

*Research Article*

## **A General Projection Method for a System of Relaxed Cocoercive Variational Inequalities in Hilbert Spaces**

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We consider a new algorithm for a generalized system for relaxed cocoercive nonlinear inequalities involving three different operators in Hilbert spaces by the convergence of projection methods. Our results include the previous results as special cases extend and improve the main results of R. U. Verma (2004), S. S. Chang et al. (2007), Z. Y. Huang and M. A. Noor (2007), and many others.

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### **1. Introduction and preliminaries**

Variational inequalities introduced by Stampacchia [1] in the early sixties have had a great impact and influence in the development of almost all branches of pure and applied sciences and have witnessed an explosive growth in theoretical advances and algorithmic development; see [1–11] and references therein. It is well known that the variational inequality problems are equivalent to the fixed point problems. This alternative equivalent formulation is very important from the numerical analysis point of view and has played a significant part in several numerical methods for solving variational inequalities and complementarity; see [2, 4]. In particular, the solution of the variational inequalities can be computed using the iterative projection methods. It is well known that the convergence of the projection method requires the operator  $T$  to be strongly monotone and Lipschitz continuous. Gabay [5] has shown that the convergence of a projection method can be proved for cocoercive operators. Note that cocoercivity is a weaker condition than strong monotonicity. Recently, Verma [8] introduced a system of nonlinear strongly monotone variational inequalities and studied the approximation solvability of this system based on a system of projection methods. Chang et al. [3] also introduced a new system of nonlinear relaxed cocoercive variational inequalities and studied the approximation solvability

of this system based on a system of projection methods. Projection methods have been applied widely to problems arising especially from complementarity, convex quadratic programming, and variational problems.

In this paper, we consider, based on the projection method, the approximation solvability of a system of nonlinear relaxed cocoercive variational inequalities with three different relaxed cocoercive mappings and three quasi-nonexpansive mappings in the framework of Hilbert spaces. Solutions of the system of nonlinear relaxed cocoercive variational inequalities are also common fixed points of three different quasi-nonexpansive mappings. Our results obtained in this paper generalize the results of Chang et al. [3], Verma [8–10], Huang and Aslam Noor [6], and some others.

Let  $H$  be a real Hilbert space whose inner product and norm are denoted by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$ , respectively. Let  $C$  be a closed convex subset of  $H$  and let  $T : C \rightarrow H$  be a nonlinear mapping. Let  $P_C$  be the projection of  $H$  onto the convex subset  $C$ . The classical variational inequality denoted by  $VI(C, T)$  is to find  $u \in C$  such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in C. \quad (1.1)$$

Recall the following definitions.

(1)  $T$  is said to be  $u$ -cocoercive [8, 10] if there exists a constant  $u > 0$  such that

$$\langle Tx - Ty, x - y \rangle \geq u \|Tx - Ty\|^2, \quad \forall x, y \in C. \quad (1.2)$$

Clearly, every  $u$ -cocoercive mapping  $T$  is  $1/u$ -Lipschitz continuous.

(2)  $T$  is called  $v$ -strongly monotone if there exists a constant  $v > 0$  such that

$$\langle Tx - Ty, x - y \rangle \geq v \|x - y\|^2, \quad \forall x, y \in C. \quad (1.3)$$

(3)  $T$  is said to be relaxed  $(u, v)$ -cocoercive if there exist two constants  $u, v > 0$  such that

$$\langle Tx - Ty, x - y \rangle \geq (-u) \|Tx - Ty\|^2 + v \|x - y\|^2, \quad \forall x, y \in C. \quad (1.4)$$

For  $u = 0$ ,  $T$  is  $v$ -strongly monotone. This class of mappings is more general than the class of strongly monotone mappings. It is easy to see that we have the following implication.

$v$ -strongly monotonicity  $\Rightarrow$  relaxed  $(u, v)$ -cocoercivity.

(4)  $S : C \rightarrow C$  is said to be quasi-nonexpansive if  $F(S) \neq \emptyset$  and

$$\|Sx - p\| \leq \|x - p\|, \quad \forall x \in C, p \in F(S). \quad (1.5)$$

Next, we denote the set of fixed points of  $S$  by  $F(S)$ . If  $x^* \in F(S) \cap VI(C, T)$ , one can easily see

$$x^* = Sx^* = P_C[x^* - \rho Tx^*] = SP_C[x^* - \rho Tx^*], \quad (1.6)$$

where  $\rho > 0$  is a constant.

This formulation is used to suggest the following iterative methods for finding a common element of the set of the common fixed points of three different quasi-nonexpansive

mappings and the set of solutions of the variational inequalities with three different relaxed cocoercive mappings.

Let  $T_1, T_2, T_3 : C \times C \times C \rightarrow H$  be three mappings. Consider a system of nonlinear variational inequality (SNVID) problems as follows.

Find  $x^*, y^*, z^* \in C$  such that

$$\langle sT_1(y^*, z^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, s > 0, \quad (1.7)$$

$$\langle tT_2(z^*, x^*, y^*) + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in C, t > 0, \quad (1.8)$$

$$\langle rT_3(x^*, y^*, z^*) + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in C, r > 0. \quad (1.9)$$

One can easily see that the SNVID problems (1.7), (1.8), and (1.9) are equivalent to the following projection formulas

$$\begin{aligned} x^* &= P_C[y^* - sT_1(y^*, z^*, x^*)], \quad s > 0, \\ y^* &= P_C[z^* - tT_2(z^*, x^*, y^*)], \quad t > 0, \\ z^* &= P_C[x^* - rT_3(x^*, y^*, z^*)], \quad r > 0, \end{aligned} \quad (1.10)$$

respectively, where  $P_C$  is the projection of  $H$  onto  $C$ .

Next, we consider some special classes of the SNVID problems (1.7), (1.8), and (1.9) as follows.

(I) If  $r = 0$ , then the SNVID problems (1.7), (1.8), and (1.9) collapse to the following SNVID problems.

Find  $x^*, y^* \in C$  such that

$$\begin{aligned} \langle sT_1(y^*, x^*, x^*) + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, s > 0, \\ \langle tT_2(x^*, x^*, y^*) + y^* - x^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, t > 0. \end{aligned} \quad (1.11)$$

(II) If  $t = r = 0$ , then the SNVID problems (1.7), (1.8), and (1.9) are reduced to the following nonlinear variational inequality NVI problems.

Find an  $x^* \in C$  such that

$$\langle T_1(x^*, x^*, x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \quad (1.12)$$

(III) If  $T_1, T_2, T_3 : C \rightarrow H$  are univariate mappings, then the SVNID problems (1.7), (1.8), and (1.9) are reduced to the following SNVID problems.

Find  $x^*, y^* \in C$  such that

$$\langle sT_1(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, s > 0, \quad (1.13)$$

$$\langle tT_2(z^*) + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in C, t > 0, \quad (1.14)$$

$$\langle rT_3(x^*) + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in C, r > 0. \quad (1.15)$$

(IV) If  $T_1 = T_2 = T_3 = T : C \rightarrow H$  are univariate mappings, then the SVNID problems (1.7), (1.8), and (1.9) are reduced to the following SNVI problems.

Find  $x^*, y^* \in C$  such that

$$\langle sT(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, s > 0, \quad (1.16)$$

$$\langle tT(z^*) + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in C, t > 0, \quad (1.17)$$

$$\langle rT(x^*) + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in C, r > 0. \quad (1.18)$$

## 2. Algorithms

In this section, we consider an introduction of the general three-step models for the projection methods, and its special form can be applied to the convergence analysis for the projection methods in the context of the approximation solvability of the SNVID problems (1.7)–(1.9), (1.13)–(1.15), and SNVI problems (1.16)–(1.18).

*Algorithm 2.1.* For any  $x_0, y_0, z_0 \in C$ , compute the sequences  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  by the iterative processes

$$\begin{aligned} z_{n+1} &= S_3P_C[x_{n+1} - rT_3(x_{n+1}, y_{n+1}, z_n)], \\ y_{n+1} &= S_2P_C[z_{n+1} - tT_2(z_{n+1}, x_{n+1}, y_n)], \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nS_1P_C[y_n - sT_1(y_n, z_n, x_n)], \end{aligned} \quad (2.1)$$

where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  for all  $n \geq 0$ , and  $S_1, S_2$ , and  $S_3$  are three quasi-non-expansive mappings.

(I) If  $T_1, T_2, T_3 : C \rightarrow H$  are univariate mappings, then Algorithm 2.1 is reduced to the following algorithm.

*Algorithm 2.2.* For any  $x_0, y_0, z_0 \in C$ , compute the sequences  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  by the iterative processes

$$\begin{aligned} z_{n+1} &= S_3P_C[x_{n+1} - rT_3(x_{n+1})], \\ y_{n+1} &= S_2P_C[z_{n+1} - tT_2(z_{n+1})], \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nS_1P_C[y_n - sT_1(y_n)], \end{aligned} \quad (2.2)$$

where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  for all  $n \geq 0$ , and  $S_1, S_2$ , and  $S_3$  are three quasi-non-expansive mappings.

(II) If  $T_1 = T_2 = T_3 = T$  and  $S_1 = S_2 = S_3 = S$  in Algorithm 2.2, then we have the following algorithm.

*Algorithm 2.3.* For any  $x_0, y_0, z_0 \in C$ , compute the sequences  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  by the iterative processes

$$\begin{aligned} z_{n+1} &= SP_C[x_{n+1} - rT(x_{n+1})], \\ y_{n+1} &= SP_C[z_{n+1} - tT(z_{n+1})], \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nSP_C[y_n - sT(y_n)], \end{aligned} \quad (2.3)$$

where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  for all  $n \geq 0$ , and  $S$  is a quasi-nonexpansive mapping.

In order to prove our main results, we need the following lemmas and definitions.

LEMMA 2.4. Assume that  $\{a_n\}$  is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n, \quad \forall n \geq n_0, \quad (2.4)$$

where  $n_0$  is some nonnegative integer,  $\{\lambda_n\}$  is a sequence in  $(0, 1)$  with  $\sum_{n=1}^{\infty} \lambda_n = \infty$ ,  $b_n = o(\lambda_n)$ , and  $\sum_{n=0}^{\infty} c_n < \infty$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

Definition 2.5. A mapping  $T : C \times C \times C \rightarrow H$  is said to be relaxed  $(u, v)$ -cocoercive in the first variable if there exist constants  $u, v > 0$  such that, for all  $x, x' \in C$ ,

$$\begin{aligned} \langle T(x, y, z) - T(x', y', z'), x - x' \rangle &\geq (-u) \|T(x, y, z) - T(x', y', z')\|^2 \\ &+ v \|x - x'\|^2, \quad \forall y, y', z, z' \in C. \end{aligned} \quad (2.5)$$

Definition 2.6. A mapping  $T : C \times C \times C \rightarrow H$  is said to be  $\mu$ -Lipschitz continuous in the first variable if there exists a constant  $\mu > 0$  such that, for all  $x, x' \in C$ ,

$$\|T(x, y, z) - T(x', y', z')\| \leq \mu \|x - x'\|, \quad \forall y, y', z, z' \in C. \quad (2.6)$$

### 3. Main results

THEOREM 3.1. Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $T_1 : C \times C \times C \rightarrow H$  be a relaxed  $(u_1, v_1)$ -cocoercive and  $\mu_1$ -Lipschitz continuous mapping in the first variable,  $T_2 : C \times C \times C \rightarrow H$  a relaxed  $(u_2, v_2)$ -cocoercive and  $\mu_2$ -Lipschitz continuous mapping in the first variable,  $T_3 : C \times C \times C \rightarrow H$  a relaxed  $(u_3, v_3)$ -cocoercive and  $\mu_3$ -Lipschitz continuous mapping in the first variable, and  $S_1, S_2, S_3 : C \rightarrow C$  three quasi-nonexpansive mappings. Suppose that  $x^*, y^*, z^* \in C$  are solutions of the SNVID problems (1.7)–(1.9),  $x^*, y^*, z^* \in F(S_1) \cap F(S_2) \cap F(S_3)$ , and  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  are the sequences generated by Algorithm 2.1. If  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  satisfying the following conditions:

- (i)  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < s, t, r < \min\{2(v_1 - u_1\mu_1^2)/\mu_1^2, 2(v_2 - u_2\mu_2^2)/\mu_2^2, 2(v_3 - u_3\mu_3^2)/\mu_3^2\}$ ,
- (iii)  $v_1 > u_1\mu_1^2$ ,  $v_2 > u_2\mu_2^2$  and  $v_3 > u_3\mu_3^2$ ,

then the sequences  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  converge strongly to  $x^*$ ,  $y^*$ , and  $z^*$ , respectively.

*Proof.* Since  $x^*$ ,  $y^*$ , and  $z^*$  are the common elements of the set of solutions of the SNVID problems (1.7)–(1.9) and the set of common fixed points of  $S_1$ ,  $S_2$ , and  $S_3$ , we have

$$\begin{aligned} x^* &= S_1 P_C [y^* - s T_1(y^*, z^*, x^*)], \quad s > 0, \\ y^* &= S_2 P_C [z^* - t T_2(z^*, x^*, y^*)], \quad t > 0, \\ z^* &= S_3 P_C [x^* - r T_3(x^*, y^*, z^*)], \quad r > 0. \end{aligned} \quad (3.1)$$

Observing (2.1), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1 - \alpha_n)x_n + \alpha_n S_1 P_C [y_n - s T_1(y_n, z_n, x_n)] - x^*\| \\ &\leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \|y_n - y^* - s [T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\|. \end{aligned} \quad (3.2)$$

By the assumption that  $T_1$  is relaxed  $(u_1, v_1)$ -cocoercive and  $\mu_1$ -Lipschitz continuous in the first variable, we obtain

$$\begin{aligned}
& \|y_n - y^* - s[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\|^2 \\
&= \|y_n - y^*\| - 2s\langle y_n - y^*, T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*) \rangle \\
&\quad + s^2 \|T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\|^2 \\
&\leq \|y_n - y^*\| - 2s[-u_1 \|T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\|^2 + v_1 \|y_n - y^*\|^2] \\
&\quad + s^2 \mu_1^2 \|y_n - y^*\|^2 \quad (3.3) \\
&\leq \|y_n - y^*\| + 2su_1 \mu_1^2 \|y_n - y^*\|^2 - 2sv_1 \|y_n - y^*\|^2 + s^2 \mu_1^2 \|y_n - y^*\|^2 \\
&= \theta_1^2 \|y_n - y^*\|^2,
\end{aligned}$$

where  $\theta_1^2 = 1 + s^2 \mu_1^2 - 2sv_1 + 2su_1 \mu_1^2$ . From the conditions (ii) and (iii), we know  $\theta_1 < 1$ . Substituting (3.3) into (3.2) yields that

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \theta_1 \|y_n - y^*\|. \quad (3.4)$$

Now, we estimate

$$\begin{aligned}
\|y_{n+1} - y^*\| &= \|S_2 P_C[z_{n+1} - tT_2(z_{n+1}, x_{n+1}, y_n)] - y^*\| \\
&\leq \|z_{n+1} - z^* - t[T_2(z_{n+1}, x_{n+1}, y_n) - T_2(z^*, x^*, y^*)]\|. \quad (3.5)
\end{aligned}$$

By the assumption that  $T_2$  is relaxed  $(u_2, v_2)$ -cocoercive and  $\mu_2$ -Lipschitz continuous in the first variable, we obtain

$$\begin{aligned}
& \|z_{n+1} - z^* - t[T_2(z_{n+1}, x_{n+1}, y_n) - T_2(z^*, x^*, y^*)]\|^2 \\
&= \|z_{n+1} - z^*\|^2 - 2t\langle z_{n+1} - z^*, T_2(z_{n+1}, x_{n+1}, y_n) - T_2(z^*, x^*, y^*) \rangle \\
&\quad + t^2 \|T_2(z_{n+1}, x_{n+1}, y_n) - T_2(z^*, x^*, y^*)\|^2 \\
&\leq \|z_{n+1} - z^*\|^2 - 2t[-u_2 \|T_2(z_{n+1}, x_{n+1}, y_n) - T_2(z^*, x^*, y^*)\|^2 + v_2 \|z_{n+1} - z^*\|^2] \\
&\quad + t^2 \mu_2^2 \|z_{n+1} - z^*\|^2 \\
&\leq \|z_{n+1} - z^*\|^2 + 2tu_2 \mu_2^2 \|z_{n+1} - z^*\|^2 - 2tv_2 \|z_{n+1} - z^*\|^2 + t^2 \mu_2^2 \|z_{n+1} - z^*\|^2 \\
&\leq \theta_2^2 \|z_{n+1} - z^*\|^2, \quad (3.6)
\end{aligned}$$

where  $\theta_2 = 1 + t^2 \mu_2^2 - 2tv_2 + 2tu_2 \mu_2^2$ . From the conditions (ii) and (iii), we know that  $\theta_2 < 1$ . Substituting (3.6) into (3.5) yields that

$$\|y_{n+1} - y^*\| \leq \theta_2 \|z_{n+1} - z^*\|, \quad (3.7)$$

which implies that

$$\|y_n - y^*\| \leq \theta_2 \|z_n - z^*\|. \quad (3.8)$$

Similarly, substituting (3.8) into (3.4), we have

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\theta_2\|z_n - z^*\|. \quad (3.9)$$

Next, we show that

$$\begin{aligned} \|z_{n+1} - z^*\| &= \|S_3 P_C[x_{n+1} - rT_3(x_{n+1}, y_{n+1}, z_n)] - z^*\| \\ &\leq \|x_{n+1} - x^* - r[T_3(x_{n+1}, y_{n+1}, z_n) - T_3(x^*, y^*, z^*)]\|. \end{aligned} \quad (3.10)$$

By the assumption that  $T_3$  is relaxed  $(u_3, v_3)$ -cocoercive and  $\mu_3$ -Lipschitz continuous in the first variable, we obtain

$$\begin{aligned} &\|x_{n+1} - x^* - r[T_3(x_{n+1}, y_{n+1}, z_n) - T_3(x^*, y^*, z^*)]\|^2 \\ &= \|x_{n+1} - x^*\|^2 - 2r\langle x_{n+1} - x^*, T_3(x_{n+1}, y_{n+1}, z_n) - T_3(x^*, y^*, z^*) \rangle \\ &\quad + r^2\|T_3(x_{n+1}, y_{n+1}, z_n) - T_3(x^*, y^*, z^*)\|^2 \\ &\leq \|x_{n+1} - x^*\|^2 - 2r[-u_3\|T_3(x_{n+1}, y_{n+1}, z_n) - T_3(x^*, y^*, z^*)\|^2 + v_3\|x_{n+1} - x^*\|^2] \\ &\quad + r^2\mu_3^2\|x_n - x^*\|^2 \\ &\leq \|x_{n+1} - x^*\|^2 + 2ru_3\mu_3^2\|x_{n+1} - x^*\|^2 - 2rv_3\|x_{n+1} - x^*\|^2 + r^2\mu_3^2\|x_{n+1} - x^*\|^2 \\ &= \theta_3^2\|x_{n+1} - x^*\|^2, \end{aligned} \quad (3.11)$$

where  $\theta_3^2 = 1 + r^2\mu_3^2 - 2rv_3 + 2ru_3\mu_3^2$ . From the conditions (ii) and (iii), we know that  $\theta_3 < 1$ . Substituting (3.11) into (3.10), we obtain

$$\|z_{n+1} - z^*\| \leq \theta_3\|x_{n+1} - x^*\|, \quad (3.12)$$

which implies

$$\|z_n - z^*\| \leq \theta_3\|x_n - x^*\|. \quad (3.13)$$

Similarly, substituting (3.13) into (3.9) yields that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\theta_2\theta_3\|x_n - x^*\| \\ &\leq [1 - \alpha_n(1 - \theta_1\theta_2\theta_3)]\|x_n - x^*\|. \end{aligned} \quad (3.14)$$

Noticing that  $\sum_{n=0}^{\infty} \alpha_n(1 - \theta_1\theta_2\theta_3) = \infty$  and applying Lemma 2.4 into (3.14), we can get the desired conclusion easily. This completes the proof.  $\square$

*Remark 3.2.* Theorem 3.1 extends the solvability of the SNVI of Chang [3] and Verma [8] to the more general SNVID (1.7)–(1.9) and improves the main results of [3, Theorem 2.1], [8, Theorem 3.3] by using an explicit iteration scheme, Algorithm 2.1. The computation workload is much less than the implicit algorithms in Chang [3] and Verma [8].

Moreover, Theorem 3.1 also extends the SNVID of Huang and Aslam Noor [6] to some extent.

From Theorem 3.1, we can get the following results immediately.

**THEOREM 3.3.** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $T_1 : C \rightarrow H$  be a relaxed  $(u_1, v_1)$ -cocoerceive and  $\mu_1$ -Lipschitz continuous mapping,  $T_2 : C \rightarrow H$  a relaxed  $(u_2, v_2)$ -cocoerceive and  $\mu_2$ -Lipschitz continuous mapping,  $T_3 : C \rightarrow H$  a relaxed  $(u_3, v_3)$ -cocoerceive and  $\mu_3$ -Lipschitz continuous mapping, and  $S_1, S_2, S_3 : C \rightarrow C$  three quasi-nonexpansive mappings. Suppose that  $x^*, y^*, z^* \in C$  are solutions of the SNVID problems (1.13)–(1.15),  $x^*, y^*, z^* \in F(S_1) \cap F(S_2) \cap F(S_3)$ , and  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  are the sequences generated by Algorithm 2.2. If  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  satisfying the following conditions:*

- (i)  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < s, t, r < \min\{2(v_1 - u_1\mu_1^2)/\mu_1^2, 2(v_2 - u_2\mu_2^2)/\mu_2^2, 2(v_3 - u_3\mu_3^2)/\mu_3^2\}$ ,
- (iii)  $v_1 > u_1\mu_1^2$ ,  $v_2 > u_2\mu_2^2$  and  $v_3 > u_3\mu_3^2$ ,

*then the sequences  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  converge strongly to  $x^*$ ,  $y^*$ , and  $z^*$ , respectively.*

**Remark 3.4.** Theorem 3.3 includes Theorem 3.5 of Huang and Aslam Noor [6] as a special case and also improves the main results of Chang et al. [3] and Verma [8] by explicit projection algorithms.

**THEOREM 3.5.** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $T : C \rightarrow H$  be a relaxed  $(u, v)$ -cocoerceive and  $\mu$ -Lipschitz continuous mapping and let  $S : C \rightarrow C$  be a quasi-nonexpansive mapping. Suppose that  $x^*, y^*, z^* \in C$  are solutions of the SNVI problems (1.16)–(1.18),  $x^*, y^*, z^* \in F(S)$ , and  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  are the sequences generated by Algorithm 2.3. If  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  satisfying the following conditions:*

- (i)  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ,
- (ii)  $0 < s, t, r < (2(v - u\mu^2))/\mu^2$ ,
- (iii)  $v > u\mu^2$ ,

*then the sequences  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  converge strongly to  $x^*$ ,  $y^*$ , and  $z^*$ , respectively.*

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