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Research Article

On the Existence and Convergence of Approximate Solutions for Equilibrium Problems in Banach Spaces

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We introduce and study a new class of auxiliary problems for solving the equilibrium problem in Banach spaces. Not only the existence of approximate solutions of the equilibrium problem is proven, but also the strong convergence of approximate solutions to an exact solution of the equilibrium problem is shown. Furthermore, we give some iterative schemes for solving some generalized mixed variational-like inequalities to illuminate our results.

1. Introduction

Let X be a real Banach space with dual X^*, let $K \subset X$ be a nonempty subset, and let $f : K \times K \to \mathbb{R} = (-\infty, +\infty)$ be a given bifunction. By the equilibrium problem introduced by Blum and Oettli in [1], we can formulate the following equilibrium problem of finding an $\hat{x} \in K$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in K, \tag{1.1}$$

where $f(x,x) = 0$ for all $x \in K$.

The following is a list of special cases of problem (1.1).

(1) If $f(x,y) = \langle N(x,x), \eta(y,x)\rangle + b(x,y) - b(x,x)$ for all $x, y \in K$, where $N : K \times K \to X^*$, $\eta : K \times K \to X$, and $b : K \times K \to \mathbb{R}$, then the problem of finding an $\hat{x} \in K$ such that

$$\langle N(\hat{x},\hat{x}), \eta(y,\hat{x})\rangle + b(\hat{x}, y) - b(\hat{x},\hat{x}) \geq 0, \quad \forall y \in K, \tag{1.2}$$

is a special case of problem (1.1). This problem is known as the generalized mixed variational-like inequality. Problem (1.2) was considered by Huang and Deng [2] in the Hilbert space setting with set-valued mappings.

(2) If $X = X^* = H$ is a Hilbert space, $N(x,y) = Tx - Ay$, and $b(x,y) = m(y)$ for all $x,y \in K$, where $T,A,m : K \to X^*$, then problem (1.2) reduces to the following mixed variational-like inequality problem, which is to find an $\hat{x} \in K$ such that

$$\langle T(\hat{x}) - A(\hat{x}), \eta(y,\hat{x})\rangle + m(y) - m(\hat{x}) \geq 0, \quad \forall y \in K. \tag{1.3}$$

This problem was introduced and studied by Ansari and Yao [3] and Ding [4].

Remark 1.1. Through appropriate choices of the mappings f, N, η, and b, it can be easily shown that problem (1.1) covers many known problems as special cases. For example, see [1–8] and the references therein.

It is well known that many interesting and complicated problems in nonlinear analysis, such as nonlinear programming, optimization, Nash equilibria, saddle points, fixed points, variational inequalities, and complementarity problems (see [1, 9–12] and the references therein), can all be cast as equilibrium problems in the form of problem (1.1).

There are several papers available in the literature which are devoted to the development of iterative procedures for solving some of these equilibrium problems in finite as well as infinite-dimensional spaces. For example, some proximal point algorithms were developed based on the Bregman functions, see [13–18]. For other related works, we refer to [10, 12] and the references therein.

In [8], Iusem and Sosa presented some iterative algorithms for solving equilibrium problems in finite-dimensional spaces. They have also established the convergence of the algorithms In [19], Chen and Wu introduced an auxiliary problem for the equilibrium problem (1.1). They then showed that the approximate solutions generated by the auxiliary problem converge to the exact solution of the equilibrium problem (1.1) in Hilbert space.

In this paper, a new class of auxiliary problems for the equilibrium problem (1.1) in Banach space is introduced. We show the existence of approximate solutions of the auxiliary problems for the equilibrium problem, and establish the strong convergence of the approximate solutions to an exact solution of the equilibrium problem. Then, we develop an iterative scheme for solving problems (1.2) and (1.3). Our results extend and improve the corresponding results reported in [3, 4, 19].

2. Preliminaries

Throughout this paper, let X be a real Banach space and X^* its dual, let $\langle \cdot, \cdot \rangle$ be the dual pair between X and X^*, and let K be a nonempty convex subset of X.

In the sequel, we give some preliminary concepts and lemmas.

Definition 2.1 (see [20, 21]). Let $\eta : K \times K \to X$. A differentiable function $h : K \to \mathbb{R}$ on a convex set K is said to be

(i) η-convex if

$$h(y) - h(x) \geq \langle h'(x), \eta(y,x) \rangle, \quad \forall x, y \in K, \tag{2.1}$$

where $h'(x)$ denotes the Fréchet derivative of h at x;

(ii) μ-η-strongly convex if there exists a constant $\mu > 0$ such that

$$h(y) - h(x) - \langle h'(x), \eta(y,x) \rangle \geq \frac{\mu}{2} \|x - y\|^2, \quad \forall x, y \in K. \tag{2.2}$$

Remark 2.2. If $\eta(x, y) = x - y$ for all $x, y \in K$, then (i)-(ii) of Definition 2.1 reduce to the definitions of convexity and strong convexity, respectively.

Remark 2.3. h' is strongly monotone with the constant $\sigma > 0$ if h is strongly convex with a constant $\sigma/2$. In fact, by the strong convexity of h, we have

$$\begin{aligned}
\langle h'(x) - h'(y), x - y \rangle &= \langle h'(x), x - y \rangle - \langle h'(y), x - y \rangle \\
&= h(y) - h(x) - \langle h'(x), y - x \rangle + h(x) - h(y) - \langle h'(y), x - y \rangle \\
&\geq \frac{\sigma}{2} \|x - y\|^2 + \frac{\sigma}{2} \|x - y\|^2 = \sigma \|x - y\|^2.
\end{aligned} \tag{2.3}$$

Definition 2.4. Let $\eta : K \times K \to X$ be a single-valued mapping. For all $x, y \in E$, the mapping $N : K \times K \to X^*$ is said to be

(i) ϱ-η-coercive with respect to the first argument if there exists a $\varrho > 0$ such that

$$\langle N(x, \cdot) - N(y, \cdot), \eta(x, y) \rangle \geq \varrho \|N(x, \cdot) - N(y, \cdot)\|^2, \quad \forall x, y \in K; \tag{2.4}$$

(ii) ς-η-strongly monotone with respect to the second argument if there exists a constant $\varsigma > 0$ such that

$$\langle N(\cdot, x) - N(\cdot, y), \eta(x, y) \rangle \geq \varsigma \|x - y\|^2, \quad \forall x, y \in K; \tag{2.5}$$

(iii) σ-Lipschitz continuous with respect to the second argument if there exists a constant $\sigma > 0$ such that

$$\|N(\cdot, x) - N(\cdot, y)\| \leq \sigma \|x - y\|, \quad \forall x, y \in K. \tag{2.6}$$

Definition 2.5. Let $\eta : K \times K \to X$. The mapping $T : K \to X^*$ is said to be

(i) α-η-coercive if there exists an $\alpha > 0$ such that

$$\langle T(x) - T(y), \eta(x, y) \rangle \geq \alpha \|T(x) - T(y)\|^2, \quad \forall x, y \in K; \tag{2.7}$$

(ii) β-η-strongly monotone if there exists a $\beta > 0$ such that

$$\langle T(x) - T(y), \eta(x, y) \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in K; \tag{2.8}$$

(iii) η-monotone if

$$\langle T(x) - T(y), \eta(x,y) \rangle \geq 0, \quad \forall x, y \in K; \tag{2.9}$$

(iv) δ-η-relaxed monotone if there exists a $\delta > 0$ such that

$$\langle T(x) - T(y), \eta(x,y) \rangle \leq -\delta \|x - y\|^2, \quad \forall x, y \in K; \tag{2.10}$$

(v) ϵ-Lipschitz continuous if there exists an $\epsilon > 0$ such that

$$\|T(x) - T(y)\| \leq \epsilon \|x - y\|, \quad \forall x, y \in K. \tag{2.11}$$

Remark 2.6. If $\eta(x,y) = x - y$ for all $x, y \in K$, then (i)–(iv) of Definition 2.5 reduce to the definitions of coerciveness, strong monotonicity, monotonicity, relaxed monotonicity, respectively. Obviously, the η-coerciveness implies η-monotonicity.

Definition 2.7. The mapping $\eta : K \times K \to X$ is said to be τ-Lipschitz continuous if there exists a $\tau > 0$ such that

$$\|\eta(x,y)\| \leq \tau \|x - y\|, \quad \forall x, y \in K. \tag{2.12}$$

Remark 2.8. It is easy to see that T is α-η-coercive if T is β-η-strongly monotone and α/β-Lipschitz continuous. On the other hand if T is α-η-coercive and η is τ-Lipschitz continuous, then T is τ/α-Lipschitz continuous.

Definition 2.9 (see [22]). A mapping $F : K \to \mathbb{R}$ is called sequentially continuous at x_0 if for any sequence $\{x_n\} \subset K$ such that $\|x_n - x_0\| \to 0$, then $F(x_n) \to F(x_0)$. F is said to be sequentially continuous on K if it is sequentially continuous at each $x_0 \in K$.

Definition 2.10. Let E be a nonempty subset of a real topological vector space X. A set-valued function $\Phi : E \to 2^X$ is said to be a *KKM mapping* if for any nonempty finite set $A \subset E$,

$$\operatorname{co}(A) \subset \bigcup_{x \in A} \Phi(x), \tag{2.13}$$

where $\operatorname{co}(A)$ denotes the convex hull of A.

Lemma 2.11 (see [23]). *Let K be a nonempty convex subset of a real Hausdorff topological vector space X, and let $\Phi : K \to 2^X$ be a KKM mapping. Suppose that $\Phi(x)$ is closed in X for every $x \in K$, and that there is a point $x_0 \in K$ such that $\Phi(x_0)$ is compact. Then,*

$$\bigcap_{x \in K} \Phi(x) \neq \varnothing. \tag{2.14}$$

Lemma 2.12. *Let $A : K \to X^*$ be sequentially continuous from the weak topology to the strong topology. Suppose that for a fixed $y \in K$, $x \mapsto \eta(y,x)$ is a sequentially continuous mapping from the weak topology to the weak topology. Define $f(x) = \langle A(x), \eta(y,x) \rangle$. Then, $f(x)$ is a sequentially continuous mapping from the weak topology to the strong topology.*

Proof. If $x_n \to x_0$ with the weak topology, then $A(x_n) \to A(x_0)$, and for any fixed $y \in K$, $\eta(y,x_n) \to \eta(y,x_0)$ with the weak topology. Clearly,

$$\begin{aligned} |f(x_n) - f(x_0)| &= |\langle A(x_n), \eta(y,x_n)\rangle - \langle A(x_0), \eta(y,x_0)\rangle| \\ &= |\langle A(x_n) - A(x_0), \eta(y,x_n)\rangle - \langle A(x_0), \eta(y,x_n) - \eta(y,x_0)\rangle| \\ &\leq \|A(x_n) - A(x_0)\| \cdot \|\eta(y,x_n)\| + |\langle A(x_0), \eta(y,x_n) - \eta(y,x_0)\rangle|. \end{aligned} \tag{2.15}$$

By the boundedness property of the weak convergence sequence, we see that $\|\eta(y,x_n)\|$ is bounded. Thus, it follows that $|f(x_n) - f(x_0)| \to 0$. This completes the proof. □

Lemma 2.13 (see [24]). *Let K be a nonempty convex subset of a topological vector space. Suppose that $\phi : K \times K \to (-\infty, +\infty]$ is a mapping such that the following conditions are satisfied.*

(1) *For each $y \in K$, $x \mapsto \phi(y,x)$ is semicontinuous on every compact subset of K.*

(2) *If $x = \sum_{i=1}^{n} \lambda_i y_i$, where $\{y_1, y_2, \ldots, y_n\}$is any nonempty finite set in K, while $\lambda_i \geq 0$, $i = 1,2,\ldots,n$, such that $\sum_{i=1}^{n} \lambda_i = 1$, then $\min_{1 \leq n \leq n} \phi(y_i,x) \leq 0$.*

(3) *There exist a nonempty compact convex subset K_0 of K and a nonempty compact subset D_0 of K such that for each $x \in K \backslash D_0$, there exists a $y \in \mathrm{co}(K_0 \cup \{x\})$ such that $\phi(y,x) > 0$.*

Then, there exists an $x_0 \in K$ such that $\phi(y,x_0) \leq 0$ for all $y \in K$.

3. Main results

In this section, we first deal with the approximate solvability of problem (1.1). Let X be a reflexive Banach space and X^* its dual, and let K be a nonempty convex subset of X. We introduce an auxiliary function $\varphi : K \to \mathbb{R}$ which is differentiable. Then, we construct the auxiliary problem for problem (1.1) as follows.

For any given $x_n \in K$, find an $x_{n+1} \in K$ such that

$$\rho f(x_n, y) - \rho f(x_n, x_{n+1}) + \langle \varphi'(x_{n+1}) - \varphi'(x_n), y - x_{n+1}\rangle \geq 0, \quad \forall y \in K, \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pair between X and X^*, $\rho > 0$ is a constant, and $\varphi'(x)$ is the Fréchet derivative of φ at x.

We note that x_n is a solution of problem (1.1) when $x_{n+1} = x_n$.

Remark 3.1. If $\rho = 1$, then the auxiliary problem for problem (3.1) reduces to the auxiliary problem studied by Chen and Wu [19].

Similarly, we can construct the auxiliary problems (3.2) and (3.3) for problems (1.2) and (1.3), respectively.

(1) If $f(x,y) = \langle N(x,x), \eta(y,x)\rangle + b(x,y) - b(x,x)$ for all $x,y \in K$, where $N : K \times K \to X^*$, $\eta : K \times K \to X$, and $b : K \times K \to \mathbb{R}$, then for any given x_n, problem (3.1) is equivalent to finding an x_{n+1} such that

$$\begin{aligned} \langle \varphi'(x_{n+1}), y - x_{n+1}\rangle \geq{}& \langle \varphi'(x_n), y - x_{n+1}\rangle - \rho\langle N(x_n,x_n), \eta(y,x_{n+1})\rangle \\ &+ \rho b(x_n,x_{n+1}) - \rho b(x_n,y), \quad \forall y \in K. \end{aligned} \tag{3.2}$$

(2) If X, T, A, m are the same as in problem (1.3), then for a given iterate x_n, problem (3.2) reduces to the following problem of finding an x_{n+1} such that

$$\langle \rho(T(x_n) - A(x_n)) + \varphi'(x_{n+1}) - \varphi'(x_n), \eta(y, x_{n+1})\rangle + \rho[m(y) - m(x_{n+1})] \geq 0, \quad \forall y \in K. \tag{3.3}$$

Now, we are in a position to state and prove the main results of the paper.

Theorem 3.2. *Let X be a reflexive Banach space with dual space X^* and let K be a nonempty convex subset of X. Suppose that $f : K \times K \to \mathbb{R}$ is a bifunction and $\varphi : K \to R$ is a differentiable function. Furthermore, for all $x, y, z \in K$, assume that the following conditions are satisfied.*

(i) *$y \mapsto f(x, y)$ is affine and weakly lower semicontinuous.*

(ii) *φ' is μ-strongly monotone and sequentially continuous from the weak topology to the strong topology.*

(iii) *There exist a compact set $C \subset K$ and a vector $y_0 \in K$ such that for any $\rho > 0$,*

$$\rho f(x_n, x) - \rho f(x_n, y_0) > \langle \varphi'(x) - \varphi'(x_n), y_0 - x\rangle, \quad \forall x \in K \backslash C. \tag{3.4}$$

Then, auxiliary problem (3.1) admits a unique solution $x_{n+1} \in K$. In addition, suppose that the following condition is also satisfied.

(iv) *There exist constants $a \leq 0$, $b > 0$, and $c \in \mathbb{R}$ such that*

$$f(x_n, x_{n+1}) - f(x_n, z) - f(z, x_{n+1}) \geq a\|x_n - x_{n+1}\|^2 + b\|x_n - z\|^2 + c\|x_n - x_{n+1}\| \cdot \|x_n - z\| \tag{3.5}$$

for all $z \in K$ and $n = 0, 1, 2, \ldots$.

If the original problem (1.1) has a solution and

$$\mu + 2a\rho \geq 0, \qquad 0 < \rho < \frac{2b\mu}{c^2 - 4ab}, \tag{3.6}$$

then the sequence $\{x_n\}$ generated by (3.1) converges to a solution of equilibrium problem (1.1).

Proof. Let

$$S(y) = \{x \in K \mid \rho f(x_n, y) - \rho f(x_n, x) + \langle \varphi'(x) - \varphi'(x_n), y - x\rangle \geq 0\}. \tag{3.7}$$

If $\bigcap_{y \in K} S(y) \neq \varnothing$, then there exists a solution to (3.1).

Since $y \in S(y)$ for all $y \in K$, $\overline{S(y)} \neq \varnothing$, it follows from (iii) that for any $x \in K \backslash C$,

$$\rho f(x_n, y_0) - \rho f(x_n, x) + \langle \varphi'(x) - \varphi'(x_n), y_0 - x\rangle < 0. \tag{3.8}$$

That is, $x \notin S(y_0)$. Thus, $S(y_0) \subset K \cap C$. Since C is compact, there exists a $y_0 \in K$ such that $\overline{S(y_0)}$ is also compact.

For any finite subset $\{t_1, t_2, \ldots, t_r\} \subset K$, let $\operatorname{co}\{t_1, \ldots, t_r\}$ be its convex hull. If $t \in \operatorname{co}(\{t_i\}_{i=1}^r)$, then $t = \sum_{i=1}^r \lambda_i t_i$ with $\lambda_i \geq 0$, $i = 1, 2, \ldots, r$, and $\sum_{i=1}^r \lambda_i = 1$. If $t \notin \bigcup_{i=1}^r S(t_i)$, then $t \notin S(t_i)$ for all $i = 1, 2, \ldots, r$. Hence,

$$\rho f(x_n, t_i) - \rho f(x_n, t) + \langle \varphi'(t) - \varphi'(x_n), t_i - t \rangle < 0, \tag{3.9}$$

and so

$$\sum_{i=1}^{r} [\lambda_i \rho f(x_n, t_i) - \lambda_i \rho f(x_n, t) + \lambda_i \langle \varphi'(t) - \varphi'(x_n), t_i - t \rangle] < 0. \tag{3.10}$$

Since $t = \sum_{i=1}^{r} \lambda_i t_i$, it follows from (i) that

$$f(x_n, t) = \sum_{i=1}^{r} \lambda_i f(x_n, t_i) < f(x_n, t), \tag{3.11}$$

which is a contradiction. Therefore,

$$\operatorname{co}\{t_1, t_2, \ldots, t_r\} \subset \bigcup_{i=1}^{r} S(t_i) \subset \bigcup_{i=1}^{r} \overline{S(t_i)}. \tag{3.12}$$

By Lemma 2.11, $\bigcap_{y \in K} \overline{S(y)} \neq \emptyset$.

Set $\overline{x} \in \bigcap_{y \in K} \overline{S(y)}$. Then, $\overline{x} \in \overline{S(y)}$ for all $y \in K$ and there exists a sequence $\{u_k\} \subset S(y)$ such that $u_k \to \overline{x}$. It follows that

$$\rho f(x_n, y) - \rho f(x_n, u_k) + \langle \varphi'(u_k) - \varphi'(x_n), y - u_k \rangle \geq 0. \tag{3.13}$$

Since $y \mapsto f(x, y)$ is weakly lower semicontinuous and φ' is sequentially continuous from the weak topology to the strong topology, as $k \to \infty$, we have

$$\rho f(x_n, y) - \rho f(x_n, \overline{x}) + \langle \varphi'(\overline{x}) - \varphi'(x_n), y - \overline{x} \rangle \geq 0, \tag{3.14}$$

which implies that $\overline{x} \in S(y)$ for all $y \in K$. Therefore, $\bigcap_{y \in K} S(y) \neq \emptyset$.

Now, we will prove that the solution of (3.1) is unique. In fact, if there exist $\overline{x}_1, \overline{x}_2 \in \bigcap_{y \in K} S(y) \subset K$ with $\overline{x}_1 \neq \overline{x}_2$, then

$$\rho f(x_n, y) - \rho f(x_n, \overline{x}_1) + \langle \varphi'(\overline{x}_1) - \varphi'(x_n), y - \overline{x}_1 \rangle \geq 0, \quad \forall y \in K, \tag{3.15}$$

$$\rho f(x_n, y) - \rho f(x_n, \overline{x}_2) + \langle \varphi'(\overline{x}_2) - \varphi'(x_n), y - \overline{x}_2 \rangle \geq 0, \quad \forall y \in K. \tag{3.16}$$

Setting $y = \overline{x}_2$ in (3.15) and $y = \overline{x}_1$ in (3.16), we get

$$\rho f(x_n, \overline{x}_2) - \rho f(x_n, \overline{x}_1) + \langle \varphi'(\overline{x}_1) - \varphi'(x_n), \overline{x}_2 - \overline{x}_1 \rangle \geq 0, \tag{3.17}$$

$$\rho f(x_n, \overline{x}_1) - \rho f(x_n, \overline{x}_2) + \langle \varphi'(\overline{x}_2) - \varphi'(x_n), \overline{x}_1 - \overline{x}_2 \rangle \geq 0. \tag{3.18}$$

Adding (3.17) to (3.18), we obtain

$$\langle \varphi'(\overline{x}_1), \overline{x}_2 - \overline{x}_1 \rangle + \langle \varphi'(\overline{x}_2), \overline{x}_1 - \overline{x}_2 \rangle \geq 0. \tag{3.19}$$

Since φ is strictly convex with constant $\mu > 0$, it holds that

$$\varphi(\overline{x}_2) - \varphi(\overline{x}_1) - \frac{\mu}{2}\|\overline{x}_2 - \overline{x}_1\|^2 + \varphi(\overline{x}_1) - \varphi(\overline{x}_2) - \frac{\mu}{2}\|\overline{x}_2 - \overline{x}_1\|^2 \geq 0, \tag{3.20}$$

that is,

$$-\mu\|\overline{x}_2 - \overline{x}_1\|^2 \geq 0. \tag{3.21}$$

This contradicts with $\mu > 0$ and $\overline{x}_1 \neq \overline{x}_2$. Hence, problem (3.1) admits a unique solution, which is denoted by x_{n+1}.

Let $\hat{x}$ be a solution of the original problem (1.1). For each $y \in K$, we define a function $\Theta : K \to R$ by

$$\Theta(y) = \varphi(\hat{x}) - \varphi(y) - \langle \varphi'(y), \hat{x} - y \rangle. \tag{3.22}$$

It follows from the strict convexity of φ that

$$\Theta(y) \geq \frac{\mu}{2}\|\hat{x} - y\|^2 \geq 0, \tag{3.23}$$

$$\begin{aligned}
\Theta(x_n) - \Theta(x_{n+1}) &= \varphi(\hat{x}) - \varphi(x_n) - \langle \varphi'(x_n), \hat{x} - x_n \rangle - \varphi(\hat{x}) + \varphi(x_{n+1}) + \langle \varphi'(x_{n+1}), \hat{x} - x_{n+1} \rangle \\
&= \varphi(x_{n+1}) - \varphi(x_n) - \langle \varphi'(x_n), \hat{x} - x_n \rangle + \langle \varphi'(x_{n+1}), \hat{x} - x_{n+1} \rangle \\
&= [\varphi(x_{n+1}) - \varphi(x_n) - \langle \varphi'(x_n), x_{n+1} - x_n \rangle] + \langle \varphi'(x_{n+1}) - \varphi'(x_n), \hat{x} - x_{n+1} \rangle \\
&\geq \frac{\mu}{2}\|x_{n+1} - x_n\|^2 + \langle \varphi'(x_{n+1}) - \varphi'(x_n), \hat{x} - x_{n+1} \rangle.
\end{aligned} \tag{3.24}$$

Setting $y = \hat{x}$ in (3.1), we have

$$\rho f(x_n, \hat{x}) - \rho f(x_n, x_{n+1}) + \langle \varphi'(x_{n+1}) - \varphi'(x_n), \hat{x} - x_{n+1} \rangle \geq 0, \tag{3.25}$$

that is,

$$\langle \varphi'(x_{n+1}) - \varphi'(x_n), \hat{x} - x_{n+1} \rangle \geq \rho f(x_n, x_{n+1}) - \rho f(x_n, \hat{x}). \tag{3.26}$$

Let $y = x_{n+1}$ in (1.1). Then, $f(\hat{x}, x_{n+1}) \geq 0$, and so

$$\rho f(\hat{x}, x_{n+1}) \geq 0. \tag{3.27}$$

By (3.24)–(3.27), we have

$$\begin{aligned}
\Theta(x_n) - \Theta(x_{n+1}) &\geq \frac{\mu}{2}\|x_{n+1} - x_n\|^2 + \rho f(x_n, x_{n+1}) - \rho f(x_n, \hat{x}) - \rho f(\hat{x}, x_{n+1}) \\
&= \frac{\mu}{2}\|x_{n+1} - x_n\|^2 + \rho Q,
\end{aligned} \tag{3.28}$$

where $Q = f(x_n, x_{n+1}) - f(x_n, \hat{x}) - f(\hat{x}, x_{n+1})$. From assumption (iv), there exist constants $a \leq 0$, $b > 0$ and $c \in \mathbb{R}$, such that

$$Q \geq a\|x_n - x_{n+1}\|^2 + b\|x_n - \hat{x}\|^2 + c\|x_n - \hat{x}\| \cdot \|x_{n+1} - x_n\|. \tag{3.29}$$

Combining (3.28) and (3.29), we have

$$\begin{aligned}
&\Theta(x_n) - \Theta(x_{n+1}) \\
&\quad \geq \left(\frac{\mu}{2} + a\rho\right)\|x_{n+1} - x_n\|^2 + b\rho\|x_n - \hat{x}\|^2 + c\rho\|x_n - \hat{x}\| \cdot \|x_{n+1} - x_n\| \\
&\quad = \left(\frac{\mu}{2} + a\rho\right)\left\{\|x_{n+1} - x_n\| + \frac{c\rho}{\mu + 2a\rho}\|x_n - \hat{x}\|\right\}^2 + \left(b\rho - \frac{(c\rho)^2}{2(\mu + 2a\rho)}\right)\|x_n - \hat{x}\|^2 \\
&\quad \geq \left(b\rho - \frac{(c\rho)^2}{2(\mu + 2a\rho)}\right)\|x_n - \hat{x}\|^2.
\end{aligned} \tag{3.30}$$

It follows from (3.6) and (3.30) that

$$\Theta(x_n) - \Theta(x_{n+1}) \geq 0. \tag{3.31}$$

From (3.31), we know that $\{\Theta(x_n)\}$ is a decreasing sequence with infimum, so it converges to some number. Hence, $\lim_{n\to\infty}[\Theta(x_n) - \Theta(x_{n+1})] = 0$. It follows from (3.30) that $\lim_{n\to\infty} x_n = \hat{x}$. This completes the proof. □

Remark 3.3. Suppose that $x \mapsto f(x,y)$ is additive (*i.e.*, $f(x+y,u) = f(x,u) + f(y,u)$ for all $x,y,u \in K$), that $y \mapsto f(x,y)$ is also additive, and that there exists a constant $\nu > 0$ such that $f(x,y) \geq \nu\|x\| \cdot \|y\|$. Then, by the fact that $f(z,z) = 0$ for all $z \in K$, we have

$$\begin{aligned}
&f(x_n,x_{n+1}) - f(x_n,z) - f(z,x_{n+1}) \\
&\quad = f(x_n - z, x_{n+1}) - f(x_n - z, z) = f(x_n - z, x_{n+1} - z) \\
&\quad = f(x_n - z, x_{n+1} - x_n) + f(x_n - z, x_n - z) \geq \nu\|x_n - z\|^2 + \nu\|x_n - z\| \cdot \|x_{n+1} - x_n\| \\
&\quad = 0 \cdot \|x_{n+1} - x_n\|^2 + \nu\|x_n - z\|^2 + \nu\|x_n - z\| \cdot \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.32}$$

Let $a = 0$, $b = c = \nu$. Then, the assumption (iv) of Theorem 3.2 holds. Therefore, our results extend, improve, and unify the corresponding results obtained by Chen and Wu in [19].

THEOREM 3.4. *Let K and X be the same as in Theorem 3.2. Let $N : K \times K \to X^*$ and $\eta : K \times K \to X$ be two mappings, and let $b : K \times K \to R$ and $\varphi : K \to R$ be two functions. Suppose that the following conditions are satistified.*

(i) *$N(\cdot,\cdot)$ is α-η-coercive with respect to the first argument and is ξ-η-strongly monotone and β-Lipschitz continuous with respect to the second argument, $y \mapsto \langle N(x,x), \eta(y,x)\rangle$ is concave, and N is sequentially continuous from the weak topology to the strong topology with respect to the first argument and the second argument.*

(ii) *$\eta(x,y) = \eta(x,z) + \eta(z,y)$ for all $x,y,z \in K$, η is λ-Lipschitz continuous and for any given $y \in K$, $x \mapsto \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.*

(iii) *$b(\cdot,\cdot)$ is linear with respect to the first argument and convex lower semicontinuous with respect to the second argument, there exists a constant $0 < \gamma < \beta$ such* *that* $b(x,y) \leq \gamma\|x\|\|y\|$ *for all* $x,y \in K$, *and* $b(x,y) - b(x,z) \leq b(x,y-z)$ *for all* $x,y,z \in K$.

(iv) φ *is* μ*-strongly convex and its Fréchet derivative* φ' *is sequentially continuous from the weak topology to the strong topology.*

Then, there exists a unique solution $x_{n+1} \in K$ *for auxiliary problem (3.2). In addition, if the original problem (1.2) has a solution and*

$$\mu - \frac{\lambda^2\rho}{2\alpha} \geq 0, \qquad 0 < \rho < \frac{2\mu\alpha(\xi-\gamma)}{\alpha(\beta\lambda+\gamma)^2 + (\xi-\gamma)\lambda^2}, \qquad \xi > \gamma > 0, \tag{3.33}$$

then the sequence $\{x_n\}$ *generated by (3.2) converges to a solution of generalized mixed variational-like inequality problem (1.2).*

Proof. Since $\eta(x,y) = \eta(x,z) + \eta(z,y)$ for all $x,y,z \in K$, it is easy to see that

$$\eta(x,x) = 0, \quad \eta(x,y) + \eta(y,x) = 0, \quad \forall x,y \in K. \tag{3.34}$$

Let $f(x,y) = \langle N(x,x), \eta(y,x)\rangle + b(x,y) - b(x,x)$ for all $x,y \in K$. Then, the following results follow.

(a) Assumptions (ii) and (iii) imply that condition (i) of Theorem 3.2 holds.

(b) From (3.33), (3.34), and assumptions (i)–(iii), we have

$$\begin{aligned}
& f(x,y) - f(x,z) - f(z,y) \\
&\quad = \langle N(z,z) - N(x,z), \eta(z,x)\rangle + \langle N(x,z) - N(x,x), \eta(z,x)\rangle \\
&\qquad - \langle N(z,z) - N(x,z), \eta(y,x)\rangle - \langle N(x,z) - N(x,x), \eta(y,x)\rangle \\
&\qquad - b(x-z, z-x) - b(x-z, x-y) \\
&\quad \geq \alpha||N(z,z) - N(x,z)||^2 + \xi||z-x||^2 - ||N(z,z) - N(x,z)|| \cdot ||\eta(y,x)|| \\
&\qquad - ||N(x,z) - N(x,x)|| \cdot ||\eta(y,x)|| - \gamma||x-z||^2 - \gamma\|x-z\| \cdot \|x-y\| \\
&\quad \geq \alpha||N(z,z) - N(x,z)||^2 - \lambda||N(z,z) - N(x,z)|| \cdot \|y-x\| + \xi\|z-x\|^2 \\
&\qquad - \beta\lambda\|z-x\| \cdot \|y-x\| - \gamma\|x-z\|^2 - \gamma\|x-z\| \cdot \|x-y\| \\
&\quad = \alpha\left[||N(z,z) - N(x,z)|| - \frac{\lambda}{2\alpha}\|x-y\|\right]^2 - \frac{\lambda^2}{4\alpha}\|x-y\|^2 \\
&\qquad + (\xi-\gamma)\|z-x\|^2 - (\beta\lambda+\gamma)\|x-z\| \cdot \|x-y\| \\
&\quad \geq a\|x-y\|^2 + b\|x-z\|^2 + c\|x-y\| \cdot \|x-z\|
\end{aligned} \tag{3.35}$$

for all $x,y,z \in K$, where

$$a = -\frac{\lambda^2}{4\alpha} < 0, \qquad b = \xi - \gamma > 0, \qquad c = -(\beta\lambda+\gamma). \tag{3.36}$$

This implies that assumption (iv) of Theorem 3.2 holds.

(c) By assumption (iv), it is easy to see that assumption (ii) of Theorem 3.2 holds.

(d) Assumption (iii) of Theorem 3.2 can be obtained by Lemmas 2.12 and 2.13, and conditions (i) and (iii) (see [2]).

Thus, the conclusions of the theorem follows from the argument similar to that given for Theorem 3.2. This completes the proof. □

Theorem 3.5. *Let K be a nonempty convex subset of a real Hilbert space H. Let $m : K \to \mathbb{R}$ be a lower semicontinuous and convex functional, let T, $A : K \to H$ be two mappings such that T is α-η-coercive and A is ξ-η-relaxed monotone and β-Lipschitz continuous. Assume that*

(i) *$\eta : K \times K \to H$ is λ-Lipschitz continuous such that*
  (a) *$\eta(x,y) = \eta(x,z) + \eta(z,y)$ for all $x,y,z \in K$,*
  (b) *$\eta(\cdot,\cdot)$ is affine in the first variable,*
  (c) *for each fixed $y \in K$, $x \mapsto \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology;*

(ii) *$\varphi : K \to R$ is μ-η-strongly convex and its derivative φ' is sequentially continuous from the weak topology to the strong topology.*

Then, there exists a unique solution $x_{n+1} \in K$ for auxiliary problem (3.3). In addition, if the original problem (1.3) has a solution and

$$\mu - \frac{\lambda^2 \rho}{2\alpha} \geq 0, \quad 0 < \rho < \frac{2\alpha\mu\xi}{\lambda^2(\xi + \alpha\beta^2)}, \tag{3.37}$$

then the sequence $\{x_n\}$ generated by (3.3) converges to a solution of problem (1.3).

Proof. Let $f(x,y) = \langle T(x) - A(x), \eta(y,x)\rangle + m(y) - m(x)$ for all $x,y \in K$. By the proof of Theorem 3.4, we can take $a = -\lambda^2/4\alpha < 0$, $b = \xi > 0$, and $c = -\beta\lambda \in \mathbb{R}$ in (3.35) and check that all conditions of Theorem 3.2 hold. This completes the proof. □

Remark 3.6. Our results extend and improve those obtained by Ansari and Yao in [3] in the following ways: (i) the mixed variational-like inequality (1.3) in a Hilbert space is extended and generalized to the equilibrium problem (1.1) in a Banach space, (ii) we do not require that K is bounded, (iii) the condition $\eta(x,y) + \eta(y,x) = 0$ for all $x,y \in K$ is removed, (iv) our method for the proof of the existence of approximate solutions is very different from theirs. Furthermore, our results also extend Ding's results in [4] in the following ways: (i) the mixed variational-like inequality (1.3) in a Hilbert space is extended and generalized to the equilibrium problem (1.1) in a Banach space, (ii) the condition $\eta(x,y) + \eta(y,x) = 0$ for all $x,y \in K$ is removed, (iii) our convergence criteria are very different from the ones used by Ding.

From Theorem 3.2, as noted by Zhu and Marcotte in [25], the solution of problem (3.1) cannot be obtained in closed form. Thus, a tradeoff between the amount of work spent on solving the auxiliary problem and the accuracy of the corresponding solution is to be decided. More precisely, we can choose preassigned tolerances, ε_n, $n = 1,2,\ldots$. Then, at step n, one can find an approximate solution of the auxiliary problem, that is, a point $x_{n+1} \in K$ such that

$$\rho f(x_n, y) - \rho f(x_n, x_{n+1}) + \langle \varphi'(x_{n+1}) - \varphi'(x_n), y - x_{n+1} \rangle \geq \varepsilon_n, \quad \forall y \in K', \tag{3.38}$$

where $\rho > 0$ is a constant and $\varphi'(x)$ is the Fréchet derivative of φ at x. If K is bounded, we take $K' = K$. Otherwise, we define

$$K' = K \cap \{x : \|x\| \leq M\}, \tag{3.39}$$

where M is a suitably large constant. We note that such a number always exists because $\{\|x_n - \hat{x}\|\}$ is bounded.

Theorem 3.7. *Suppose that all conditions of Theorem 3.2 are satisfied and that $\{\varepsilon_n\}$ is a sequence such that*

$$\varepsilon_n \geq 0, \qquad \lim_{n\to\infty} \varepsilon_n = 0. \tag{3.40}$$

Then, the sequence $\{x_n\}$ generated by (3.38) converges to a solution $\hat{x}$ of equilibrium problem (1.1).

Proof. From the proof of Theorem 3.2, we have

$$\Theta(x_n) - \Theta(x_{n+1}) \geq r\left\{||x_{n+1} - x_n|| + \frac{c\rho}{\mu + 2a\rho}||x_n - \hat{x}||\right\}^2 + s||x_n - \hat{x}||^2 + \varepsilon_n, \tag{3.41}$$

where $r = \mu/2 + a\rho$ and $s = b\rho - (c\rho)^2/2(\mu + 2a\rho)$. It follows from (3.6), (3.40), and (3.41) that

$$\Theta(x_n) - \Theta(x_{n+1}) \geq s||x_n - \hat{x}||^2 + \varepsilon_n \geq 0, \tag{3.42}$$

that is, $\{\Theta(x_n)\}$ is strictly (unless $x_n = \hat{x}$) decreasing and is nonnegative by (3.23), and so it converges to some number. Hence, $\lim_{n\to\infty}[\Theta(x_n) - \Theta(x_{n+1})] = 0$, and by (3.42), we have

$$\lim_{n\to\infty} (s||x_n - \hat{x}||^2 + \varepsilon_n) = 0. \tag{3.43}$$

Therefore,

$$\begin{aligned} &\lim_{n\to\infty} s||x_n - \hat{x}||^2 \\ &\quad = \lim_{n\to\infty} [(s||x_n - \hat{x}||^2 + \varepsilon_n) - \varepsilon_n] = \lim_{n\to\infty} (s||x_n - \hat{x}||^2 + \varepsilon_n) - \lim_{n\to\infty} \varepsilon_n = 0, \end{aligned} \tag{3.44}$$

that is, $\{x_n\}$ converges strongly to $\hat{x}$, a solution of problem (1.1). This completes the proof. □

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