Diamond_\alpha t = \int_{\ell}^{b} h(t)\Diamond_\alpha t. \tag{3.4}$$

Then

$$\int_a^b f(t)g(t)\Diamond_\alpha t = \int_a^\gamma \big(f(t)h(t) - [f(t) - f(\gamma)][h(t) - g(t)]\big)\Diamond_\alpha t + \int_\gamma^b [f(t) - f(\gamma)]g(t)\Diamond_\alpha t, \tag{3.5}$$

$$\int_a^b f(t)g(t)\Diamond_\alpha t = \int_a^\ell [f(t) - f(\ell)]g(t)\Diamond_\alpha t + \int_\ell^b \big(f(t)h(t) - [f(t) - f(\ell)][h(t) - g(t)]\big)\Diamond_\alpha. \tag{3.6}$$

Proof. We prove the integral identity (3.5). By direct computation, we have

$$\begin{aligned}
&\int_a^\gamma \big(f(t)h(t) - [f(t) - f(\gamma)][h(t) - g(t)]\big)\Diamond_\alpha t - \int_a^b f(t)g(t)\Diamond_\alpha t \\
&\quad = \int_a^\gamma \big(f(t)h(t) - f(t)g(t) - [f(t) - f(\gamma)][h(t) - g(t)]\big)\Diamond_\alpha t \\
&\qquad + \int_a^\gamma f(t)g(t)\Diamond_\alpha t - \int_a^b f(t)g(t)\Diamond_\alpha t \\
&\quad = \int_a^\gamma f(\gamma)[h(t) - g(t)]\Diamond_\alpha t - \int_\gamma^b f(t)g(t)\Diamond_\alpha t \\
&\quad = f(\gamma)\left(\int_a^\gamma h(t)\Diamond_\alpha t - \int_a^\gamma g(t)\Diamond_\alpha t\right) - \int_\gamma^b f(t)g(t)\Diamond_\alpha t.
\end{aligned} \tag{3.7}$$

If we apply assumption

$$\int_a^\gamma h(t)\Diamond_\alpha t = \int_a^b g(t)\Diamond_\alpha t \tag{3.8}$$

to (3.7), we obtain

$$\begin{aligned}
&f(\gamma)\left(\int_a^\gamma h(t)\Diamond_\alpha t - \int_a^\gamma g(t)\Diamond_\alpha t\right) - \int_\gamma^b f(t)g(t)\Diamond_\alpha t \\
&\quad = f(\gamma)\left(\int_a^b g(t)\Diamond_\alpha t - \int_a^\gamma g(t)\Diamond_\alpha t\right) - \int_\gamma^b f(t)g(t)\Diamond_\alpha t \\
&\quad = f(\gamma)\int_\gamma^b g(t)\Diamond_\alpha t - \int_\gamma^b f(t)g(t)\Diamond_\alpha t \\
&\quad = \int_\gamma^b [f(\gamma) - f(t)]g(t)\Diamond_\alpha t.
\end{aligned} \tag{3.9}$$

By combining the integral identities (3.7) and (3.9), we have integral identity (3.5). The proof of identity (3.6) is similar to that of integral identity (3.5) and is omitted. □

THEOREM 3.4. *Let $a, b \in \mathbb{T}^\kappa_\kappa$ with $a < b$ and f, g and $h : [a,b]_\mathbb{T} \to \mathbb{R}$ be $\Diamond_\alpha$-integrable functions, f of one sign and decreasing and $0 \le g(t) \le h(t)$ on $[a,b]_\mathbb{T}$. Assume $\ell, \gamma \in [a,b]_\mathbb{T}$ such that*

$$\int_a^\gamma h(t)\Diamond_\alpha t = \int_a^b g(t)\Diamond_\alpha t = \int_\ell^b h(t)\Diamond_\alpha t. \tag{3.10}$$

Then

$$\begin{aligned}\int_{\ell}^{b} f(t)h(t)\Diamond_\alpha t &\le \int_{\ell}^{b} \left(f(t)h(t) - [f(t)-f(\ell)][h(t)-g(t)]\right)\Diamond_\alpha t\\ &\le \int_{a}^{b} f(t)g(t)\Diamond_\alpha t\\ &\le \int_{a}^{\gamma} \left(f(t)h(t) - [f(t)-f(\gamma)][h(t)-g(t)]\right)\Diamond_\alpha t\\ &\le \int_{a}^{\gamma} f(t)h(t)\Diamond_\alpha t.\end{aligned} \tag{3.11}$$

Proof. In view of the assumptions that the function f is decreasing on $[a,b]_{\mathbb{T}}$ and that $0 \le g(t) \le h(t)$, we conclude that

$$\int_{a}^{\ell} [f(t)-f(\ell)]g(t)\Diamond_\alpha t \ge 0, \tag{3.12}$$

$$\int_{\ell}^{b} [f(\ell)-f(t)][h(t)-g(t)]\Diamond_\alpha t \ge 0. \tag{3.13}$$

Using the integral identity (3.6) together with the integral inequalities (3.12) and (3.13), we have

$$\int_{\ell}^{b} f(t)h(t)\Diamond_\alpha t \le \int_{\ell}^{b} \left(f(t)h(t) - [f(t)-f(\ell)][h(t)-g(t)]\right)\Diamond_\alpha t \le \int_{a}^{b} f(t)g(t)\Diamond_\alpha t. \tag{3.14}$$

In the same way as above, we can prove that

$$\begin{aligned}\int_{a}^{b} f(t)g(t)\Diamond_\alpha t &\le \int_{a}^{\gamma} \left(f(t)h(t) - [f(t)-f(\gamma)][h(t)-g(t)]\right)\Diamond_\alpha t\\ &\le \int_{a}^{\gamma} f(t)h(t)\Diamond_\alpha t.\end{aligned} \tag{3.15}$$

The proof of Theorem 3.4 is completed by combining the inequalities (3.14) and (3.15). □

Theorem 3.5. *Let $a,b \in \mathbb{T}_\kappa^\kappa$ with $a<b$ and f, g, h and $\varphi : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\Diamond_\alpha$-integrable functions, f of one sign and decreasing and $0 \le \varphi(t) \le g(t) \le h(t)-\varphi(t)$ on $[a,b]_{\mathbb{T}}$. Assume ℓ,γ is given by*

$$\int_{a}^{\gamma} h(t)\Diamond_\alpha t = \int_{a}^{b} g(t)\Diamond_\alpha t = \int_{\ell}^{b} h(t)\Diamond_\alpha t \tag{3.16}$$

such that $\ell,\gamma \in [a,b]_{\mathbb{T}}$. Then

$$\begin{aligned}&\int_{\ell}^{b} f(t)h(t)\Diamond_\alpha t + \int_{a}^{b} \left|[f(t)-f(\ell)]\varphi(t)\right|\Diamond_\alpha t\\ &\quad\le \int_{a}^{b} f(t)g(t)\Diamond_\alpha t \le \int_{a}^{\gamma} f(t)h(t)\Diamond_\alpha t - \int_{a}^{b} \left|[f(t)-f(\gamma)]\varphi(t)\right|\Diamond_\alpha t.\end{aligned} \tag{3.17}$$

Proof. By the assumptions that the function f is decreasing on $[a,b]_{\mathbb{T}}$ and that

$$0 \le \varphi(t) \le g(t) \le h(t) - \varphi(t) \quad (t \in [a,b]_{\mathbb{T}}), \tag{3.18}$$

it follows that

$$\begin{aligned}
&\int_a^{\gamma} [f(t) - f(\gamma)][h(t) - g(t)]\Diamond_\alpha t + \int_{\gamma}^{b} [f(\gamma) - f(t)]g(t)\Diamond_\alpha t \\
&\quad = \int_a^{\gamma} |f(t) - f(\gamma)|[h(t) - g(t)]\Diamond_\alpha t + \int_{\gamma}^{b} |f(\gamma) - f(t)|g(t)\Diamond_\alpha t \\
&\quad \ge \int_a^{\gamma} |f(t) - f(\gamma)|\varphi(t)\Diamond_\alpha t + \int_{\gamma}^{b} |f(\gamma) - f(t)|\varphi(t)\Diamond_\alpha t \\
&\quad = \int_a^{b} |[f(t) - f(\gamma)]\varphi(t)|\Diamond_\alpha t.
\end{aligned} \tag{3.19}$$

Similarly, we find that

$$\int_a^{\ell} [f(t) - f(\ell)]g(t)\Diamond_\alpha t + \int_{\ell}^{b} [f(\ell) - f(t)][h(t) - g(t)]\Diamond_\alpha t \ge \int_a^{b} |[f(t) - f(\ell)]\varphi(t)|\Diamond_\alpha t. \tag{3.20}$$

By combining the integral identities (3.5) and (3.6) and the inequalities (3.19) and (3.20), we have inequality (3.17). □

Remark 3.6. When $\alpha = 0$ and setting $h(t) = 1$ and $\varphi(t) = 0$, inequality (3.17) reduces to [3, inequality (3.1)].

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