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Research Article

On the (p,q)-Boundedness of Nonisotropic Spherical Riesz Potentials

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We introduced the concept of nonisotropic spherical Riesz potential operators generated by the λ-distance of variable order on λ-sphere and its (p,q)-boundedness were investigated.

1. Introduction

Let

$$\mathbb{R}^n = \{x = (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R},\ 1 \le i \le n\}. \tag{1.1}$$

In $\mathbb{R}^n$ spaces, L_p and L_∞ are defined as follows:

$$\begin{gathered} L_p = L_p(\Omega_{n,\lambda}) = \left\{ f(x) : \|f\|_p = \left(\int_{\Omega_{n,\lambda}} |f(x)|^p dx \right)^{1/p} < \infty \right\}, \quad 1 \le p < \infty \\ L_\infty = L_\infty(\Omega_{n,\lambda}) = \left\{ f(x) : \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega_{n,\lambda}} |f(x)| < \infty \right\}, \end{gathered} \tag{1.2}$$

where $\Omega_{n,\lambda}$ is the n-dimensional unite λ-sphere of $\mathbb{R}^n$ which is dependent on the λ-distance. The λ-distance between points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ is defined by the following formula given in [1–10]:

$$|x - y|_\lambda := \left(|x_1 - y_1|^{1/\lambda_1} + |x_2 - y_2|^{1/\lambda_2} + \cdots + |x_n - y_n|^{1/\lambda_n} \right)^{|\lambda|/n}, \tag{1.3}$$

where $x,y \in \Omega_{n,\lambda}$, $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)$, $\lambda_k > 0$, $k = 1,2,\ldots,n$, $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Note that this distance has the following properties of homogeneity for any positive t,

$$\left(\left| t^{\lambda_1} x_1 \right|^{1/\lambda_1} + \cdots + \left| t^{\lambda_n} x_n \right|^{1/\lambda_n} \right)^{|\lambda|/n} = t^{|\lambda|/n} |x|_\lambda. \tag{1.4}$$

This equality give us that nonisotropic λ-distance is the order of a homogeneous function $|\lambda|/n$. So the nonisotropic λ-distance has the following properties:

(1) $|x|_\lambda = 0 \Leftrightarrow x = \theta$,

(2) $|t^\lambda x|_\lambda = |t|^{|\lambda|/n} |x|_\lambda$,

(3) $|x+y|_\lambda \leq 2^{(1+1/\lambda_{\min})|\lambda|/n} (|x|_\lambda + |y|_\lambda)$.

Here we consider λ-spherical coordinates by the following formulas:

$$x_1 = (\rho \cos\theta_1)^{2\lambda_1}, \ldots, x_n = (\rho \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1})^{2\lambda_n}. \tag{1.5}$$

We obtained that $|x|_\lambda = \rho^{2|\lambda|/n}$. It can be seen that the Jacobian $J_\lambda(\rho,\varphi)$ of this transformation is $J_\lambda(\rho,\theta) = \rho^{2|\lambda|-1} W_\lambda(\theta)$, where $W_\lambda(\theta)$ is the bounded function, which only depends on angles $\theta_1,\theta_2,\ldots,\theta_{n-1}$. It is clear that if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1/2$, then the λ-distance is the Euclidean distance.

We define angle

$$\cos|x-y|_\lambda = x \cdot y, \tag{1.6}$$

where x and y are vectors on the n-dimensional unite λ-sphere.

For $f \in L(\Omega_{n,\lambda})$, $0 < \alpha(x) < n$, we will consider the following nonisotropic spherical Riesz potential operator generated by the λ-distance of variable order:

$$I_\lambda^{\alpha(x)} f(x) = \int_{\Omega_{n,\lambda}} |x-y|_\lambda^{\alpha(x)-n} f(y)\,dy, \quad x \in \Omega_{n,\lambda}. \tag{1.7}$$

The aim of this paper to show that the well-known properties of classical Riesz potentials may be formulated for our generalization (1.7). We will study the (p,q)-boundedness of operators (1.7). Note that our results are the generalization of corresponding results for classical Riesz potentials, given in [11]. The important properties of the nonisotropic Riesz potentials and theirs generalizations were studied by many authors. We refer to papers [1–9, 12]. The nonisotropic spherical Riesz potential generated by λ-distance is the classical Riesz potential for $\lambda_i = 1/2$, $i = 1,2,\ldots,n$ and $\alpha(x) = \alpha$. Here particular importance of the nonisotropic kernel is that it does not have the classical triangle inequality.

It is well known that the classical Riesz potentials $I_\alpha\varphi = \varphi * |x|^{\alpha-n}$ are bounded operators from $L_p(R^n)$ to $L_q(R^n)$ for $1/q = 1/p - \alpha/n$, $0 < \alpha < n$, $1 \leq p < q < \infty$ [10].

Lemma 1.1. *Let* $J_\lambda(x) = \int_{\Omega^1_{n,\lambda}} f(x) K(x,y)\,dy$, $x \in \Omega^2_{n,\lambda}$,

$$k_1 = \sup_{y\in\Omega^1_{n,\lambda}} \left(\int_{\Omega^2_{n,\lambda}} |K(x,y)|^q dx \right)^{1/r} < \infty, \qquad k_2 = \sup_{x\in\Omega^2_{n,\lambda}} \left(\int_{\Omega^1_{n,\lambda}} |K(x,y)|^q dy \right)^{1/q-1/r} < \infty \tag{1.8}$$

and the following conditions are carried out: $1 \leq p \leq r \leq \infty$, $1 - 1/p + 1/r = 1/q$, $f \in L_p(\Omega^1_{n,\lambda})$. *Then*

$$||J_\lambda||_{L_p(\Omega^1_{n,\lambda})} \leq ||f||_{L_r(\Omega^2_{n,\lambda})} k_1 k_2. \tag{1.9}$$

Proof. Let λ, μ, ν be positive numbers such that $1/\lambda + 1/\mu + 1/\nu = 1$. We write

$$J_\lambda(x) = \int_{\Omega^1_{n,\lambda}} f^{p(1/p-1/\mu)}(x) f^{p/\mu}(x) K^{q(1/q-1/\nu)}(x,y) K^{q/\nu}(x,y)\,dy. \tag{1.10}$$

By Hölder's inequality with exponents λ, μ, and ν, we obtain

$$J_\lambda(x) \leq \left(\int_{\Omega^1_{n,\lambda}} f(y)^{p\lambda(1/p-1/\mu)} K(x,y)^{\lambda q(1/q-1/\nu)}\,dy\right)^{1/\lambda} \left(\int_{\Omega^1_{n,\lambda}} f(y)^p\,dy\right)^{1/\mu} \left(\int_{\Omega^1_{n,\lambda}} K(x,y)^q\,dy\right)^{1/\nu}. \tag{1.11}$$

Since we want to have f^p and K^q in the integrand above, we note that we can choose λ, μ, ν in such a way

$$\frac{1}{\lambda} = \left(\frac{1}{p} - \frac{1}{\mu}\right), \qquad \frac{1}{\lambda} = \left(\frac{1}{q} - \frac{1}{\nu}\right), \qquad \frac{1}{\lambda} = \frac{1}{r}. \tag{1.12}$$

With these choices of λ, μ and ν, we can rewrite expression last inequality,

$$J_\lambda(x) \leq \|f\|^{p/\mu}_{L_p(\Omega^1_{n,\lambda})} \left(\int_{\Omega^1_{n,\lambda}} K(x,y)^q\,dy\right)^{1/\nu} \left(\int_{\Omega^1_{n,\lambda}} f(y)^p K(x,y)^q\,dy\right)^{1/\lambda}. \tag{1.13}$$

Taking rth powers and integrating in x,

$$\begin{aligned}
&\int_{\Omega^2_{n,\lambda}} |J_\lambda(x)|^r\,dx \\
&\quad\leq \|f\|^{rp/\mu}_{L_p(\Omega^1_{n,\lambda})} \int_{\Omega^2_{n,\lambda}} \left(\int_{\Omega^1_{n,\lambda}} K(x,y)^q\,dy\right)^{r/\nu} \left(\int_{\Omega^1_{n,\lambda}} f(y)^p K(x,y)^q\,dy\right)^{r/\lambda} dx \\
&\quad\leq \|f\|^{rp/\mu}_{L_p(\Omega^1_{n,\lambda})} \sup_{x\in\Omega^2_{n,\lambda}} \left(\int_{\Omega^1_{n,\lambda}} K(x,y)^q\,dy\right)^{r/\nu} \int_{\Omega^2_{n,\lambda}} \sup_{y\in\Omega^1_{n,\lambda}} K(x,y)^q \left(\int_{\Omega^1_{n,\lambda}} f(y)^p\,dy\right) dx \\
&\quad\leq \|f\|^{rp/\mu}_{L_p(\Omega^1_{n,\lambda})} \|f\|^{p}_{L_p(\Omega^1_{n,\lambda})} \sup_{y\in\Omega^1_{n,\lambda}} \left(\int_{\Omega^2_{n,\lambda}} K(x,y)^q\,dx\right) \sup_{x\in\Omega^2_{n,\lambda}} \left(\int_{\Omega^1_{n,\lambda}} K(x,y)^q\,dy\right)^{r/\nu}.
\end{aligned} \tag{1.14}$$

Hence

$$||J_\lambda(x)||^r_{L_r(\Omega^2_{n,\lambda})} \leq \|f\|^{p(r/\mu+1)}_{L_p(\Omega^1_{n,\lambda})} \sup_{y\in\Omega^1_{n,\lambda}} \left(\int_{\Omega^2_{n,\lambda}} K(x,y)^q\,dx\right) \sup_{x\in\Omega^2_{n,\lambda}} \left(\int_{\Omega^1_{n,\lambda}} K(x,y)^q\,dy\right)^{r/\nu}. \tag{1.15}$$

Taking rth roots, we have the following inequality:

$$\begin{aligned}
&||J_\lambda(x)||_{L_r(\Omega^2_{n,\lambda})}\\
&\quad\le \|f\|^{p(1/\mu+1/r)}_{L_p(\Omega^1_{n,\lambda})}\sup_{y\in\Omega^1_{n,\lambda}}\left(\int_{\Omega^2_{n,\lambda}}K(x,y)^q dx\right)^{1/r}\sup_{x\in\Omega^2_{n,\lambda}}\left(\int_{\Omega^1_{n,\lambda}}K(x,y)^q dy\right)^{1/\nu}\\
&\quad\le \|f\|_{L_p(\Omega^1_{n,\lambda})}\sup_{y\in\Omega^1_{n,\lambda}}\left(\int_{\Omega^2_{n,\lambda}}K(x,y)^q dx\right)^{1/r}\sup_{x\in\Omega^2_{n,\lambda}}\left(\int_{\Omega^1_{n,\lambda}}K(x,y)^q dy\right)^{1/q-1/r}\\
&\quad\le \|f\|_{L_p(\Omega^1_{n,\lambda})}k_1k_2.
\end{aligned}\tag{1.16}$$

□

THEOREM 1.2 (Riesz-Therin interpolation theorem, [13]). *Suppose T is simultaneously of weak types (p_0,q_0) and (p_1,q_1), $1\le p_i,q_i\le\infty$. If $0<t<1$ and $1/p_t=(1-t)/p_0+t/p_1$, $1/q_t=(1-t)/q_0+t/q_1$, then T is of type (p_t,q_t), and*

$$\|T\|_{(p_t,q_t)}\le\|T\|^{1-t}_{(p_0,q_0)}\|T\|^t_{(p_1,q_1)}.\tag{1.17}$$

The following theorem gives the condition of absolute convergence of the potential $I_\lambda^{\alpha(x)}f$.

THEOREM 1.3. *Let $0<m\le\alpha(x)<n$, $f\in L_1(\Omega_{n,\lambda})$. Then the integral (1.7) is absolutely convergent for almost every x.*

Proof. Let $L_{y,\theta}=\{x\in\Omega_{n,\lambda}: y\cdot x=\cos\theta\}$, $|L_{y,\theta}|=|\Omega_{n-1,\lambda}|\sin^{2|\lambda|-1}\theta$. Hence we have

$$\begin{aligned}
&\int_{\Omega_{n,\lambda}}|I_\lambda^{\alpha(x)}f(x)|\,dx\\
&\quad\le\iint_{\Omega_{n,\lambda}}\frac{|f(y)|}{|x-y|_\lambda^{n-\alpha(x)}}dy\,dx\\
&\quad=\int_{\Omega_{n,\lambda}}|f(y)|\int_{\Omega_{n,\lambda}}\frac{1}{|x-y|_\lambda^{n-\alpha(x)}}dx\,dy\\
&\quad=\int_{\Omega_{n,\lambda}}|f(y)|\left[\int_0^\pi\left(\int_{L_{y,\theta}}\frac{1}{\theta^{(2|\lambda|/n)(n-\alpha(x))}}dL_{y,\theta}(x)\right)d\theta\right]dy\\
&\quad=\int_{\Omega_{n,\lambda}}|f(y)|\left[\left(\int_0^1+\int_1^\pi\right)\left(\int_{L_{y,\theta}}\frac{1}{\theta^{(2|\lambda|/n)(n-\alpha(x))}}dL_{y,\theta}(x)\right)d\theta\right]dy\\
&\quad\le\int_{\Omega_{n,\lambda}}|f(y)|\left[\int_0^1\left(\int_{L_{y,\theta}}\frac{1}{\theta^{(2|\lambda|/n)(n-m)}}dL_{y,\theta}(x)\right)d\theta+\int_1^\pi\left(\int_{L_{y,\theta}}dL_{y,\theta}(x)\right)d\theta\right]dy\\
&\quad\le\int_{\Omega_{n,\lambda}}|f(y)|\left[\int_0^1\frac{|\Omega_{n-1,\lambda}|\sin^{2|\lambda|-1}\theta}{\theta^{(2|\lambda|/n)(n-m)}}d\theta+\int_1^\pi|\Omega_{n-1,\lambda}|\sin^{2|\lambda|-1}\theta\,d\theta\right]dy\\
&\quad\le|\Omega_{n-1,\lambda}|\int_{\Omega_{n,\lambda}}|f(y)|\left[\int_0^1\frac{1}{\theta^{1-(2|\lambda|/n)m}}d\theta+\int_1^\pi d\theta\right]dy\le M\|f\|_1<\infty.
\end{aligned}\tag{1.18}$$

The proof is completed. □

THEOREM 1.4. *Let* $0 < m \leq \alpha(x) < n$, $1 \leq p < \infty$. *Then* $I_\lambda^{\alpha(x)} f$ *is of type* (p,p), *that is,*

$$||I_\lambda^{\alpha(x)} f||_p \leq M \| f \|_p, \tag{1.19}$$

where the constant M *is dependent on* λ, m, *and* n.

Proof. Let

$$S_\theta f(x) = \frac{1}{|L_{x,\theta}|} \int_{L_{x,\theta}} f(y) dL_{x,\theta}(y). \tag{1.20}$$

Thus we have

$$||S_\theta f||_p \leq M \| f \|_p. \tag{1.21}$$

By the Minkowsky inequality for integrals, we have the following inequality:

$$\begin{aligned}
&\left(\int_{\Omega_{n,\lambda}} | I_\lambda^{\alpha(x)} f(x) |^p dx \right)^{1/p} \\
&\quad = \left(\int_{\Omega_{n,\lambda}} \left| \int_{\Omega_{n,\lambda}} \frac{|f(y)|}{|x-y|_\lambda^{n-\alpha(x)}} dy \right|^p dx \right)^{1/p} \\
&\quad \leq \left(\int_{\Omega_{n,\lambda}} \left| \int_0^\pi \frac{|\Omega_{n-1,\lambda}| \sin^{2\|\lambda|-1} \theta}{\theta^{(2|\lambda/n)(n-\alpha(x))}} \frac{1}{|L_{x,\theta}|} \int_{L_{x,\theta}} f(y) dL_{x,\theta} d\theta \right|^p dx \right)^{1/p} \\
&\quad \leq M \left(\int_{\Omega_{n,\lambda}} \left| \int_0^\pi \frac{1}{\theta^{1-(2|\lambda|/n)\alpha(x)}} |S_\theta(f)| d\theta \right|^p dx \right)^{1/p} \\
&\quad \leq M \left(\int_0^1 + \int_1^\pi \right) \left(\int_{\Omega_{n,\lambda}} \frac{1}{\theta^{(1-(2|\lambda|/n)\alpha(x))p}} |S_\theta(f)|^p dx \right)^{1/p} d\theta \\
&\quad \leq M \int_0^1 \frac{1}{\theta^{1-(2|\lambda|/n)\alpha(x)}} \left(\int_{\Omega_{n,\lambda}} |S_\theta(f)|^p dx \right)^{1/p} d\theta + M \int_1^\pi \left(\int_{\Omega_{n,\lambda}} |S_\theta(f)|^p dx \right)^{1/p} d\theta \\
&\quad \leq M \int_0^1 \frac{1}{\theta^{1-(2|\lambda|/n)\alpha(x)}} ||S_\theta(f)||_p d\theta + M \int_1^\pi ||S_\theta(f)||_p d\theta \\
&\quad \leq M \| f \|_p \left(\int_0^1 \frac{d\theta}{\theta^{1-(2|\lambda|/n)\alpha(x)}} + \int_1^\pi d\theta \right) \leq M \| f \|_p.
\end{aligned} \tag{1.22}$$

The proof is completed. □

The following theorem is an expanded form of Theorem 1.4.

THEOREM 1.5. *Let* $0 < m \leq \alpha(x) < n$, $1 < p \leq r$, $n/p - n/r < m$. *Then* $I_\lambda^{\alpha(x)} f$ *is of type* (p,r), *that is,*

$$||I_\lambda^{\alpha(x)} f||_r \leq M \| f \|_p, \tag{1.23}$$

where the constant M *is dependent on* λ, m, *and* n.

Proof. Let $q = pr/(pr+p-r)$, $1/q+1/q' = 1$. We show that $I_\lambda^{\alpha(x)} f$ is of type $(1,q)$ and (q',∞). By the Minkowsky inequality for integrals, we have the following inequality:

$$\begin{aligned}
&||I_\lambda^{\alpha(x)} f||_q \\
&\quad\le \left(\int_{\Omega_{n,\lambda}} \left|\int_{\Omega_{n,\lambda}} \frac{|f(y)|}{|x-y|_\lambda^{n-\alpha(x)}} dy\right|^q dx\right)^{1/q} \\
&\quad\le \int_{\Omega_{n,\lambda}} \left(\int_{\Omega_{n,\lambda}} \frac{|f(y)|^q}{|x-y|_\lambda^{(n-\alpha(x))q}} dx\right)^{1/q} dy \\
&\quad= \int_{\Omega_{n,\lambda}} |f(y)| \left(\int_{\Omega_{n,\lambda}} \frac{1}{|x-y|_\lambda^{(n-\alpha(x))q}} dx\right)^{1/q} dy \\
&\quad\le \int_{\Omega_{n,\lambda}} |f(y)| \left[\int_0^\pi \left(\int_{L_{y,\theta}} \frac{1}{\theta^{(2|\lambda|/n)(n-\alpha(x))q}} dL_{y,\theta}(x)\right) d\theta\right] dy \\
&\quad\le \int_{\Omega_{n,\lambda}} |f(y)| \left[\int_0^1 \left(\int_{L_{y,\theta}} \frac{1}{\theta^{(2|\lambda|/n)(n-m)q}} dL_{y,\theta}(x)\right) d\theta + \int_1^\pi |\Omega_{n-1,\lambda}| \sin^{2|\lambda|-1}\theta d\theta\right] dy \\
&\quad\le \int_{\Omega_{n,\lambda}} |f(y)| \left[\int_0^1 \frac{|\Omega_{n-1,\lambda}| \sin^{2|\lambda|-1}\theta}{\theta^{(2|\lambda|/n)(n-m)q}} d\theta + M\right] dy \\
&\quad\le \int_{\Omega_{n,\lambda}} |f(y)| \left[|\Omega_{n-1,\lambda}| \int_0^1 \frac{1}{\theta^{(2|\lambda|/n)(n-m)q-2|\lambda|+1}} d\theta + M\right] dy \le M\|f\|_1.
\end{aligned} \tag{1.24}$$

Thus the last integral is convergence where

$$\begin{gathered}
\frac{pr}{pr+p-r} < \frac{n}{n-m} \quad \text{for } \frac{n}{p} - \frac{n}{r} < m, \\
q < \frac{n}{n-m} \Longrightarrow \frac{2|\lambda|}{n}(n-m)q + 1 - 2|\lambda| < 1.
\end{gathered} \tag{1.25}$$

This shows that $I_\lambda^{\alpha(x)} f$ is of type $(1,q)$. On the other hand, from Hölder's inequality, we have

$$\begin{aligned}
|I_\lambda^{\alpha(x)} f| &< \int_{\Omega_{n,\lambda}} \frac{|f(y)|}{|x-y|_\lambda^{n-\alpha(x)}} dy \\
&\le \left(\int_{\Omega_{n,\lambda}} |f(y)|^{q'} dy\right)^{1/q'} \left(\int_{\Omega_{n,\lambda}} \frac{1}{|x-y|_\lambda^{(n-\alpha(x))q}} dy\right)^{1/q} \le M\|f\|_{q'}.
\end{aligned} \tag{1.26}$$

Therefore we have

$$||I_\lambda^{\alpha(x)} f||_\infty \le M\|f\|_{q'}. \tag{1.27}$$

This shows that $I_\lambda^{\alpha(x)} f$ is of type (q',∞).

Let $t = q(1-1/p)$, then from Theorem 1.2, $I_\lambda^{\alpha(x)} f$ is of type (p,r) where $1/p = (1-t)/1 + t/q'$, $1/r = (1-t)/q'$. The proof is completed. □

THEOREM 1.6. *Let $0 < m \leq \alpha(x) < n$, $1 < p < r$, $n/p - n/r = m$. Then $I_\lambda^{\alpha(x)} f$ is of type (p,r).*

Proof. Firstly, for a constant m we will consider the $\alpha(x) = m$. Thus, by using Lemma 1.1 for $K(x,y) = |x-y|_\lambda^{m-n}$, we obtain the following inequality:

$$\left\|I_\lambda^m f\right\|_r < M\|f\|_p. \tag{1.28}$$

This shows that I_λ^m is of (p,r) type.

Let

$$\Omega_{n,\lambda,x} = \{y \in \Omega_{n,\lambda} : |x-y|_\lambda \geq 1\}, \qquad \overline{\Omega}_{n,\lambda,x} = \Omega_{n,\lambda} \backslash \Omega_{n,\lambda,x}. \tag{1.29}$$

Then

$$\begin{aligned} \left| I_\lambda^{\alpha(x)} f \right| &\leq \int_{\Omega_{n,\lambda}} \frac{|f(y)|}{|x-y|_\lambda^{n-\alpha(x)}} dy \leq \int_{\Omega_{n,\lambda,x}} |f(y)|\, dy + \int_{\overline{\Omega}_{n,\lambda,x}} \frac{|f(y)|}{|x-y|_\lambda^{n-m}} dy \\ &\leq \int_{\Omega_{n,\lambda,x}} |f(y)|\, dy + I_\lambda^m f(x) \leq M' \|f\|_p + I_\lambda^m f(x). \end{aligned} \tag{1.30}$$

Therefore we have

$$\begin{aligned} \left\|I_\lambda^{\alpha(x)} f\right\|_r &\leq \left\|M' \|f\|_p + I_\lambda^m f(x)\right\|_r \leq M' \|f\|_p + \left\|I_\lambda^m f(x)\right\|_r \\ &\leq M' \|f\|_p + M\|f\|_p = C\|f\|_p. \end{aligned} \tag{1.31}$$

Thus $I_\lambda^{\alpha(x)} f$ is of type (p,r).

The proof is completed. □

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