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Research Article

Spectrum of Class $wF(p,r,q)$ Operators

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Dedicated to Professor Daoxing Xia on his 77th birthday with respect and affection

Recommended by Jozsef Szabados

This paper discusses some spectral properties of class $wF(p,r,q)$ operators for $p>0$, $r>0$, $p+r\le 1$, and $q\ge 1$. It is shown that if T is a class $wF(p,r,q)$ operator, then the Riesz idempotent E_λ of T with respect to each nonzero isolated point spectrum λ is selfadjoint and $E_\lambda\mathcal{H}=\ker(T-\lambda)=\ker(T-\lambda)^*$. Afterwards, we prove that every class $wF(p,r,q)$ operator has SVEP and property (β), and Weyl's theorem holds for $f(T)$ when $f\in H(\sigma(T))$.

1. Introduction

A capital letter (such as T) means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. For $p>0$, an operator T is said to be p-hyponormal if $(T^*T)^p\ge(TT^*)^p$, where T^* is the adjoint operator of T. An invertible operator T is said to be log-hyponormal if $\log(T^*T)\ge\log(TT^*)$. If $p=1$, T is called hyponormal, and if $p=1/2$ T is called semi-hyponormal. Log-hyponormality is sometimes regarded as 0-hyponormal since $(X^p-1)/p\to\log X$ as $p\to 0$ for $X>0$.

See Martin and Putinar [1] and Xia [2] for basic properties of hyponormal and semi-hyponormal operators. Log-hyponormal operators were introduced by Tanahashi [3], Aluthge and Wang [4], and Fujii et al. [5] independently. Aluthge [6] introduced p-hyponormal operators.

As generalizations of p-hyponormal and log-hyponormal operators, many authors introduced many classes of operators. Aluthge and Wang [4] introduced w-hyponormal operators defined by $|\widetilde{T}|\ge|T|\ge|(\widetilde{T})^*|$, where the polar decomposition of T is $T=U|T|$ and $\widetilde{T}=|T|^{1/2}U|T|^{1/2}$ is called Aluthge transformation of T. For $p>0$ and $r>0$, Ito [7] introduced class $wA(p,r)$ defined by

$$\left(|T^*|^r|T|^{2p}|T^*|^r\right)^{r/(p+r)} \geq |T^*|^{2r}, \qquad \left(|T|^p|T^*|^{2r}|T|^p\right)^{s/(p+r)} \leq |T|^{2p}. \tag{1.1}$$

Note that the two exponents $r/(p+r)$ and $p/(p+r)$ in the formula above satisfy $r/(p+r)+p/(p+r)=1$, Yang and Yuan [8] introduced class $wF(p,r,q)$.

Definition 1.1 (see [8, 9]). For $p>0$, $r>0$, and $q\geq 1$, an operator T belongs to class $wF(p,r,q)$ if

$$\left(|T^*|^r|T|^{2p}|T^*|^r\right)^{1/q} \geq |T^*|^{2(p+r)/q}, \qquad |T|^{2(p+r)(1-1/q)} \geq \left(|T|^p|T^*|^{2r}|T|^p\right)^{(1-1/q)}. \tag{1.2}$$

Denote $(1-q^{-1})^{-1}$ by q^* when $q>1$ because q and $(1-q^{-1})^{-1}$ are a couple of conjugate exponents. It is clear that class $wA(p,r)$ equals class $wF(p,r,(p+r)/r)$.

w-hyponormality equals $wA(1/2,1/2)$ [7]. Ito and Yamazaki [10] showed that class $wA(p,r)$ coincides with class $A(p,r)$ (introduced by Fujii et al. [11]) for each $p>0$ and $r>0$. Consequently, class $wA(1,1)$ equals class A (i.e., $|T^2|\geq|T|^2$, introduced by Furuta et al. [12]). Reference [9] showed that class $wF(p,r,q)$ coincides with class $F(p,r,q)$ (introduced by Fujii and Nakamoto [13]) when $rq\leq p+r$.

Recently, there are great developments in the spectral theory of the classes of operators above. We cite [8, 14–22]. In this paper, we will discuss several spectral properties of class $wF(p,r,q)$ for $p>0$, $r>0$, $p+r\leq 1$, and $q\geq 1$.

In Section 2, we prove that Riesz idempotent E_λ of T with respect to each nonzero isolated point spectrum λ is selfadjoint and $E_\lambda\mathscr{H}=\ker(T-\lambda)=\ker(T-\lambda)^*$. In Section 3, we will show that each class $wF(p,r,q)$ operator has SVEP (single-valued extension property) and Bishop's property (β). In Section 4, we show that Weyl's theorem holds for class $wF(p,r,q)$.

2. Riesz idempotent

Let $\sigma(T)$, $\sigma_p(T)$, $\sigma_{jp}(T)$, $\sigma_a(T)$, $\sigma_{ja}(T)$, and $\sigma_r(T)$ mean the spectrum, point spectrum, joint point spectrum, approximate point spectrum, joint approximate point spectrum, and residual spectrum of an operator T, respectively (cf. [8, 23]). $\sigma_r^{\text{Xia}}(T)$ and $\sigma_{\text{iso}}(T)$ mean the set $\sigma(T)-\sigma_a(T)$ and the set of isolated points of $\sigma(T)$, see [23, 2].

If $\lambda\in\sigma_{\text{iso}}(T)$, the Riesz idempotent E_λ of T with respect λ is defined by

$$E_\lambda=\int_{\partial\mathscr{D}}(z-T)^{-1}dz, \tag{2.1}$$

where $\mathscr{D}$ is an open disk which is far from the rest of $\sigma(T)$ and $\partial\mathscr{D}$ means its boundary. Stampfli [24] showed that if T is hyponormal, then E_λ is selfadjoint and $E_\lambda\mathscr{H}=\ker(T-\lambda)=\ker(T-\lambda)^*$. The recent developments of this result are shown in [16, 17, 20, 22], and so on.

In this section, it is shown that when $\lambda\neq 0$, this result holds for class $wF(p,r,q)$ with $p+r\leq 1$ and $q\geq 1$. It is always assumed that $\lambda\in\sigma_{\text{iso}}(T)$ when the idempotent E_λ is considered.

Theorem 2.1. *Let T belong to class $wF(p,r,q)$ with $p+r \leq 1$, $\lambda = |\lambda|e^{i\theta} \in \mathscr{C}$, and $\lambda_{p+r} = |\lambda|^{p+r}e^{i\theta}$, then the following assertions hold.*

(1) *If $\lambda \neq 0$, then $E_\lambda = E_\lambda(p,r)$ and $E_\lambda\mathscr{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$, where $E_\lambda(p,r)$ is the Riesz idempotent of $T(p,r) = |T|^pU|T|^r$ (the generalized Aluthge transformation of T) with respect to λ_{p+r}.*

(2) *If $\lambda = 0$, then $\ker T = E_0\mathscr{H} = E_0(p,r)\mathscr{H} = \ker(T(p,r))$.*

Reference [21] gave an example that the operator T is w-hyponormal, E_0 is not selfadjoint, and $\ker T \neq \ker T^*$.

An operator T is said to be isoloid if $\sigma_{\text{iso}}(T) \subseteq \sigma_p(T)$, is said to be reguloid if $(T-\lambda)\mathscr{H}$, is closed for each $\lambda \in \sigma_{\text{iso}}(T)$.

Theorem 2.2. *If T belongs to class $wF(p,r,q)$ with $p+r \leq 1$, then T is isoloid and reguloid.*

To give proofs, we prepare the following results.

Theorem 2.3 (see [14]). *Let $\lambda \neq 0$, and let $\{x_n\}$ be a sequence of vectors. Then the following assertions are equivalent.*

(1) $(T-\lambda)x_n \to 0$ *and* $(T^* - \overline{\lambda})x_n \to 0$.

(2) $(|T| - |\lambda|)x_n \to 0$ *and* $(U - e^{i\theta})x_n \to 0$.

(3) $(|T|^* - |\lambda|)x_n \to 0$ *and* $(U^* - e^{-i\theta})x_n \to 0$.

Theorem 2.4 (see [8]). *If T is a class $wF(p,r,q)$ operator for $p+r \leq 1$ and $q \geq 1$, then the following assertions hold.*

(1) *If $Tx = \lambda x$, $\lambda \neq 0$, then $T^*x = \overline{\lambda}x$.*

(2) $\sigma_a(T) - \{0\} = \sigma_{ja}(T) - \{0\}$.

(3) *If $Tx = \lambda x$, $Ty = \mu y$ and $\lambda \neq \mu$, then $(x,y) = 0$.*

Theorem 2.5 (see [9]). *If T is a class $wF(p,r,q)$ operator, then there exists $\alpha_0 > 0$, which satisfies*

$$\left|T(p,r)\right|^{2\alpha_0} \geq |T|^{2\alpha_0(p+r)} \geq \left|(T(p,r))^*\right|^{2\alpha_0}. \tag{2.2}$$

Lemma 2.6. *If T belongs to class $wF(p,r,q)$ for $p+r \leq 1$, $\lambda = |\lambda|e^{i\theta} \in \mathscr{C}$, and $\lambda_{p+r} = |\lambda|^{p+r}e^{i\theta}$, then $\ker(T-\lambda) = \ker(T(p,r) - \lambda_{p+r})$.*

Proof. We only prove $\ker(T-\lambda) \supseteq \ker(T(p,r)-\lambda_{p+r})$ because $\ker(T-\lambda) \subseteq \ker(T(p,r) - \lambda_{p+r})$ is obvious by Theorems 2.3-2.4.

If $\lambda \neq 0$, let $0 \neq x \in \ker(T(p,r) - \lambda_{p+r})$. By Theorem 2.5, $T(p,r)$ is α_0-hyponormal and we have

$$\begin{gathered} \left|T(p,r)\right|x = |\lambda|^{p+r}x = \left|(T(p,r))^*\right|x, \\ \left|T(p,r)\right|^{2\alpha_0} - \left|(T(p,r))^*\right|^{2\alpha_0} \geq \left|T(p,r)\right|^{2\alpha_0} - |T|^{2\alpha_0(p+r)} \geq 0. \end{gathered} \tag{2.3}$$

Hence $(|T(p,r)|^{2\alpha_0} - |T|^{2\alpha_0(p+r)})x = 0$,

$$\begin{aligned} &\left\| |T|^{2\alpha_0(p+r)}x - |\lambda|^{2\alpha_0(p+r)}x \right\| \\ &\quad \leq \left\| |T|^{2\alpha_0(p+r)}x - \left|T(p,r)\right|^{2\alpha_0}x \right\| + \left\| \left|T(p,r)\right|^{2\alpha_0}x - |\lambda|^{2\alpha_0(p+r)}x \right\| = 0. \end{aligned} \tag{2.4}$$

On the other hand, $(T(p,r))^*x = |\lambda|^{p+r}e^{-i\theta}x$ implies that $|T|^rU^*x = |\lambda|^r e^{-i\theta}x$, $T^* = |\lambda|e^{-i\theta}x$. Therefore,

$$\begin{aligned}||(T-\lambda)x||^2 &= \|Tx\|^2 - \lambda(x,Tx) - \overline{\lambda}(Tx,x) + |\lambda|^2\|x\|^2 \\ &= |||T|x||^2 - \lambda(T^*x,x) - \overline{\lambda}(x,T^*x) + |\lambda|^2\|x\|^2 = 0.\end{aligned} \tag{2.5}$$

If $\lambda = 0$, let $0 \neq x \in \ker T(p,r)$, then $x \in \ker|T| = \ker T$ by Theorem 2.5 so that $\ker(T - \lambda) \supseteq \ker(T(p,r) - \lambda_{p+r})$. □

Lemma 2.7 (see [18, 25]). *If A is normal, then for every operator B, $\sigma(AB) = \sigma(BA)$.*

Let $\mathcal{F}$ be the set of all strictly monotone increasing continuous nonnegative functions on $\mathcal{R}^+ = [0,\infty)$. Let $\mathcal{F}_0 = \{\Psi \in \mathcal{F} : \Psi(0) = 0\}$. For $\Psi \in \mathcal{F}_0$, the mapping $\widetilde{\Psi}$ is defined by $\widetilde{\Psi}(\rho e^{i\theta}) = e^{i\theta}\Psi(\rho)$ and $\widetilde{\Psi}(T) = U\Psi(|T|)$.

Theorem 2.8 (see [26]). *If $\Psi \in \mathcal{F}_0$, then for every operator T, $\sigma_{ja}(\widetilde{\Psi}(T)) = \widetilde{\Psi}(\sigma_{ja}(T))$.*

Lemma 2.9. *Let T belong to class $wF(p,r,q)$ with $p+r \leq 1$, $\lambda = |\lambda|e^{i\theta} \in \mathcal{C}$, $T(t) = U|T|^{1-t+t(p+r)}$, and $\tau_t(\rho e^{i\theta}) = e^{i\theta}\rho^{1+t(p+r-1)}$, where $t \in [0,1]$. Then*

$$\sigma_a(T(t)) = \tau_t(\sigma_a(T)), \qquad \sigma_r^{\text{Xia}}(T(t)) = \tau_t(\sigma_r^{\text{Xia}}(T)), \qquad \sigma(T(t)) = \tau_t(\sigma(T)). \tag{2.6}$$

Proof. We only need to show that $\sigma_a(T(t)) = \tau_t(\sigma_a(T))$ by homotopy property of the spectrum [2, page 19].

Since T belongs to class $wF(p,r,q)$ with $p+r \leq 1$, $T(t)$ belongs to class $wF(p/(1+t(p+r-1)),r/(1+t(p+r-1),q))$ with $p/(1+t(p+r-1)) + r/(1+t(p+r-1)) \leq 1$. By Theorems 2.4(2) and 2.8,

$$\sigma_a(T(t)) - \{0\} = \sigma_{ja}(T(t)) - \{0\} = \tau_t(\sigma_{ja}(T) - \{0\}) = \tau_t(\sigma_a(T)) - \{0\}. \tag{2.7}$$

On the other hand, if $0 \in \sigma_a(T)$, then there exists a sequence $\{x_n\}$ of unit vectors such that $U|T|x_n \to 0$. Hence $|T|x_n = U^*U|T|x_n \to 0$, so that $|T|^{1/(2^m)}x_n \to 0$ for each positive integer m by induction. Take a positive integer $m(t)$ such that $1/(2^{m(t)}) \leq 1 + t(p+r-1)$, then

$$|T|^{1+t(p+r-1)}x_n = |T|^{1+t(p+r-1)-1/(2^{m(t)})}|T|^{1/(2^{m(t)})}x_n \longrightarrow 0 \tag{2.8}$$

and $0 \in \sigma_a(T(t))$. It is obvious that if $0 \in \sigma_a(T(t))$, then $0 \in \sigma_a(T)$ because of $p+r \leq 1$. Therefore $\sigma_a(T(t)) = \tau_t(\sigma_a(T))$. □

Theorem 2.10 (see [15]). *If T is p-hyponormal or log-hyponormal, then E_λ is selfadjoint and $E_\lambda\mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$.*

Riesz and Sz.-Nagy [27] gave the the formula $E_\lambda\mathcal{H} = \{x \in \mathcal{H} : \|(T-\lambda)^n x\|^{1/n} \to 0\}$.

Lemma 2.11. *For any operator T, $|T|^p\ker(T-\lambda) \subseteq |T|^pE_\lambda\mathcal{H} \subseteq E_\lambda(p,r)\mathcal{H}$ for $p+r = 1$.*

Proof. Let $x \in E_\lambda$, by the formula above we have

$$\left\|(T(p,r)-\lambda)^n|T|^p x\right\|^{1/n} = \left\||T|^p(T-\lambda)^n x\right\|^{1/n} \longrightarrow 0. \tag{2.9}$$

Hence $|T|^p x \in E_\lambda(p,r)\mathcal{H}$. □

LEMMA 2.12. *If T belongs to class $wF(p,r,q)$ with $p+r \le 1$, then*

$$\ker T = E_0\mathcal{H} = E_0(p,r)\mathcal{H} = \ker\left(T(p,r)\right). \tag{2.10}$$

Note that $\lambda_{p+r} \in \sigma_{\text{iso}}(T(t))$ if $\lambda \in \sigma_{\text{iso}}(T)$ by Lemma 2.9, so the notion $E_0(p,r)$ in Lemma 2.11 is reasonable.

Proof. Since $T(p,r)$ is α_0-hyponormal by Theorem 2.5, we only need to prove that $E_0\mathcal{H} \subseteq E_0(p,r)\mathcal{H}$ for $E_0\mathcal{H} \supseteq E_0(p,r)\mathcal{H}$ holds by Lemma 2.6 and Theorem 2.10. We may also assume that $p+r=1$ by Lemma 2.6.

It follows from Lemma 2.11 that

$$|T|^p E_0(p,r)\mathcal{H} \subseteq |T|^p E_0\mathcal{H} \subseteq E_0(p,r)\mathcal{H}, \tag{2.11}$$

thus $E_0(p,r)\mathcal{H}$ is reduced by $|T|^p$.

Let $x \in E_0\mathcal{H}$ and $x = x_1 + x_2 \in E_0(p,r)\mathcal{H} \oplus (E_0(p,r)\mathcal{H})^\perp$. Then $|T|^p x \in |T|^p E_0\mathcal{H} \subseteq E_0(p,r)\mathcal{H}$, $|T|^p x_1 \in E_0(p,r)\mathcal{H}$, $|T|^p x_2 \in (E_0(p,r)\mathcal{H})^\perp$ by (2.11), and $E_0(p,r)\mathcal{H}$ is reduced by $|T|^p$.

Thus $|T|^p x_2 = |T|^p x - |T|^p x_1 \in E_0(p,r)\mathcal{H}$, $|T|^p x_2 \in E_0(p,r)\mathcal{H} \cap (E_0(p,r)\mathcal{H})^\perp$ so that $x_2 \in \ker|T|^p \subseteq \ker(T(p,r)) = E_0(p,r)\mathcal{H}$, $x \in E_0(p,r)\mathcal{H}$. □

Proof of Theorem 2.1. We only need to prove (1) for (2) holds by Lemma 2.12.

Since $\sigma(T(p,r)) = \sigma(U|T|^{p+r}) = \{e^{i\theta}\rho^{p+r} : e^{i\theta}\rho \in \sigma(T)\}$ by Lemmas 2.7 and 2.9, $\lambda_{p+r} \in \sigma_{\text{iso}}(T(p,r))$. Hence

$$\left(E_\lambda(p,r)\mathcal{H}\right)^\perp = \ker\left(E_\lambda(p,r)\right) = \left(I - E_\lambda(p,r)\right)\mathcal{H} \tag{2.12}$$

by Theorem 2.10, so $\lambda_{p+r} \notin \sigma(T(p,r)|_{(E_\lambda(p,r)\mathcal{H})^\perp})$. By Theorem 2.4(1) and Lemma 2.6, we have $T = \lambda \oplus T_{22}$ on $\mathcal{H} = E_\lambda(p,r)\mathcal{H} \oplus (E_\lambda(p,r)\mathcal{H})^\perp$, where $T_{22} = T|_{(\ker(T-\lambda))^\perp}$.

Since $\ker(T-\lambda)$ is reduced by T, T_{22} also belongs to class $wF(p,r,q)$ and $T_{22}(p,r) = T(p,r)|_{(E_\lambda(p,r)\mathcal{H})^\perp}$ so that $\lambda \notin \sigma(T_{22})$ because $\lambda_{p+r} \notin \sigma(T_{22}(p,r))$. Hence $T - \lambda = 0 \oplus (T_{22} - \lambda)$ and $\ker(T-\lambda)^* = \ker(T-\lambda) \oplus \ker(T_{22}-\lambda)^* = \ker(T-\lambda)$.

Meanwhile, $E_\lambda = \int_{\partial\mathcal{D}} (z-\lambda)^{-1} \oplus (z - T_{22})^{-1} dz = 1 \oplus 0 = E_\lambda(p,r)$. □

Proof of Theorem 2.2. We only need to prove that T is reguloid for T being isoloid follows by Theorem 2.1 easily.

If $\lambda \in \sigma_{\text{iso}}(T)$, then $\mathcal{H} = E_\lambda\mathcal{H} + (I - E_\lambda)\mathcal{H}$, where $E_\lambda\mathcal{H}$, and $(I-E_\lambda)\mathcal{H}$ are topologically complemented [28, page 94]. By $T = T|_{E_\lambda\mathcal{H}} + T|_{(I-E_\lambda)\mathcal{H}}$ on $\mathcal{H} = E_\lambda\mathcal{H} + (I-E_\lambda)\mathcal{H}$ and Theorem 2.1, we have

$$(T-\lambda)\mathcal{H} = \left(T\big|_{(I-E_\lambda)\mathcal{H}} - \lambda\right)(I - E_\lambda)\mathcal{H}. \tag{2.13}$$

Therefore $(T-\lambda)\mathcal{H}$ is closed because $\sigma(T|_{(I-E_\lambda)\mathcal{H}}) = \sigma(T) - \{\lambda\}$. □

3. SVEP and Bishop's property (β)

Definition 3.1. An operator T is said to have SVEP at $\lambda \in \mathscr{C}$ if for every open neighborhood G of λ, the only function $f \in H(G)$ such that $(T-\lambda)f(\mu) = 0$ on G is $0 \in H(G)$, where $H(G)$ means the space of all analytic functions on G.

When T have SVEP at each $\lambda \in \mathscr{C}$, say that T has SVEP.

This is a good property for operators. If T has SVEP, then for each $\lambda \in \mathscr{C}$, $\lambda - T$ is invertible if and only if it is surjective (cf. [29, 18]).

Definition 3.2. An operator T is said to have Bishop's property (β) at $\lambda \in \mathscr{C}$ if for every open neighborhood G of λ, the function $f_n \in H(G)$ with $(T-\lambda)f_n(\mu) \to 0$ uniformly on every compact subset of G implies that $f_n(\mu) \to 0$ uniformly on every compact subset of G.

When T has Bishop's property (β) at each $\lambda \in \mathscr{C}$, simply say that T has property (β).

This is a generalization of SVEP and it is introduced by Bishop [30] in order to develop a general spectral theory for operators on Banach space.

Theorem 3.3. *Let p and r be positive numbers. If $p+r=1$, then T has SVEP if and only if $T(p,r)$ has SVEP, T has property (β) if and only if $T(p,r)$ has property (β). In particular, every class $wF(p,r,q)$ operator T with $p+r \le 1$ has SVEP and property (β).*

This result is a generalization of [18]. Lemma 3.4 and the relations between T and its transformation $T(p,r)$ are important:

$$\begin{aligned} T(p,r)|T|^p &= |T|^p U|T|^r|T|^p = |T|^p T, \\ U|T|^r T(p,r) &= U|T|^r|T|^p U|T|^r = TU|T|^r. \end{aligned} \tag{3.1}$$

Lemma 3.4 (see [18]). *Let G be open subset of complex plane $\mathscr{C}$ and let $f_n \in H(G)$ be functions such that $\mu f_n(\mu) \to 0$ uniformly on every compact subset of G, then $f_n(\mu) \to 0$ uniformly on every compact subset of G.*

Proof of Theorem 3.3. We only prove that T has property (β) if and only if $T(p,r)$ has property (β) because the assertion that T has SVEP if and only if $T(p,r)$ has SVEP can be proved similarly.

Suppose that $T(p,r)$ has property (β). Let G be an open neighborhood of λ and let $f_n \in H(G)$ be functions such that $(\mu - T)f_n(\mu) \to 0$ uniformly on every compact subset of G. By (3.1), $(T(p,r)-\mu)|T|^p f_n(\mu) = |T|^p(T-\mu)f_n(\mu) \to 0$ uniformly on every compact subset of G. Hence $Tf_n(\mu) = U|T|^r|T|^p f_n(\mu) \to 0$ uniformly on every compact subset of G for $T(p,r)$ has property (β), so that $\mu f_n(\mu) \to 0$ uniformly on every compact subset of G, and T having property (β) follows by Lemma 3.4.

Suppose that T has property (β). Let G be an open neighborhood of λ and let $f_n \in H(G)$ be functions such that $(\mu - T(p,r))f_n(\mu) \to 0$ uniformly on every compact subset of G. By (3.1), $(\mu - T)(U|T|^r f_n(\mu)) = U|T|^r(\mu - T(p,r))f_n(\mu) \to 0$ uniformly on every compact subset of G. Hence $T(p,r)f_n(\mu) \to 0$ uniformly on every compact subset of G for T has property (β) so that $\mu f_n(\mu) \to 0$ uniformly on every compact subset of G, and $T(p,r)$ having property (β) follows by Lemma 3.4. □

4. Weyl spectrum

For a Fredholm operator T, $\operatorname{ind} T$ means its (Fredholm) index. A Fredholm operator T is said to be Weyl if $\operatorname{ind} T = 0$.

Let $\sigma_e(T)$, $\sigma_w(T)$, and $\pi_{00}(T)$ mean the essential spectrum, Weyl spectrum, and the set of all isolated eigenvalues of finite multiplicity of an operator T, respectively (cf. [28, 17]).

According to Coburn [31], we say that Weyl's theorem holds for an operator T if $\sigma(T) - \sigma_w(T) = \pi_{00}(T)$. Very recently, the theorem was shown to hold for several classes of operators including w-hyponormal operators and paranormal operators (cf. [17, 32, 20]).

In this section, we will prove that Weyl's theorem and Weyl spectrum mapping theorem hold for class $wF(p,r,q)$ operator T with $p+r \leq 1$. We also assume that $p+r=1$ because of the inclusion relations among class $wF(p,r,q)$ [9].

Theorem 4.1. *Let T belong to class $wF(p,r,q)$ with $p+r=1$ and let $H(\sigma(T))$ be the space of all functions f analytic on some open set G containing $\sigma(T)$, then the following assertions hold.*

(1) *Weyl's theorem holds for T.*
(2) $\sigma_w(f(T)) = f(\sigma_w(T))$ *when* $f \in H(\sigma(T))$.
(3) *Weyl's theorem holds for $f(T)$ when $f \in H(\sigma(T))$.*

This is a generalization of the related assertions of [17].

Theorem 4.2. *Let T belong to class $wF(p,r,q)$ with $p+r=1$, then the following assertions hold.*

(1) *If $m_2(\sigma(T)) = 0$ where m_2 means the planar Lebesgue measure, then T is normal.*
(2) *If $\sigma_w(T) = 0$, then T is compact and normal.*

Theorem 4.2(1) is a generalization of [26] and (2) is a generalization of [24].

To give proofs, the following results are needful.

Theorem 4.3 [9]. *Let $p>0$, $r>0$, and $q \geq 1$, $s \geq p$, $t \geq r$. If T is a class $wF(p,r,q)$ operator and $T(s,t)$ is normal, then T is normal.*

Lemma 4.4. *If T belongs to class $wF(p,r,q)$ with $p+r=1$ and is Fredholm, then* $\operatorname{ind} T \leq 0$.

This result can be regarded as a good complement of Theorem 2.1.

Proof. Since T is Fredholm, $|T|^p$ is also Fredholm and $\operatorname{ind}(|T|^p) = 0$. By (3.1),

$$\operatorname{ind} T = \operatorname{ind}\left(|T|^p T\right) = \operatorname{ind}\left(T(p,r)|T|^p\right) = \operatorname{ind}\left(T(p,r)\right). \tag{4.1}$$

Hence, $\operatorname{ind} T \leq 0$ for $\operatorname{ind}(T(p,r)) \leq 0$ by Theorem 2.5. □

Proof of Theorem 4.1. (1) Let $\lambda \in \sigma(T) - \sigma_w(T)$, then $T - \lambda$ is Fredholm, $\operatorname{ind}(T-\lambda) = 0$, and $\dim\ker(T-\lambda) > 0$.

If λ is an interior point of $\sigma(T)$, there would be an open subset $G \subseteq \sigma(T)$ including λ such that $\text{ind}(T-\mu) = \text{ind}(T-\lambda) = 0$ for all $\mu \in G$ [28, page 357]. So $\dim\ker(T-\mu) > 0$ for all $\mu \in G$, this is impossible for T has SVEP by Theorem 3.3 [29, Theorem 10]. Thus $\lambda \in \partial\sigma(T) - \sigma_w(T)$, $\lambda \in \sigma_{\text{iso}}(T)$ by [28, Theorem 6.8, page 366], and $\lambda \in \pi_{00}(T)$ follows.

Let $\lambda \in \pi_{00}(T)$, then the Riesz idempotent E_λ has finite rank by Theorem 2.1, and $\lambda \in \sigma(T) - \sigma_w(T)$ follows.

(2) We only need to prove that $\sigma_w(f(T)) \supseteq f(\sigma_w(T))$ since $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$ is always true for any operators.

Assume that $f \in H(\sigma(T))$ is not constant. Let $\lambda \notin \sigma_w(f(T))$ and $f(z) - \lambda = (z - \lambda_1)\cdots(z-\lambda_k)g(z)$, where $\{\lambda_i\}_1^k$ are the zeros of $f(z)-\lambda$ in G (listed according to multiplicity) and $g(z) \neq 0$ for each $z \in G$. Thus

$$f(T) - \lambda = (T-\lambda_1)\cdots(T-\lambda_k)g(T). \tag{4.2}$$

Obviously, $\lambda \in f(\sigma_w(T))$ if and only if $\lambda_i \in \sigma_w(T)$ for some i. Next we prove that $\lambda_i \notin \sigma_w(T)$ for every $i \in \{1,\dots,k\}$, thus $\lambda \notin f(\sigma_w(T))$ and $\sigma_w(f(T)) \supseteq f(\sigma_w(T))$.

In fact, for each i, $T-\lambda_i$ is also Fredholm because $f(T)-\lambda$ is Fredholm. By Theorem 2.1 and Lemma 4.4, $\text{ind}(T-\lambda_i) \leq 0$ for each i. Since $0 = \text{ind}(f(T)-\lambda) = \text{ind}(T-\lambda_1) + \cdots + \text{ind}(T-\lambda_k)$, $\text{ind}(T-\lambda_i) = 0$ and $\lambda_i \notin \sigma_w(T)$ for each i.

(3) By Theorem 2.2, T is isoloid and it follows from [33] that

$$\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T) - \pi_{00}(T)). \tag{4.3}$$

On the other hand, $f(\sigma(T) - \pi_{00}(T)) = f(\sigma_w(T)) = \sigma_w(f(T))$ by (1)-(2). The proof is complete. □

Proof of Theorem 4.2. (1) By α_0-hyponormality of $T(p,r)$ and Putnam's inequality for α_0-hyponormal operators [26], $T(p,r)$ is normal. Hence, (1) follows by Theorem 4.3.

(2) Since $\sigma_w(T) = 0$, $\sigma(T) - \{0\} = \pi_{00}(T) \subseteq \sigma_{\text{iso}}(T)$ by Theorem 4.1(1). Hence $m_2(\sigma(T)) = 0$ and T is normal by (1).

Next to prove that T is compact, we may assume that $\sigma(T) - \{0\}$ is a countable infinite set for $\sigma(T) - \{0\} \subseteq \sigma_{\text{iso}}(T)$. Let $\sigma(T) - \{0\} = \{\lambda_n\}_1^\infty$ with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq 0$ and $\lambda_0 = \lim_{n\to\infty}|\lambda_n|$, then $\lambda_0 = 0$. Since every E_{λ_n} has finite rank by Theorems 2.1 and 4.1, for every $\varepsilon > 0$, $\bigoplus_{|\lambda_n|>\varepsilon} E_{\lambda_n}$ also has finite rank. Therefore T is compact [28, page 271]. □

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