

Research Article

Existence and Asymptotic Behavior of Positive Solutions to $p(x)$ -Laplacian Equations with Singular Nonlinearities

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This paper investigates the $p(x)$ -Laplacian equations with singular nonlinearities $-\Delta_{p(x)}u = \lambda/u^{y(x)}$ in Ω , $u(x) = 0$ on $\partial\Omega$, where $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$ -Laplacian. The existence of positive solutions is given, and the asymptotic behavior of solutions near boundary is discussed.

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1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)$ -growth conditions is a new and interesting topic. We refer to [1, 2], the background of these problems. Many results have been obtained on this kind of problems, for example, [2–13]. In [4, 7], Fan and Zhao give the regularity of weak solutions for differential equations with nonstandard $p(x)$ -growth conditions. On the existence of solutions for $p(x)$ -Laplacian problems in bounded domain, we refer to [5, 11, 12].

In this paper, we consider the $p(x)$ -Laplacian equations with singular nonlinearities:

$$\begin{aligned} -\Delta_{p(x)}u &= \frac{\lambda}{u^{y(x)}} \quad \text{in } \Omega, \\ u(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{P}$$

where $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$ -Laplacian, $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$. If $p(x) \equiv p$ (a constant), then (P) is the well-known p -Laplacian problem. There are many results on the existence of positive solutions for p -Laplacian problems with singular nonlinearities (see [14–18]), but the results on the existence of positive solutions for $p(x)$ -Laplacian problems with singular nonlinearities

are rare. Our aim is to give the existence of positive solutions for problem (P), and give the asymptotic behavior of positive solutions near boundary.

Throughout the paper, we assume that $0 < \gamma(x) \in C(\bar{\Omega})$ and $p(x)$ satisfy (H₁) $p(x) \in C^1(\bar{\Omega})$, $1 < p^- \leq p^+ < +\infty$, where $p^- = \inf_{\Omega} p(x)$, $p^+ = \sup_{\Omega} p(x)$.

Because of the nonhomogeneity of $p(x)$ -Laplacian, $p(x)$ -Laplacian problems are more complicated than those of p -Laplacian ones, many results and methods for p -Laplacian problems are invalid for $p(x)$ -Laplacian problems (see [6]), and another difficulty of this paper is that $f(x, u) = 1/u^{\gamma(x)}$ cannot be represented as $h(x)f(u)$. Our results partially generalized the results of [18].

2. Preliminary

In order to deal with $p(x)$ -Laplacian problems, we need some theories on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$ -Laplacian which we will use later (see [3, 8]). Let

$$L^{p(x)}(\Omega) = \left\{ u \mid u \text{ is a measurable real-valued function, } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}, \tag{2.1}$$

$$C_0^+(\Omega) = \{ u \in C(\bar{\Omega}) \mid u > 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega \}.$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf \left\{ \mu > 0 \mid \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1. \right\}. \tag{2.2}$$

The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space. We call it generalized Lebesgue space. The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, reflexive, and uniform convex Banach space (see [3, Theorems 1.10, Theorem 1.14]).

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega) \}, \tag{2.3}$$

and it can be equipped with the norm

$$|u| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega). \tag{2.4}$$

$W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive, and uniform convex Banach spaces (see [3, Theorem 2.1]).

If $u \in W_{loc}^{1,p(x)}(\Omega) \cap C_0^+(\Omega)$, u is called a positive solution of (P) if $u(x)$ satisfies

$$\int_Q |\nabla u|^{p(x)-2} \nabla u \nabla q dx - \int_Q \frac{\lambda}{u^{\gamma(x)}} q dx = 0, \quad \forall q \in W_0^{1,p(x)}(Q), \tag{2.5}$$

for any domain $Q \Subset \Omega$.

Let $W_{0,\text{loc}}^{1,p(x)}(\Omega) = \{u \mid \text{there is an open domain } Q \Subset \Omega \text{ s.t. } u \in W_0^{1,p(x)}(Q)\}$, and define $A : W_{\text{loc}}^{1,p(x)}(\Omega) \cap C_0^+(\Omega) \rightarrow (W_{0,\text{loc}}^{1,p(x)}(\Omega))^*$ as

$$\langle Au, \varphi \rangle = \int_{\Omega} \left(|\nabla u|^{p(x)-2} \nabla u \nabla \varphi - \frac{\lambda}{u^{y(x)}} \varphi \right) dx, \quad (2.6)$$

where $u \in W_{\text{loc}}^{1,p(x)}(\Omega) \cap C_0^+(\Omega)$, $\varphi \in W_{0,\text{loc}}^{1,p(x)}(\Omega)$; then we have the following lemma.

LEMMA 2.1 (see [5, Theorem 3.1]). $A : W_{\text{loc}}^{1,p(x)}(\Omega) \cap C_0^+(\Omega) \rightarrow (W_{0,\text{loc}}^{1,p(x)}(\Omega))^*$ is strictly monotone.

Let $g \in (W_{0,\text{loc}}^{1,p(x)}(\Omega))^*$, if $\langle g, \varphi \rangle \geq 0$, for all $\varphi \in W_{0,\text{loc}}^{1,p(x)}(\Omega)$, $\varphi \geq 0$ a.e. in Ω , then denote $g \geq 0$ in $(W_{0,\text{loc}}^{1,p(x)}(\Omega))^*$; correspondingly, if $-g \geq 0$ in $(W_{0,\text{loc}}^{1,p(x)}(\Omega))^*$, then denote $g \leq 0$ in $(W_{0,\text{loc}}^{1,p(x)}(\Omega))^*$.

Definition 2.2. Let $u \in W_{\text{loc}}^{1,p(x)}(\Omega) \cap C_0^+(\Omega)$. If $Au \geq 0$ ($Au \leq 0$) in $(W_{0,\text{loc}}^{1,p(x)}(\Omega))^*$, then u is called a weak supersolution (weak subsolution) of (P) .

Copying the proof of [10], we have the following lemma.

LEMMA 2.3 (comparison principle). Let $u, v \in W_{\text{loc}}^{1,p(x)}(\Omega) \cap C(\Omega)$ be positive and satisfy $Au - Av \geq 0$ in $(W_{0,\text{loc}}^{1,p(x)}(\Omega))^*$. Let $\varphi(x) = \min\{u(x) - v(x), 0\}$. If $\varphi(x) \in W_{0,\text{loc}}^{1,p(x)}(\Omega)$ (i.e., $u \geq v$ on $\partial\Omega$), then $u \geq v$ a.e. in Ω .

LEMMA 2.4 (see [7]). If $g(x, u)$ is continuous on $\bar{\Omega} \times \mathbb{R}$, $u \in W^{1,p(x)}(\Omega)$ is a bounded weak solution of $-\Delta_{p(x)}u + g(x, u) = 0$ in Ω , $u = w_0$ on $\partial\Omega$, where $w_0 \in W^{1,p(x)}(\Omega)$, then $u \in C_{\text{loc}}^{1,\alpha}(\Omega)$, where $\alpha \in (0, 1)$ is a constant.

3. Existence of positive solutions

In order to deal with the existence of positive solutions, let us consider the problem

$$\begin{aligned} -\Delta_{p(x)}u &= \frac{\lambda}{(|u| + a_n)^{y(x)}} \quad \text{in } \Omega, \\ u(x) &= 0 \quad \text{for } x \in \partial\Omega, \end{aligned} \quad (3.1)$$

where $\{a_n\}$ is a positive strictly decreasing sequence and $\lim_{n \rightarrow +\infty} a_n = 0$. We have the following lemma.

LEMMA 3.1. For any $n = 1, 2, \dots$, problem (3.1) possesses a weak positive solution $\bar{w}_n \in C(\bar{\Omega})$.

Proof. The relative functional of (3.1) is

$$\varphi = \int_{\Omega} \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx - \int_{\Omega} F_n(x, u) dx, \quad (3.2)$$

where $F_n(x, u) = \int_0^u \lambda / (|t| + a_n)^{y(x)} dt$. Since φ is coercive in $W_0^{1,p(x)}(\Omega)$, then φ possesses a nontrivial minimum point \bar{w}_n , then $|\bar{w}_n|$ is also a nontrivial minimum point of problem (3.1), then (3.1) possesses a weak positive solution. The proof is completed. \square

Here and hereafter, we will use the notation $d(x, \partial\Omega)$ to denote the distance of $x \in \Omega$ to the boundary of Ω . Denote $d(x) = d(x, \partial\Omega)$ and $\partial\Omega_\epsilon = \{x \in \Omega \mid d(x) < \epsilon\}$. Since $\partial\Omega$ is C^2 regularly, then there exists a positive constant σ such that $d(x) \in C^2(\overline{\partial\Omega}_{2\sigma})$, and $|\nabla d(x)| \equiv 1$. Let $\delta \in (0, (1/3)\sigma)$ be a small enough constant. Denote

$$v_1(x) = \begin{cases} d(x), & d(x) < \delta, \\ \delta + \int_\delta^{d(x)} \left(\frac{2\delta - t}{\delta}\right)^{2/(p^- - 1)} dt, & \delta \leq d(x) < 2\delta, \\ \delta + \int_\delta^{2\delta} \left(\frac{2\delta - t}{\delta}\right)^{2/(p^- - 1)} dt, & 2\delta \leq d(x). \end{cases} \quad (3.3)$$

Obviously, $v_1(x) \in C^1(\overline{\Omega}) \cap C_0^+(\Omega)$.

LEMMA 3.2. *If $\lambda > 0$ is large enough, then $v_1(x)$ is a subsolution of (P).*

Proof. Since $|\nabla d(x)| \equiv 1$, when $\lambda > 0$ is large enough, we have

$$-\Delta_{p(x)} v_1 = -\Delta d(x) \leq \frac{\lambda}{[v_1(x)]^{p(x)}}, \quad \forall x \in \Omega, d(x) < \delta. \quad (3.4)$$

By computation, when $\delta < d(x) < 2\delta$, we have

$$\begin{aligned} -\Delta_{p(x)} v_1 &= -\operatorname{div} \left\{ \left[\left(\frac{2\delta - d(x)}{\delta} \right)^{2/(p^- - 1)} \right]^{p(x)-1} \nabla d(x) \right\} \\ &= - \left[\left(\frac{2\delta - d(x)}{\delta} \right)^{2/(p^- - 1)} \right]^{p(x)-1} \Delta d(x) \\ &\quad - \left[\left(\frac{2\delta - d(x)}{\delta} \right)^{2/(p^- - 1)} \right]^{p(x)-1} [\nabla d(x) \nabla p(x)] \ln \left(\frac{2\delta - d(x)}{\delta} \right)^{2/(p^- - 1)} \\ &\quad + \frac{2}{\delta} \frac{(p(x) - 1)}{p^- - 1} \left[\frac{2\delta - d(x)}{\delta} \right]^{(2(p(x)-1)/(p^- - 1)) - 1}. \end{aligned} \quad (3.5)$$

When $\lambda > 0$ is large enough, it is easy to see that

$$\begin{aligned} -\Delta_{p(x)} v_1 &\leq \frac{\lambda}{[v_1(x)]^{p(x)}}, \quad \forall x \in \Omega, \delta < d(x) < 2\delta, \\ -\Delta_{p(x)} v_1 &= 0 \leq \frac{\lambda}{[v_1(x)]^{p(x)}}, \quad \forall x \in \Omega, 2\delta < d(x). \end{aligned} \quad (3.6)$$

From (3.4) and (3.6), we can conclude that $v_1(x)$ is a subsolution of (P). □

THEOREM 3.3. *If $\lambda > 0$ is a large enough constant, then problem (P) possesses only one positive solution u_λ , and u_λ is increasing with respect to λ .*

Proof. Denote $u_n = \bar{\omega}_n + a_n$, where $\bar{\omega}_n$ is a solution of (3.1). Since $\{u_n\}$ is a sequence of positive solutions of

$$\begin{aligned} -\Delta_{p(x)}u &= \frac{\lambda}{u^{y(x)}} \quad \text{in } \Omega, \\ u(x) &= a_n \quad \text{for } x \in \partial\Omega, \end{aligned} \tag{II}$$

then every u_n is subsolution and supersolution of $-\Delta_{p(x)}u = \lambda/u^{y(x)}$ in Ω . According to comparison principle, we have that $u_n \geq u_{n+1}$ for $n = 1, 2, \dots$. Since $v_1(x)$ is a subsolution of (P) and $v_1(x) = 0$ on $\partial\Omega$, then $u_n \geq u_{n+1} \geq v_1$ for $n = 1, 2, \dots$. According to Lemma 2.4, we have that $\{u_n\}$ has uniform $C^{1,\alpha}$ local regularity property, and hence we can choose a subsequence, which we denoted by $\{u_n^1\}$, such that $u_n^1 \rightarrow w$ and $\nabla u_n^1 \rightarrow h$ in Ω . In fact, $h = \nabla w$ in Ω .

For any domain $D \Subset \Omega$, for any $\varphi \in W_0^{1,p(x)}(D)$. The $C^{1,\alpha}$ regularity result implies that the sequences $\{u_n\}$ and $\{\nabla u_n\}$ are equicontinuous in D ; from the $C^{1,\alpha}$ estimate we conclude that $\nabla w \in C^\alpha(D)$ for some $0 < \alpha < 1$. Thus $w \in W^{1,p(x)}(D) \cap C^{1,\alpha}(D)$. From the $C^{1,\alpha}$ regularity result, we see that $|\nabla u_n^1|^{p-1}|\nabla\varphi| \leq C|\nabla\varphi|$ on D , and since the function $\xi \rightarrow |\xi|^{p-2}\xi$ is continuous on \mathbb{R}^n , it follows that $|\nabla u_n^1|^{p-2}\nabla u_n^1(x) \cdot \nabla\varphi(x) \rightarrow |\nabla w(x)|^{p-2}\nabla w(x) \cdot \nabla\varphi(x)$ for $x \in D$. Thus, by the dominated convergence theorem, for any $\varphi \in W_0^{1,p(x)}(D)$, we can see that

$$\int_D |\nabla u_n^1(x)|^{p-2}\nabla u_n^1(x) \cdot \nabla\varphi(x)dx \longrightarrow \int_D |\nabla w(x)|^{p-2}\nabla w(x) \cdot \nabla\varphi(x)dx. \tag{3.7}$$

Furthermore, since $0 \leq \lambda/([u_n^1(x)]^{y(x)}) \leq \lambda/([u_{n+1}^1(x)]^{y(x)})$, and $\lambda/([u_n^1(x)]^{y(x)}) \rightarrow \lambda/([w(x)]^{y(x)})$ for each $x \in D$, by the monotone convergence theorem we obtain

$$\int_D \frac{\lambda}{[u_n^1(x)]^{y(x)}}\varphi dx \longrightarrow \int_D \frac{\lambda}{[w(x)]^{y(x)}}\varphi dx, \quad \forall \varphi \in W_0^{1,p(x)}(D). \tag{3.8}$$

Therefore, it follows that

$$\int_D |\nabla w(x)|^{p-2}\nabla w(x) \cdot \nabla\varphi(x)dx - \int_D \frac{\lambda}{[w(x)]^{y(x)}}\varphi dx = 0, \quad \forall \varphi \in W_0^{1,p(x)}(D), \tag{3.9}$$

and hence w is a weak solution of $-\Delta_{p(x)}w = \lambda/([w(x)]^{y(x)})$ on D .

Obviously, w is a solution of (P), and satisfies $w \geq v_1$. According to comparison principle, it is easy to see that (P) possesses only one positive solution, and u_λ is increasing with respect to λ . \square

4. Asymptotic behavior of positive solutions

In the following, we will use C_i to denote positive constants.

THEOREM 4.1. *If u is a positive weak solution of problem (P), then $C_2d(x) \leq u(x)$ as $x \rightarrow \partial\Omega$.*

Proof. Similar to the proof of Lemma 3.2, there exists a positive constant C_2 such that when $\delta > 0$ is small enough, then $v_2(x) = C_2d(x)$ is a subsolution of (P) on $\partial\bar{\Omega}_\delta$. Thus $u(x) \geq v_2(x) = C_2d(x)$ on $\partial\bar{\Omega}_\delta$. The proof is completed. \square

Denote $\gamma^* = \max_{x \in \partial\bar{\Omega}_{2\sigma}} \gamma(x)$ and $\gamma_* = \min_{x \in \partial\bar{\Omega}_{2\sigma}} \gamma(x)$.

THEOREM 4.2. *If $1 \leq \gamma_* < \gamma^*$, for any weak solution u of problem (P), we have*

$$C_3[d(x)]^{\theta_1} \leq u(x) \leq C_4[d(x)]^{\theta_2} \quad \text{as } x \rightarrow \partial\Omega, \tag{4.1}$$

where $\theta_1 = \max_{d(x) \leq \sigma} (p(x)/(p(x) - 1 + \gamma(x)))$, $\theta_2 = \min_{d(x) \leq \sigma} (p(x)/(p(x) - 1 + \gamma(x)))$.

Proof. From Theorem 4.1 we only consider (P) in the case of $1 < \gamma_* < \gamma^*$. Denote

$$v_3(x) = \begin{cases} a(d(x))^\theta, & d(x) < \delta, \\ a\delta^\theta + \int_\delta^{d(x)} a\theta\delta^{\theta-1} \left(\frac{2\delta-t}{\delta}\right)^{2/(p^- - 1)} dt, & \delta \leq d(x) < 2\delta, \\ a\delta^\theta + \int_\delta^{2\delta} a\theta\delta^{\theta-1} \left(\frac{2\delta-t}{\delta}\right)^{2/(p^- - 1)} dt, & 2\delta \leq d(x), \end{cases} \tag{4.2}$$

where a and θ are positive constants and satisfy $\theta \in (0, 1)$, $0 < \delta$ is small enough.

Obviously, $v_3(x) \in C^1(\Omega) \cap C_0^+(\Omega)$. By computation,

$$-\Delta_{p(x)} v_3(x) = -(a\theta)^{p(x)-1} (\theta - 1) (p(x) - 1) (d(x))^{(\theta-1)(p(x)-1)-1} (1 + \Pi(x)), \quad d(x) < \delta, \tag{4.3}$$

where

$$\Pi(x) = d \frac{(\nabla p \nabla d) \ln a \theta}{(\theta - 1)(p(x) - 1)} + d \frac{(\nabla p \nabla d) \ln d}{(p(x) - 1)} + d \frac{\Delta d}{(\theta - 1)(p(x) - 1)}. \tag{4.4}$$

Obviously $|\Pi(x)| \leq 1/2$, when $\delta > 0$ is small enough. Let $\theta = \theta_1$ and $a \in (0, 1)$ is small enough, when $\delta \in (0, a)$ is small enough, we can conclude that

$$-\Delta_{p(x)} v_3(x) \leq \frac{\lambda}{[v_2(x)]^{\gamma(x)}}, \quad d(x) < \delta. \tag{4.5}$$

By computation, when $\delta < d(x) < 2\delta$, we have

$$\begin{aligned}
 -\Delta_{p(x)} v_3 &= -\operatorname{div} \left\{ \left[a\theta\delta^{\theta-1} \left(\frac{2\delta - d(x)}{\delta} \right)^{2/(p^- - 1)} \right]^{p(x)-1} \nabla d(x) \right\} \\
 &= - \left[a\theta\delta^{\theta-1} \left(\frac{2\delta - d(x)}{\delta} \right)^{2/(p^- - 1)} \right]^{p(x)-1} \\
 &\quad \times [\nabla d(x) \nabla p(x)] \ln a\theta\delta^{\theta-1} \left(\frac{2\delta - d(x)}{\delta} \right)^{2/(p^- - 1)} \\
 &\quad - \left[a\theta\delta^{\theta-1} \left(\frac{2\delta - d(x)}{\delta} \right)^{2/(p^- - 1)} \right]^{p(x)-1} \Delta d(x) \\
 &\quad + \frac{2}{\delta} \frac{(p(x) - 1)}{p^- - 1} (a\theta\delta^{\theta-1})^{p(x)-1} \left[\frac{2\delta - d(x)}{\delta} \right]^{(2(p(x)-1)/(p^- - 1)) - 1}.
 \end{aligned} \tag{4.6}$$

Thus, there exists a positive constant C^* such that

$$| -\Delta_{p(x)} v_3 | \leq C^* \delta^{(\theta-1)(p(x)-1)-1}, \quad \delta < d(x) < 2\delta. \tag{4.7}$$

Obviously,

$$v_3(x) \leq a(\theta + 1)\delta^\theta, \quad \delta < d(x) < 2\delta. \tag{4.8}$$

Let $\theta = \theta_1$, when $a \in (0, 1)$ is small enough, $\delta \in (0, a)$ is small enough, then

$$-\Delta_{p(x)} v_3(x) \leq \frac{\lambda}{[v_2(x)]^{\gamma(x)}}, \quad \delta < d(x) < 2\delta. \tag{4.9}$$

It is easy to see that

$$-\Delta_{p(x)} v_3(x) = 0 \leq \frac{\lambda}{[v_2(x)]^{\gamma(x)}}, \quad 2\delta < d(x). \tag{4.10}$$

Combining (4.5), (4.9), and (4.10), it is easy to see that when $\theta = \theta_1$, $a \in (0, 1)$ is small enough and $\delta \in (0, a)$ is small enough, then $v(x)$ is a subsolution of (P), then $u(x) \geq C_3 [d(x)]^{\theta_1}$ on $\overline{\partial\Omega}_\delta$.

Similarly, when $\delta > 0$ is small enough, $\theta = \theta_2$, and $a \geq \max_{x \in \overline{\partial\Omega}_\delta} (u(x)/\delta^\theta)$ is large enough, we can see that $v(x)$ is a supersolution of (P) on $\overline{\partial\Omega}_\delta$, and $u(x) \leq a[d(x)]^{\theta_2}$ on $\overline{\partial\Omega}_\delta$. The proof is completed. \square

THEOREM 4.3. *If $\lim_{d(x) \rightarrow 0} p(x) = p_0$ and $\lim_{d(x) \rightarrow 0} p(x)/(p(x) - 1 + \gamma(x)) = s$, where $s \leq 1$ is a positive constant, u is a solution of (P), then*

$$\lim_{d(x) \rightarrow 0} \frac{u(x)}{C(d(x))^s} = 1, \quad C = \lim_{d(x) \rightarrow 0} \left[\frac{\lambda}{\theta^{p(x)-1} (1 - \theta)(p(x) - 1)} \right]^{1/(p(x)-1+\gamma(x))}. \tag{4.11}$$

Proof. It can be obtained easily from Theorem 4.2. \square

THEOREM 4.4. *If $1 \geq \gamma^*$, for any positive constant $\theta \in (0, 1)$, u is a weak solution of problem (P), then there exists a positive constant C_5 such that $C_1 d(x) \leq u(x) \leq C_5 (d(x))^\theta$ as $x \rightarrow \partial\Omega$.*

Proof. According to Theorem 4.1, it only needs to prove $u(x) \leq C_5 (d(x))^\theta$ as $x \rightarrow \partial\Omega$. Define a function on $\overline{\partial\Omega}_\delta$ as $v_4(x) = C_5 (d(x))^\theta$, where $C_5 \geq (1/\delta^\theta) \max_{x \in \overline{\partial\Omega}_\delta} u(x)$. Similar to the proof of Theorem 4.2, when $\delta > 0$ is small enough, then $v_4(x)$ is a supersolution of (P) on $\overline{\partial\Omega}_\delta$, then $u(x) \leq v_4(x) = C_5 (d(x))^\theta$ on $\overline{\partial\Omega}_\delta$. The proof is completed. \square

THEOREM 4.5. *If $\gamma_* < 1 < \gamma^*$, u is a weak solution of problem (P), then there exists a positive constant C_6 such that $C_1 d(x) \leq u(x) \leq C_6 (d(x))^\theta$ as $x \rightarrow \partial\Omega$, where $\theta = \min_{d(x) \leq \delta} (p(x)/(p(x) - 1 + \gamma(x)))$.*

Proof. According to Theorem 4.1, it only needs to prove $u(x) \leq C_6 (d(x))^\theta$ as $x \rightarrow \partial\Omega$. Similar to the proof of Theorem 4.2, when $\delta > 0$ is small enough, then $v_5(x) = C_6 (d(x))^\theta$ is a supersolution of (P) on $\overline{\partial\Omega}_\delta$, then $u(x) \leq v_5(x) = C_6 (d(x))^\theta$ on $\overline{\partial\Omega}_\delta$. The proof is completed. \square

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