Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 26765, 14 pages
doi:10.1155/2007/26765

Research Article

L^2-Boundedness of Marcinkiewicz Integrals along Surfaces with Variable Kernels: Another Sufficient Condition

Qingying Xue and Kôzô Yabuta

Received 18 December 2006; Accepted 23 April 2007

Recommended by Shusen Ding

We give the L^2 estimates for the Marcinkiewicz integral with rough variable kernels associated to surfaces. More precisely, we give some other sufficient conditions which are different from the conditions known before to warrant that the L^2-boundedness holds. As corollaries of this result, we show that similar properties still hold for parametric Littlewood-Paley area integral and parametric g_λ^* functions with rough variable kernels. Some of the results are extensions of some known results.

1. Introduction

In order to study the elliptic partial differential equations of order two with variable coefficients, Calderón and Zygmund [3] defined and studied the L^2-boundedness of singular integral T with variable kernels. In 1980, Aguilera and Harboure [4] studied the problem of pointwise convergence of singular integral and the L^2-bounds of Hardy-Littlewood maximal function with variable kernels. In 2002, Tang and Yang [1] gave the L^2 boundedness of the singular integrals with rough variable kernels associated to surfaces of the form $\{x = \Phi(|y|)y'\}$, where $y' = y/|y|$ for any $y \in \mathbb{R}^n \setminus \{0\}$ $(n \geq 2)$. That is, they considered the variable Calderón-Zygmund singular integral operator T_Φ defined by

$$T_\Phi(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} k(x,y) f(x - \Phi(|y|)y')\,dy, \tag{1.1}$$

and its truncated maximal operator T_Φ^* defined by

$$T_\Phi^*(f)(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} k(x,y) f(x - \Phi(|y|)y')\,dy \right|, \tag{1.2}$$

for $f \in C_0^\infty(\mathbb{R}^n)$, where $k(x,y) = \Omega(x,y)/|y|^n : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, $\Omega(x,y)$ is positively homogeneous in y of degree 0, namely $\Omega(x,\lambda y) = \Omega(x,y)$ for any $\lambda > 0$, and

$$\int_{S^{n-1}} \Omega(x,y')d\sigma(y') = 0 \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{1.3}$$

where S^{n-1} is the unit sphere of $\mathbb{R}^n$ equipped with Lebesgue measure $d\sigma = d\sigma(x')$.

They gave the following result.

Theorem 1.1 (see [1]). *Suppose $k(x,y)$ is as above and satisfies, for some $q > 2(n-1)/n$,*

$$\|\Omega\|_{L^\infty \times L^q(S^{n-1})} = \sup_{x \in \mathbb{R}^n} \left(\int_{S^{n-1}} |\Omega(x,y')|^q d\sigma(y') \right)^{1/q} < \infty. \tag{1.4}$$

Let $\Phi(t)$ be a nonnegative (or nonpositive) C^1 function on $(0,\infty)$ satisfying

(a) *Φ is strictly increasing (or decreasing);*

(b) *$\Phi(t)/t = C_2\Phi'(t)\varphi(t)$ for all $t \in (0,\infty)$, φ is defined on $(0,\infty)$ which is a monotonic and uniformly bounded function.*

Then T_Φ^* is bounded on $L^2(\mathbb{R}^n)$ and T_Φ can be uniquely extended to be a bounded operator on $L^2(\mathbb{R}^n)$. Moreover, for all $f \in L^2(\mathbb{R}^n)$,

$$\|T_\Phi(f)\|_2 \le C\|f\|_2, \qquad \|T_\Phi^*\|_2 \le C\|f\|_2, \tag{1.5}$$

where the constant C is independent of f.

On the other hand, as a related vector-valued singular integral with variable kernel, the Marcinkiewicz integral with rough variable kernel associated with surfaces of the form $\{x = \Phi(|y|)y'\}$ is considered. It is defined by

$$\mu_\Omega^\Phi(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \tag{1.6}$$

where

$$F_{\Omega,t}(x) = \int_{|y|\le t} \frac{\Omega(x,y)}{|y|^{n-1}} f(x - \Phi(|y|)y')dy. \tag{1.7}$$

If $\Phi(|y|) = |y|$, we put $\mu_\Omega^\Phi = \mu_\Omega$. Then μ_Ω with convolution type of kernel is just the Marcinkiewicz integral of higher dimension which was first defined and studied by Stein [5] in 1958. Since then, many works have been done about this integral (see, e.g., [6–8]). In 2005, Ding et al. [9] studied the L^2 boundedness of the operator μ_Ω.

Theorem 1.2 (see [9]). *Suppose that $\Omega(x,y)$ is positively homogeneous in y of degree 0, and satisfies (1.3) and (1.4) for some $q > 2(n-1)/n$. Then there is a constant C such that $\|\mu_\Omega(f)\|_2 \le C\|f\|_2$, where the constant C is independent of f.*

So, we have considered that it is natural to ask if the results in Theorem 1.1 still hold or not for the Marcinkiewicz integral with rough variable kernels along surfaces, and got in our paper [2] the following answer.

Theorem 1.3 (see [2]). *Suppose that $\Omega(x,y)$ is positively homogeneous in y of degree* 0, *and satisfies (1.3) and (1.4) for some $q > 2(n-1)/n$. Let Φ be a positive and strictly increasing (or negative and decreasing) C^1 function and let it satisfy $\Phi(t)/t = \Phi'(t)\varphi(t)$ for all $t \in (0,\infty)$, where φ is a function defined on $(0,\infty)$ and there exist two constants δ, M such that $0 < \delta \le |\varphi(t)| \le M$. Suppose moreover φ satisfies one of the following conditions:*

(i) *$t\varphi'(t)$ is bounded;*

(ii) *φ is a monotonic function.*

Then there is a constant C such that $\|\mu_\Omega^\Phi(f)\|_2 \le C\|f\|_2$, where constant C is independent of f.

In this paper, we will give another sufficient condition, relating to a recent paper by Al-Qassem [10].

Theorem 1.4. *Suppose that $\Omega(x,y)$ is positively homogeneous in y of degree* 0, *and satisfies (1.3) and (1.4) for some $q > 2(n-1)/n$. Let Φ be a positive and monotonic (or negative and monotonic) C^1 function on $(0,\infty)$ and let it satisfy the following conditions:*

(i) *$\delta \le |\Phi(t)/(t\Phi'(t))| \le M$ for some $0 < \delta \le M < \infty$;*

(ii) *$\Phi'(t)$ is monotonic on $(0,\infty)$.*

Then there is a constant C such that $\|\mu_\Omega^\Phi(f)\|_2 \le C\|f\|_2$, where constant C is independent of f.

Remark 1.5. There is no including relationship between condition (ii) and conditions (i), (ii) in Theorem 1.3, this can be seen from the example given in [2, Section 2 and Examples 2 and 3].

Remark 1.6. If $\Phi(t)$ is a positive and monotonic function on $(0,\infty)$ and $\Phi'(t)$ is monotonic, then the following (i) and (ii) are equivalent.

(i) $\delta \le |\Phi(t)/(t\Phi'(t))| \le M$ $(0 < t < \infty)$ for some $0 < \delta \le M < \infty$;

(ii) $\eta \le \max\{g(2t)/g(t), g(t)/g(2t)\} \le L$ on $(0,\infty)$ for some $1 < \eta \le L < \infty$.

This can be checked by elementary consideration, using convexity or concavity.

Condition (ii) is used to give L^p boundedness of Marcinkiewicz integrals along surfaces with convolution type of kernel by Al-Qassem [10].

Furthermore, our results can be extended to the parametric Marcinkiewicz integrals, parametric area integral, and parametric μ_λ^* function, which are defined by

$$\begin{aligned}
\mu_\Omega^{\Phi,\sigma}(f)(x) &= \left(\int_0^\infty \left|\int_{|y|\le t} \frac{\Omega(x,y)}{|y|^{n-\sigma}} f(x-\Phi(|y|)y')\,dy\right|^2 \frac{dt}{t^{1+2\sigma}}\right)^{1/2},\\
\mu_S^{\Phi,\sigma}(f)(x) &= \left(\iint_{\Gamma(x)} \left|\frac{1}{t^\sigma}\int_{|z|<t} \frac{\Omega(y,z)}{|z|^{n-\sigma}} f(y-\Phi(|z|)z')\,dz\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2},\\
\mu_{\lambda,\Phi}^{*,\sigma}(f)(x) &= \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\frac{1}{t^\sigma}\int_{|z|<t} \frac{\Omega(y,z)}{|z|^{n-\sigma}} f(y-\Phi(|z|)z')\,dz\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2},
\end{aligned} \tag{1.8}$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t\}$ and $\lambda > 1$.

We get the following result.

THEOREM 1.7. *Let $\sigma > 0$. Then Theorem 1.4 still holds for the parametric operators $\mu_\Omega^{\Phi,\sigma}$, $\mu_S^{\Phi,\sigma}$, and $\mu_{\lambda,\Phi}^{*,\sigma}$.*

Throughout this paper, the letter C will denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Proof of Theorem 1.4

We begin with recalling a known lemma. This lemma can be obtained from [11, (2.19), page 152], and [11, Theorem 3.10, page 158], see also [1].

LEMMA 2.1 (see [11]). *Let $n \geq 2$, $k \geq 0$, and let $P(y)$ be a spherical harmonic of degree k. Then*

$$\int_{S^{n-1}} P(y')e^{-ix\cdot y'}\,d\sigma(y') = (-i)^k(2\pi)^{n/2}\frac{J_{n/2+k-1}(|x|)}{|x|^{n/2-1}}P\left(\frac{x}{|x|}\right). \tag{2.1}$$

The first part of the next lemma is given in [2, page 372].

LEMMA 2.2. (1) *Let $g(t)$ be a nonnegative (positive) and nondecreasing (strictly increasing) function on $(0,\infty)$ such that there exists $\varphi(t)$ satisfying*

$$\frac{g(t)}{t} = g'(t)\varphi(t). \tag{2.2}$$

If there exists $\delta > 0$ such that $0 < \delta \leq \varphi(t)$ on $(0,\infty)$, then $[g^{-1}(t)]^\sigma/t^\varepsilon$ is nondecreasing (strictly increasing) on $(0,\infty)$ for $0 < \varepsilon \leq \sigma\delta$ $(0 < \varepsilon < \sigma\delta)$. Conversely, if $[g^{-1}(t)]^\sigma/t^\varepsilon$ is nondecreasing (strictly increasing) for some $\varepsilon > 0$, then $\varphi(t) \geq \varepsilon/\sigma$ $(\varphi(t) > \varepsilon/\sigma)$.

(2) *Let $g(t)$ be a nonnegative (positive) and nonincreasing (strictly decreasing) function on $(0,\infty)$ such that there exists $\varphi(t)$ satisfying*

$$\frac{g(t)}{t} = g'(t)\varphi(t). \tag{2.3}$$

If there exists $\delta > 0$ such that $0 < \delta \leq -\varphi(t)$ on $(0,\infty)$, then $[g^{-1}(t)]^\sigma t^\varepsilon$ is non-increasing (strictly decreasing) on $(0,\infty)$ for $0 < \varepsilon \leq \sigma\delta$ $(0 < \varepsilon < \sigma\delta)$. Conversely, if $[g^{-1}(t)]^\sigma/t^\varepsilon$ is non-increasing (strictly decreasing) for some $\varepsilon > 0$, then $-\varphi(t) \geq \varepsilon/\sigma$ $(-\varphi(t) > \varepsilon/\sigma)$.

One can prove this in an elementary calculation. Case (1) is given in [2, page 372], and Case (2) is shown similarly. We also note that if $\varphi(t)$ in Lemma 2.2 is bounded (without boundedness from below), it follows $\lim_{t\to 0} g(t) = 0$ and $\lim_{t\to\infty} g(t) = +\infty$ in Case (1), and $\lim_{t\to 0} g(t) = +\infty$ and $\lim_{t\to\infty} g(t) = 0$ in Case (2). (Cf. [12] for the proof.)

Below we give one example.

Example 2.3. Take a nondecreasing function $\psi(t) \in C^\infty(\mathbb{R})$ satisfying $0 \leq \psi(t) \leq 1$ $(t \in \mathbb{R})$, $\psi(t) = 0$ on $(-\infty,0)$, $\psi(t) = 1$ on $[1,\infty)$, and $0 < \psi'(t) < 2$ $(0 < t < 1)$. Set

$$\varphi(t) = \frac{1}{5}\left(2 + \psi(t) - \sum_{j=1}^{\infty} 2^{-j}\psi\left(2^{2j}(t-2^j)\right)\right). \tag{2.4}$$

Then, we have $2/5 \le \varphi(t) \le 3/5$ on $(0,\infty)$, $0 < \varphi'(t) = \psi'(t)/5 < 2/5$ $(0 < t < 1)$, $\varphi'(t) \le 0$ $(t \ge 1)$, $\varphi'(t) < 0$ $(2^j < t < 2^j + 2^{-2j},\ j = 1,2,\dots)$ (hence $\varphi(t)$ is not monotonic on $(0,\infty)$), and $\limsup_{t\to+\infty} |t\varphi'(t)| = +\infty$. Put $g(t) = \exp(\int_1^t (ds/s\varphi(s)))$. Then $g(t)$ is positive and increasing on $(0,\infty)$, and $g'(t) = g(t)/(t\varphi(t))$ (i.e., $g(t)/t = g'(t)\varphi(t)$), and $g''(t) = (1 - \varphi(t) - t\varphi'(t))/(t\varphi(t))^2$. By the definition of $\varphi(t)$ we have, for $0 < t < 1$

$$1 - \varphi(t) - t\varphi'(t) \ge 1 - \frac{3}{5} - t\varphi'(t) > \frac{2}{5} - \frac{2}{5} = 0, \tag{2.5}$$

and for $t \ge 1$, because of $\varphi'(t) \le 0$ $(t \ge 1)$

$$1 - \varphi(t) - t\varphi'(t) \ge 1 - \frac{3}{5} = \frac{2}{5}. \tag{2.6}$$

Hence $g''(t) > 0$ on $(0,\infty)$, and so $g'(t)$ is strictly increasing. This $g(t)$ satisfies conditions (i) and (ii) in Theorem 1.4. But, $\varphi(t) = g(t)/(tg'(t))$ is not monotonic nor $t\varphi'(t)$ is bounded.

Next, we prepare two more lemmas. Denote by J_ν the Bessel function of order ν of the first kind. The following lemma is given by L. Lorch and P. Szego, the old version of this type inequality is due to A. P. Calderón and A. Zygmund.

Lemma 2.4 (see [13]). *Suppose ν and λ satisfy $\nu - \lambda > -1$, and $|\nu| > 1/2$, $\lambda \ge -1/2$ or $\nu > -1$, $\lambda \ge 0$. Then,*

$$\left| \int_0^r \frac{J_\nu(t)}{t^\lambda} dt \right| \le \frac{C}{|\nu|^\lambda}, \quad \textit{for } 0 < r < \infty. \tag{2.7}$$

Lemma 2.5 (see[4]). *Suppose $m \ge 1$ and $\lambda > 0$. Then*

$$\left| \frac{1}{r} \int_0^r \frac{J_{m+\lambda}}{t^\lambda} dt \right| \le \frac{C}{m^{\lambda+1}}, \quad \textit{for } 0 < r < \infty. \tag{2.8}$$

Now we turn to the proof of Theorem 1.4.

Let $\mathcal{H}_k$ be the space of surface spherical harmonics of degree k on S^{n-1} with dimension D_k. By the same argument as in [3], one can reduce the proof of Theorem 1.4 to the case as follows:

$$f \in C_0^\infty, \qquad \Omega(x,y') = \sum_{k\ge1} \sum_{j=1}^{D_k} a_{k,j}(x) Y_{k,j}(y') \text{ is a finite sum}, \tag{2.9}$$

where $\{Y_{k,j}\}$ $(k \ge 1,\ j = 1,2,\dots,D_k)$ denotes the complete system of normalized surface spherical harmonics. Set

$$a_k(x) = \left(\sum_{j=1}^{D_k} |a_{k,j}(x)|^2 \right)^{1/2}, \qquad b_{k,j}(x) = \frac{a_{k,j}(x)}{a_k(x)}. \tag{2.10}$$

Then we get

$$\sum_{j=1}^{D_k} b_{k,j}^2(x)=1,\qquad \Omega(x,y')=\sum_{k\geq 1}a_k(x)\sum_{j=1}^{D_k} b_{k,j}(x)Y_{k,j}(y'). \tag{2.11}$$

Note that if we take $0<\varepsilon<1$ sufficiently close to 1, then by [3, (4.4), page 230] we have

$$\left(\sum_{k\geq 1}k^{-\varepsilon}a_k^2(x)\right)^{1/2}\leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})}=:C\|\Omega\|. \tag{2.12}$$

By Hölder's inequality, the above estimates, and Fourier transform, we get

$$\begin{aligned}
\|\mu_\Omega^\Phi(f)\|_2^2&=\int_{\mathbb{R}^n}\int_0^\infty\left|\int_{|y|\leq t}\sum_{k\geq 1}a_k(x)\sum_{j=1}^{D_k}b_{k,j}(x)\frac{Y_{k,j}(y')}{|y|^{n-1}}f(x-\Phi(|y|)y')dy\right|^2\frac{dt}{t^3}dx\\
&\leq\int_{\mathbb{R}^n}\left(\sum_{k\geq 1}k^{-\varepsilon}a_k^2(x)\right)\sum_{k\geq 1}k^{\varepsilon}\int_0^\infty\left(\sum_{j=1}^{D_k}b_{k,j}^2(x)\right)\\
&\quad\times\sum_{j=1}^{D_k}\left|\int_{|y|\leq t}\frac{Y_{k,j}(y')}{|y|^{n-1}}f(x-\Phi(|y|)y')dy\right|^2\frac{dt}{t^3}dx\\
&\leq C\|\Omega\|^2\sum_{k\geq 1}k^{\varepsilon}\sum_{j=1}^{D_k}\int_0^\infty\int_{\mathbb{R}^n}\left|\int_{|y|\leq t}\frac{Y_{k,j}(y')}{|y|^{n-1}}f(x-\Phi(|y|)y')dy\right|^2dx\frac{dt}{t^3}\\
&\leq C\|\Omega\|^2\sum_{k\geq 1}k^{\varepsilon}\sum_{j=1}^{D_k}\int_0^\infty\int_{\mathbb{R}^n}\left|\left(\int_{|y|\leq t}\frac{Y_{k,j}(y')}{|y|^{n-1}}f(\cdot-\Phi(|y|)y')dy\right)^{\wedge}(\xi)\right|^2d\xi\frac{dt}{t^3}\\
&=C\|\Omega\|^2\sum_{k\geq 1}k^{\varepsilon}\int_{\mathbb{R}^n}\sum_{j=1}^{D_k}[\mu_\Omega^\Phi(Y_{k,j})(\xi)]^2|\hat{f}(\xi)|^2d\xi,
\end{aligned} \tag{2.13}$$

where

$$\mu_\Omega^\Phi(Y_{k,j})(\xi)=\left(\int_0^\infty\left|\frac{1}{t}\int_0^t\int_{S^{n-1}}e^{-i\Phi(r)\xi\cdot y'}Y_{k,j}(y')d\sigma(y')dr\right|^2\frac{dt}{t}\right)^{1/2}. \tag{2.14}$$

So by Lemma 2.1, we only need to show

$$\sum_{j=1}^{D_k}\int_0^\infty\left|\frac{1}{t}\int_0^t\frac{J_{n/2+k-1}(\Phi(r)|\xi|)}{(\Phi(r)|\xi|)^{n/2-1}}dr\,Y_{k,j}(\xi')\right|^2\frac{dt}{t}\leq Ck^{-2}. \tag{2.15}$$

Denote

$$N_t(\xi)=\frac{1}{t}\int_0^t\frac{J_{n/2+k-1}(\Phi(r)|\xi|)}{(\Phi(r)|\xi|)^{n/2-1}}dr. \tag{2.16}$$

In the sequel, we set $\varphi(t)=\Phi(t)/(t\Phi'(t))$ and $\nu=n/2+k-1$. We note that $\rho/|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))=\varphi(\Phi^{-1}(\rho/|\xi|))\Phi^{-1}(\rho/|\xi|)$.

We will treat the following two cases. (A) $\Phi(t)$ is positive and increasing, and (B) $\Phi(t)$ is positive and decreasing. We do not need to treat the case where $\Phi(t)$ is negative.

(A) We treat first the case where $\Phi(t)$ is positive and increasing. Setting $\rho = \Phi(r)|\xi|$, we have

$$\begin{aligned} N_t(\xi) &= \frac{1}{t}\int_0^{\Phi(t)|\xi|} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} (\Phi^{-1})'\left(\frac{\rho}{|\xi|}\right)\frac{d\rho}{|\xi|} \\ &= \frac{1}{t}\int_0^{\Phi(t)|\xi|} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}. \end{aligned} \tag{2.17}$$

Setting $s = \Phi(t)|\xi|$, we have

$$\begin{aligned} \int_0^\infty |N_t(\xi)|^2\frac{dt}{t} &= \int_0^\infty \frac{\varphi(\Phi^{-1}(s/|\xi|))}{\Phi^{-1}(s/|\xi|)^2}\left|\int_0^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}\right|^2\frac{ds}{s} \\ &\le M\int_0^\infty \frac{1}{\Phi^{-1}(s/|\xi|)^2}\left|\int_0^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}\right|^2\frac{ds}{s}. \end{aligned} \tag{2.18}$$

Since $J_{n/2+k-1}(\rho) > 0$ for $0 < \rho < n/2+k-1$ and $\Phi(t)$ is positive and increasing on $(0,\infty)$, together with Lemma 2.5 and $\rho/|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|)) = \varphi(\Phi^{-1}(\rho/|\xi|))\Phi^{-1}(\rho/|\xi|)$, we have, for $0 < s < \nu$,

$$\begin{aligned} &\left|\int_0^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}\rho/|\xi|)}\right| \\ &\quad = \left|\int_0^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2}}\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\varphi\left(\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)\right)d\rho\right| \\ &\quad \le \left(\frac{1}{s}\int_0^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2}}d\rho\right)s\Phi^{-1}\left(\frac{s}{|\xi|}\right)\|\varphi\|_\infty \le \frac{Cs\Phi^{-1}(s/|\xi|)}{(k-1)^{n/2+1}}. \end{aligned} \tag{2.19}$$

To treat the case where s is big, we fix ε with $0 < \varepsilon < \min\{1/4, \delta\}$. Then, by Lemma 2.2(1), $\Phi^{-1}(\rho/|\xi|)/\rho^\varepsilon$ is increasing on $(0,\infty)$. We consider the two cases where $\Phi'(t)$ is increasing and decreasing on $(0,\infty)$.

(A1) The case where $\Phi'(t)$ is decreasing.

(A1-1) For $0 < s \le \nu$, by (2.19), we have

$$\left|\int_0^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}\right| \le C\frac{\Phi^{-1}(s/|\xi|)}{s^\varepsilon}\frac{1}{(n/2+k-1)^{n/2-\varepsilon}}. \tag{2.20}$$

(A1-2)

$$\begin{aligned} \left|\int_0^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}\right| &\le \left|\int_0^\nu \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}\right| \\ &\quad + \left|\int_\nu^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}\right| \\ &= I_1 + I_2. \end{aligned} \tag{2.21}$$

By (A1-1) and using the increasingness of $\Phi^{-1}(\rho/|\xi|)/\rho^{\varepsilon}$, we know that

$$I_1 \le C\frac{\Phi^{-1}(\nu/|\xi|)}{\nu^{\varepsilon}}\frac{1}{(n/2+k-1)^{n/2-\varepsilon}} \le C\frac{\Phi^{-1}(s/|\xi|)}{s^{\varepsilon}}\frac{1}{(n/2+k-1)^{n/2-\varepsilon}}. \tag{2.22}$$

As for I_2, since $\rho^{1-\varepsilon}/|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))$ is positive and increasing, by using the second mean-value theorem, and Lemma 2.4, we get, for some $\nu \le s' \le s$

$$\begin{aligned} I_2 &= \left|\int_{\nu}^{s}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-\varepsilon}}\frac{\rho^{1-\varepsilon}d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}\right| = \left|\int_{s'}^{s}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-\varepsilon}}d\rho\right|\frac{s^{1-\varepsilon}}{\Phi'(\Phi^{-1}(s/|\xi|))|\xi|} \\ &\le \frac{C}{(n/2+k-1)^{n/2-\varepsilon}}\frac{s/|\xi|}{\Phi'(\Phi^{-1}(s/|\xi|))\Phi^{-1}(s/|\xi|)}\frac{s^{1-\varepsilon}}{s}\Phi^{-1}\left(\frac{s}{|\xi|}\right) \\ &\le C\|\varphi\|_{\infty}\frac{\Phi^{-1}(s/|\xi|)}{s^{\varepsilon}}\frac{1}{k^{n/2-\varepsilon}}. \end{aligned} \tag{2.23}$$

(A2) The case where $\Phi'(t)$ is increasing.

(A2-1) For $0 < s \le \nu$, the same conclusion as (A1-1) holds:

$$\left|\int_{0}^{s}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}}\frac{d/\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}\right| \le C\frac{\Phi^{-1}(s/|\xi|)}{s^{\varepsilon}}\frac{1}{(n/2+k-1)^{n/2-\varepsilon}}. \tag{2.24}$$

(A2-2) For $\nu < s \le 2\nu$,

$$\begin{aligned} \left|\int_{0}^{s}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}}\frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}\right| &\le \left|\int_{0}^{\nu}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}}\frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}\right| \\ &\quad + \left|\int_{\nu}^{s}\frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}}\frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}\right| \\ &= I_1 + I_2. \end{aligned} \tag{2.25}$$

By (A2-1) and using the increasingness of $\Phi^{-1}(\rho/|\xi|)/\rho^{\varepsilon}$, we know that

$$I_1 \le C\frac{\Phi^{-1}(\nu/|\xi|)}{\nu^{\varepsilon}}\frac{1}{(n/2+k-1)^{n/2-\varepsilon}} \le C\frac{\Phi^{-1}(s/|\xi|)}{s^{\varepsilon}}\frac{1}{k^{n/2-\varepsilon}}. \tag{2.26}$$

As for I_2, by using the second mean-value theorem twice, and Lemma 2.4, we get, for some $\nu \le s' \le s'' \le s$,

$$
\begin{aligned}
I_2 &= \left| \int_\nu^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-\varepsilon}} \frac{\rho^{1-\varepsilon} d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| \\
&= \left| \int_{s'}^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-\varepsilon}} \frac{d\rho}{\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| \frac{s^{1-\varepsilon}}{|\xi|} \\
&= \left| \int_{s'}^{s''} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-\varepsilon}} d\rho \right| \frac{s^{1-\varepsilon}}{\Phi'(\Phi^{-1}(s'/|\xi|))|\xi|} \\
&\le \frac{C}{(n/2+k-1)^{n/2-\varepsilon}} \frac{s'/|\xi|}{\Phi'(\Phi^{-1}(s'/|\xi|))\Phi^{-1}(s'/|\xi|)} \frac{s^{1-\varepsilon}}{s'} \Phi^{-1}\left(\frac{s'}{|\xi|}\right) \\
&\le C \frac{\Phi^{-1}(s/|\xi|)}{s^\varepsilon} \frac{1}{(n/2+k-1)^{n/2-\varepsilon}}.
\end{aligned}
\tag{2.27}
$$

(A2-3) For $2\nu \le s < \nu^3$,

$$
\begin{aligned}
\left| \int_0^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| &\le \left| \int_0^{2\nu} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| \\
&\quad + \left| \int_{2\nu}^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| \\
&= I_3 + I_4.
\end{aligned}
\tag{2.28}
$$

By (A2-2) and using the increasingness of $\Phi^{-1}(\rho/|\xi|)/\rho^\varepsilon$, we see that

$$
I_3 \le C \frac{\Phi^{-1}(2\nu/|\xi|)}{(2\nu)^\varepsilon} \frac{1}{(n/2+k-1)^{n/2-\varepsilon}} \le C \frac{\Phi^{-1}(s/|\xi|)}{s^\varepsilon} \frac{1}{(n/2+k-1)^{n/2-\varepsilon}}.
\tag{2.29}
$$

As for I_4, since $|J_\nu(x)| \le 1$ (see [14, page 406]), it is easy to see that $|J'_\nu(\rho)| \le |J_{\nu-1}(\rho) - J_{\nu+1}(\rho)|/2 \le 1$ (see also [14, pages 45 and 406]). Hence, noting that $\rho/|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|)) = \varphi(\Phi^{-1}(\rho/|\xi|))\Phi^{-1}(\rho/|\xi|)$, we get

$$
\begin{aligned}
&\left| \int_{2\nu}^s \frac{J'_{n/2+k-1}(\rho)}{\rho^{(n-1)/2-1}(\rho^2-\nu^2)} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| \\
&\quad = \left| \int_{2\nu}^s \frac{J'_{n/2+k-1}(\rho)}{\rho^{(n-1)/2} - \varepsilon(\rho^2-\nu^2)} \frac{\Phi^{-1}(\rho/|\xi|)}{\rho^\varepsilon} \varphi(\Phi^{-1}(\rho/|\xi|)) d\rho \right| \\
&\quad \le \frac{\Phi^{-1}(s/|\xi|)\|\varphi\|_\infty}{s^\varepsilon} \int_{2\nu}^s \frac{1}{\rho^{(n-1)/2-\varepsilon}(\rho^2-\nu^2)} d\rho \\
&\quad \le C \frac{\Phi^{-1}(s/|\xi|)}{s^\varepsilon} \frac{1}{k^{(n+1)/2} - \varepsilon}.
\end{aligned}
\tag{2.30}
$$

On the other hand, since $r^2/r^{(n-1)/2-\varepsilon}(r^2-\nu^2)$ is decreasing on $[2\nu,\infty)$, by using the second mean-value theorem twice, we have, for some $2\nu \le s' \le s'' \le s$,

$$\begin{aligned}
&\int_{2\nu}^{s} \frac{\rho J''_{n/2+k-1}(\rho)}{\rho^{(n-1)/2-1}(\rho^2-\nu^2)} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \\
&\quad = \int_{2\nu}^{s} \frac{J''_{n/2+k-1}(\rho)}{\rho^{(n-1)/2-2-\varepsilon}(\rho^2-\nu^2)} \frac{1}{\Phi'(\Phi^{-1}(\rho/|\xi|))\Phi^{-1}(\rho/|\xi|)} \frac{\Phi^{-1}(\rho/|\xi|)}{|\xi|\rho^{\varepsilon}} d\rho \\
&\quad = \int_{s'}^{s''} J''_{n/2+k-1}(\rho) d\rho \frac{2\nu/|\xi|}{(2\nu)^{(n-1)/2-1-\varepsilon}((2\nu)^2-\nu^2)\Phi'(\Phi^{-1}(2\nu/|\xi|))\Phi^{-1}(2\nu/|\xi|)} \\
&\qquad \times \frac{\Phi^{-1}(s/|\xi|)}{s^{\varepsilon}}.
\end{aligned} \tag{2.31}$$

Hence, we have

$$\left| \int_{2\nu}^{s} \frac{\rho J''_{n/2+k-1}(\rho)}{\rho^{(n-1)/2-1}(\rho^2-\nu^2)} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| \le C \frac{\Phi^{-1}(s/|\xi|)}{s^{\varepsilon}} \frac{1}{k^{(n+1)/2-\varepsilon}}. \tag{2.32}$$

Thus by (2.30), (2.32), and the fact that

$$\frac{J_\nu(\rho)}{\rho^{n/2}} = -\frac{J'_\nu(\rho)}{\rho^{(n-1)/2}(\rho^2-\nu^2)} - \frac{\rho J''_\nu(\rho)}{\rho^{(n-1)/2}(\rho^2-\nu^2)}, \tag{2.33}$$

we get

$$I_4 \le C \frac{\Phi^{-1}(s/|\xi|)}{s^{\varepsilon}} \frac{1}{k^{(n+1)/2-\varepsilon}}. \tag{2.34}$$

(A2-4) For $\nu^3 < s$,

$$\begin{aligned}
\left| \int_0^s \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| &\le \left| \int_0^{\nu^3} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| \\
&\quad + \left| \int_{\nu^3}^{s} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| \\
&= I_5 + I_6.
\end{aligned} \tag{2.35}$$

By (A2-3) and using the increasingness of $\Phi^{-1}(\rho/|\xi|)/\rho^{\varepsilon}$, we see that

$$I_5 \le C \frac{\Phi^{-1}(\nu^3/|\xi|)}{(\nu^3)^{\varepsilon}} \frac{1}{k^{n/2-\varepsilon}} \le C \frac{\Phi^{-1}(s/|\xi|)}{s^{\varepsilon}} \frac{1}{k^{n/2-\varepsilon}}. \tag{2.36}$$

Using the following inequality (see [14, page 447]):

$$|J_\nu(x)| \le \frac{\sqrt{2/\pi}}{(x^2-\nu^2)^{1/4}}, \quad \text{for } x \ge \nu \ge \frac{1}{2}, \tag{2.37}$$

we see that $|J_\nu(\rho)| \le C/\sqrt{\rho}$ for $\rho > 2\nu$. Hence

$$\begin{aligned} I_6 &= \left| \int_{\nu^3}^{s} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-\varepsilon}} \frac{\rho/|\xi|}{\Phi'(\Phi^{-1}(\rho/|\xi|))\Phi^{-1}(\rho/|\xi|)} \frac{\Phi^{-1}(\rho/|\xi|)}{\rho^{\varepsilon}} d\rho \right| \\ &\le C \frac{\Phi^{-1}(s/|\xi|)}{s^{\varepsilon}} \int_{\nu^3}^{s} \frac{1}{\rho^{n/2-\varepsilon+1/2}} d\rho \\ &\le C \frac{\Phi^{-1}(s/|\xi|)}{s^{\varepsilon}} \frac{1}{k^{n/2-\varepsilon}}. \end{aligned} \tag{2.38}$$

Here, in the last inequality we have used $0 < \varepsilon < 1/4$. By (2.19), (A1-1), (A1-2) and (A2-1)–(A2-4) above, we have, in the case $\Phi(t)$ is positive and increasing,

$$\int_0^\infty |N_t(\xi)|^2 \frac{dt}{t} \le C \int_0^\nu \frac{s^2}{(k-1)^{n+2}} \frac{ds}{s} + C \int_\nu^\infty \left(\frac{1}{s^{2\varepsilon} k^{n-2\varepsilon}} + \frac{1}{s^{2\varepsilon} k^{n+1-2\varepsilon}} \right) \frac{ds}{s} \le C \frac{1}{k^n}. \tag{2.39}$$

(B) Next we consider the case $\Phi(t)$ is positive and decreasing. In this case, from the monotonicity of $\Phi'(t)$, it follows that $\Phi'(t)$ is nondecreasing. We take $\varepsilon > 0$ so that $\varepsilon < \min(1/4, \delta)$. So, by Lemma 2.2(2), we have $t^{\varepsilon}\Phi^{-1}(t)$ is decreasing on $(0, \infty)$.

Setting $\rho = \Phi(r)|\xi|$, we have

$$N_t(\xi) = -\frac{1}{t} \int_{\Phi(t)|\xi|}^{\infty} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))}. \tag{2.40}$$

Setting $s = \Phi(t)|\xi|$, we have

$$\begin{aligned} \int_0^\infty |N_t(\xi)|^2 \frac{dt}{t} &= \int_0^\infty \frac{\varphi(\Phi^{-1}(s/|\xi|))}{\Phi^{-1}(s/|\xi|)^2} \left| \int_s^\infty \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right|^2 \frac{ds}{s} \\ &\le M \int_0^\infty \frac{1}{\Phi^{-1}(s/|\xi|)^2} \left| \int_s^\infty \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right|^2 \frac{ds}{s}. \end{aligned} \tag{2.41}$$

(B-1) The case $\nu \le s < \infty$. Since $-1/\Phi'(\Phi^{-1}(\rho/|\xi|))$ is positive and decreasing, for any $h > s$, we have, by using the second mean-value theorem and Lemma 2.4

$$\begin{aligned} &\left| \int_s^h \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| \\ &\quad = \left| \int_s^{h'} \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} d\rho \frac{1}{|\xi|\Phi'(\Phi^{-1}(s/|\xi|))} \right| \\ &\quad \le \frac{C}{(n/2+k-1)^{n/2-1}} \frac{s/|\xi|}{|\Phi'(\Phi^{-1}(s/|\xi|))|\Phi^{-1}(s/|\xi|)} \frac{\Phi^{-1}(s/|\xi|)}{s} \\ &\quad \le \frac{C\|\varphi\|_\infty}{k^{n/2-1}} \frac{\Phi^{-1}(s/|\xi|)}{s}. \end{aligned} \tag{2.42}$$

Letting $h \to \infty$, we get

$$\left| \int_s^\infty \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| \le \frac{C\|\varphi\|_\infty}{k^{n/2-1}} \frac{\Phi^{-1}(s/|\xi|)}{s}. \tag{2.43}$$

(B-2) The case $0 < s < \nu$. We have

$$\begin{aligned}
&\left| \int_s^\infty \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| \\
&\quad \le \left| \int_s^\nu \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| + \left| \int_\nu^\infty \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2-1}} \frac{d\rho}{|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|))} \right| \\
&\quad = I_1 + I_2.
\end{aligned} \tag{2.44}$$

By (B-1) and the decreasingness of $t^\varepsilon \Phi^{-1}(t/|\xi|)$, we see that

$$I_2 \le C\frac{\Phi^{-1}(\nu/|\xi|)}{\nu}\frac{1}{k^{n/2-1}} = \frac{C}{k^{n/2-1}\nu^{1+\varepsilon}}\nu^\varepsilon\Phi^{-1}(\nu/|\xi|) \le \frac{C}{k^{n/2+\varepsilon}}s^\varepsilon\Phi^{-1}(s/|\xi|). \tag{2.45}$$

As for I_1, since $J_{n/2+k-1}(\rho) > 0$ for $0 < \rho < n/2 + k - 1$ and $t^\varepsilon\Phi^{-1}(t)$ is positive and decreasing on $(0, \infty)$, together with Lemma 2.4 and $\rho/|\xi|\Phi'(\Phi^{-1}(\rho/|\xi|)) = \varphi(\Phi^{-1}(\rho/|\xi|))\Phi^{-1}(\rho/|\xi|)$, we get

$$\begin{aligned}
I_1 &= \left| \int_s^\nu \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\varepsilon}} \frac{\rho/|\xi|}{\Phi'(\Phi^{-1}(\rho/|\xi|))\Phi^{-1}(\rho/|\xi|)}\rho^\varepsilon\Phi^{-1}\left(\frac{\rho}{|\xi|}\right)d\rho \right| \\
&= \|\varphi\|_\infty s^\varepsilon\Phi^{-1}\left(\frac{s}{|\xi|}\right)\left| \int_s^\nu \frac{J_{n/2+k-1}(\rho)}{\rho^{n/2+\varepsilon}} \right| \\
&\le \frac{C}{k^{n/2+\varepsilon}}s^\varepsilon\Phi^{-1}\left(\frac{s}{|\xi|}\right).
\end{aligned} \tag{2.46}$$

Thus, using (B-1) and (B-2) we obtain

$$\int_0^\infty |N_t(\xi)|^2\frac{dt}{t} \le C\int_0^\nu \frac{s^{2\varepsilon}}{k^{n+2\varepsilon}}\frac{ds}{s} + C\int_\nu^\infty \frac{1}{s^2k^{n-2}}\frac{ds}{s} \le C\frac{1}{k^n}. \tag{2.47}$$

Therefore, in both cases (A) and (B) by the fact $\sum_{j=1}^{D_k}|Y_{k,j}(\xi')|^2 = w^{-1}D_k \sim k^{n-2}$ (see [15, (2.6), page 255]), where w denotes the area of S^{n-1}, we get

$$\sum_{j=1}^{D_k}\int_0^\infty |N_t(\xi)(Y_{k,j})(\xi')|^2\frac{dt}{t} \le Ck^{-2}. \tag{2.48}$$

Thus, inequality (2.15) holds and the proof of Theorem 1.4 is finished.

3. Proof of Theorem 1.7

We will give the proof of Theorem 1.7.

First, we know that $\mu_S^{\Phi,\sigma}(f)(x) \le 2^{\lambda n}\mu_{\lambda,\Phi}^{*,\sigma}(f)(x)$. On the other hand,

$$\begin{aligned}
&\|\mu_{\lambda,\Phi}^{*,\sigma}(f)\|_2^2 \\
&\quad = \int_{\mathbb{R}^n}\iint_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\sigma}\int_{|z|<t}\frac{\Omega(y,z)}{|z|^{n-\sigma}}f(y-\Phi(|z|)z')\,dz\right|^2\frac{dy\,dt}{t^{n+1}}dx \\
&\quad = \int_0^\infty\int_{\mathbb{R}^n}\left(\frac{1}{t^n}\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}dx\right)\left|\frac{1}{t^\sigma}\int_{|z|<t}\frac{\Omega(y,z)}{|z|^{n-\sigma}}f(y-\Phi(|z|)z')\,dz\right|^2\frac{dy\,dt}{t} \\
&\quad \le C\|\mu_\Omega^{\Phi,\sigma}(f)\|_2^2.
\end{aligned} \tag{3.1}$$

Hence, we only need to give the estimates for $\mu_\Omega^{\Phi,\sigma}(f)$. Similarly as (2.13), we get

$$\|\mu_\Omega^{\Phi,\sigma}(f)\|_2^2 \le C\|\Omega\|^2\sum_{k\ge1}k^\varepsilon\sum_{j=1}^{D_k}\int_{\mathbb{R}^n}\left[\mu_\Omega^{\Phi,\sigma}(Y_{k,j})(\xi)\right]^2|\hat{f}(\xi)|^2d\xi, \tag{3.2}$$

where

$$\mu_\Omega^{\Phi,\sigma}(Y_{k,j})(\xi) = \left(\int_0^\infty\left|\frac{1}{t^\sigma}\int_0^t\int_{S^{n-1}}r^{\sigma-1}e^{-i\Phi(r)\xi\cdot y'}Y_{k,j}(y')\,d\sigma(y')\,dr\right|^2\frac{dt}{t}\right)^{1/2}. \tag{3.3}$$

By Lemma 2.1, we have

$$\frac{1}{t^\sigma}\int_0^t\int_{S^{n-1}}r^{\sigma-1}e^{-i\Phi(r)\xi\cdot y'}Y_{k,j}(y')\,d\sigma(y')\,dr = \frac{1}{t^\sigma}\int_0^t r^{\sigma-1}\frac{J_{n/2+k-1}(\Phi(r)|\xi|)}{(\Phi(r)|\xi|)^{n/2-1}}dr\,Y_{k,j}(\xi'). \tag{3.4}$$

For any $\sigma>0$, if we take $0<\varepsilon<\min\{1/4,\sigma\delta\}$, then we see by Lemma 2.2 that $[\Phi^{-1}(t)]^\sigma/t^\varepsilon$ is strictly increasing on $(0,\infty)$. Thus, Theorem 1.7 follows from repeating the steps in the proof of Theorem 1.4.

References

[1] L. Tang and D. Yang, “Boundedness of singular integrals of variable rough Calderón-Zygmund kernels along surfaces,” *Integral Equations and Operator Theory*, vol. 43, no. 4, pp. 488–502, 2002.

[2] Q. Xue and K. Yabuta, “L^2-boundedness of Marcinkiewicz integrals along surfaces with variable kernels,” *Scientiae Mathematicae Japonicae*, vol. 63, no. 3, pp. 369–382, 2006.

[3] A. P. Calderón and A. Zygmund, “On a problem of Mihlin,” *Transactions of the American Mathematical Society*, vol. 78, no. 1, pp. 209–224, 1955.

[4] N. E. Aguilera and E. O. Harboure, “Some inequalities for maximal operators,” *Indiana University Mathematics Journal*, vol. 29, no. 4, pp. 559–576, 1980.

[5] E. M. Stein, “On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz,” *Transactions of the American Mathematical Society*, vol. 88, no. 2, pp. 430–466, 1958.

[6] Y. Ding, D. Fan, and Y. Pan, “Weighted boundedness for a class of rough Marcinkiewicz integrals,” *Indiana University Mathematics Journal*, vol. 48, no. 3, pp. 1037–1055, 1999.

[7] Y. Ding, D. Fan, and Y. Pan, “L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernels,” *Acta Mathematica Sinica*, vol. 16, no. 4, pp. 593–600, 2000.

[8] D. Fan and S. Sato, "Weak type (1, 1) estimates for Marcinkiewicz integrals with rough kernels," *Tohoku Mathematical Journal*, vol. 53, no. 2, pp. 265–284, 2001.

[9] Y. Ding, C.-C. Lin, and S. Shao, "On the Marcinkiewicz integral with variable kernels," *Indiana University Mathematics Journal*, vol. 53, no. 3, pp. 805–821, 2004.

[10] H. M. Al-Qassem, "On weighted inequalities for parametric Marcinkiewicz integrals," *Journal of Inequalities and Applications*, vol. 2006, Article ID 91541, 17 pages, 2006.

[11] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Mathematical Series, no. 32, Princeton University Press, Princeton, NJ, USA, 1971.

[12] Q. Xue and K. Yabuta, "Correction and addition to "L^2-boundedness of Marcinkiewicz integrals along surfaces with variable kernels"," *Scientiae Mathematicae Japonicae*, vol. 65, no. 2, pp. 291–298, 2007.

[13] L. Lorch and P. Szego, "A singular integral whose kernel involves a Bessel function," *Duke Mathematical Journal*, vol. 22, no. 3, pp. 407–418, 1955.

[14] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, London, UK, 2nd edition, 1966.

[15] A. P. Calderón and A. Zygmund, "On singular integrals with variable kernels," *Applicable Analysis*, vol. 7, no. 3, pp. 221–238, 1978.

Qingying Xue: School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
Email address: qyxue@bnu.edu.cn

Kôzô Yabuta: School of Science and Technology, Kwansei Gakuin University, Gakuen 2-1, Sanda 669-1337, Japan
Email address: yabuta@ksc.kwansei.ac.jp