

Research Article

On General Summability Factor Theorems

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The goal of this paper is to obtain sufficient and (different) necessary conditions for a series $\sum a_n$, which is absolutely summable of order k by a triangular matrix method A , $1 < k \leq s < \infty$, to be such that $\sum a_n \lambda_n$ is absolutely summable of order s by a triangular matrix method B . As corollaries, we obtain two inclusion theorems.

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In the recent papers [1, 2], the author obtained necessary and sufficient conditions for a series $\sum a_n$ which is absolutely summable of order k by a weighted mean method, $1 < k \leq s < \infty$, to be such that $\sum a_n \lambda_n$ is absolutely summable of order s by a triangular matrix method. In this paper, we obtain sufficient and (different) necessary conditions for a series $\sum a_n$ which is absolutely summable $|A|_k$ to imply the series $\sum a_n \lambda_n$ which is absolutely summable $|B|_s$.

Let T be a lower triangular matrix, $\{s_n\}$ a sequence.

Then

$$T_n := \sum_{\nu=0}^n t_{n\nu} s_\nu. \quad (1)$$

A series $\sum a_n$ is said to be summable $|T|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty. \quad (2)$$

We may associate with T two lower triangular matrices \bar{T} and \hat{T} as follows:

$$\begin{aligned} \bar{t}_{n\nu} &= \sum_{r=\nu}^n t_{nr}, \quad n, \nu = 0, 1, 2, \dots, \\ \hat{t}_{n\nu} &= \bar{t}_{n\nu} - \bar{t}_{n-1, \nu}, \quad n = 1, 2, 3, \dots \end{aligned} \tag{3}$$

With $s_n := \sum_{i=0}^n a_i \lambda_i$,

$$\begin{aligned} y_n &:= \sum_{i=0}^n t_{ni} s_i = \sum_{i=0}^n t_{ni} \sum_{\nu=0}^i a_\nu \lambda_\nu = \sum_{\nu=0}^n a_\nu \lambda_\nu \sum_{i=\nu}^n t_{ni} = \sum_{\nu=0}^n \bar{t}_{n\nu} a_\nu \lambda_\nu, \\ Y_n &:= y_n - y_{n-1} = \sum_{\nu=0}^n (\bar{t}_{n\nu} - \bar{t}_{n-1, \nu}) \lambda_\nu a_\nu = \sum_{\nu=0}^n \hat{t}_{n\nu} \lambda_\nu a_\nu. \end{aligned} \tag{4}$$

We will call T as a triangle if T is lower triangular and $t_{nn} \neq 0$ for each n . The notation $\Delta_\nu \hat{a}_{n\nu}$ means $\hat{a}_{n\nu} - \hat{a}_{n, \nu+1}$. The notation $\lambda \in (|A|_k, |B|_s)$ will be used to represent the statement that if $\sum a_n$ is summable $|A|_k$, then $\sum a_n \lambda_n$ is summable $|B|_s$.

THEOREM 1. *Let $1 < k \leq s < \infty$. Let $\{\lambda_n\}$ be a sequence of constants, A and B triangles satisfying*

- (i) $|b_{nn} \lambda_n| / |a_{nn}| = O(\nu^{1/s-1/k})$,
 - (ii) $(n|X_n|)^{s-k} = O(1)$,
 - (iii) $|a_{nn} - a_{n+1, n}| = O(|a_{nn} a_{n+1, n+1}|)$,
 - (iv) $\sum_{\nu=0}^{n-1} |\Delta_\nu(\hat{b}_{n\nu} \lambda_\nu)| = O(|b_{nn} \lambda_n|)$,
 - (v) $\sum_{n=\nu+1}^\infty (n|b_{nn} \lambda_n|)^{s-1} |\Delta_\nu(\hat{b}_{n\nu} \lambda_\nu)| = O(\nu^{s-1} |b_{\nu\nu} \lambda_\nu|^s)$,
 - (vi) $\sum_{\nu=0}^{n-1} |b_{\nu\nu} \lambda_{\nu+1}| |\hat{b}_{n, \nu+1} \lambda_{\nu+1}| = O(|b_{nn} \lambda_{n+1}|)$,
 - (vii) $\sum_{n=\nu+1}^\infty (n|b_{nn} \lambda_{n+1}|)^{s-1} |\hat{b}_{n, \nu+1} \lambda_{\nu+1}| = O((\nu |b_{\nu\nu} \lambda_{\nu+1}|)^{s-1})$,
 - (viii) $\sum_{n=1}^\infty n^{s-1} |\sum_{\nu=2}^n \hat{b}_{n\nu} \lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i|^s = O(1)$,
- then $\lambda \in (|A|_k, |B|_s)$.

Proof. If y_n denotes the n th term of the B -transform of a sequence $\{s_n\}$, then

$$\begin{aligned} y_n &= \sum_{i=0}^n b_{ni} s_i = \sum_{i=0}^n b_{ni} \sum_{\nu=0}^i \lambda_\nu a_\nu = \sum_{\nu=0}^n \lambda_\nu a_\nu \sum_{i=\nu}^n b_{ni} = \sum_{\nu=0}^n \bar{b}_{n\nu} \lambda_\nu a_\nu, \\ y_{n-1} &= \sum_{\nu=0}^{n-1} \bar{b}_{n-1, \nu} \lambda_\nu a_\nu, \end{aligned} \tag{5}$$

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^n \hat{b}_{n\nu} \lambda_\nu a_\nu, \tag{6}$$

where $s_n = \sum_{i=0}^n \lambda_i a_i$.

Let x_n denote the n th term of the A -transform of a series $\sum a_n$, then as in (6),

$$X_n := x_n - x_{n-1} = \sum_{\nu=0}^n \hat{a}_{n\nu} a_\nu. \tag{7}$$

Since \hat{A} is a triangle, it has a unique two-sided inverse, which we will denote by A' . Thus we may solve (7) for a_n to obtain

$$a_n = \sum_{\nu=0}^n \hat{a}'_{n\nu} X_\nu. \quad (8)$$

Substituting (8) into (6) yields

$$\begin{aligned} Y_n &= \sum_{\nu=0}^n \hat{b}_{n\nu} \lambda_\nu a_\nu = \sum_{\nu=0}^n \hat{b}_{n\nu} \lambda_\nu \left(\sum_{i=0}^{\nu} \hat{a}'_{\nu i} X_i \right) = \sum_{\nu=0}^n \hat{b}_{n\nu} \lambda_\nu \left(\sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i + \hat{a}'_{\nu, \nu-1} X_{\nu-1} + \hat{a}'_{\nu\nu} X_\nu \right) \\ &= \sum_{\nu=0}^n \hat{b}_{n\nu} \lambda_\nu \hat{a}'_{\nu\nu} X_\nu + \sum_{\nu=1}^n \hat{b}_{n\nu} \lambda_\nu \hat{a}'_{\nu, \nu-1} X_{\nu-1} + \sum_{\nu=2}^n \hat{b}_{n\nu} \lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\ &= \hat{b}_{nn} \lambda_n \hat{a}'_{nn} X_n + \sum_{\nu=0}^{n-1} \hat{b}_{n\nu} \lambda_\nu \hat{a}'_{\nu\nu} X_\nu + \sum_{\nu=0}^{n-1} \hat{b}_{n, \nu+1} \lambda_{\nu+1} \hat{a}'_{\nu+1, \nu} X_\nu + \sum_{\nu=2}^n \hat{b}_{n\nu} \lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\ &= \frac{\hat{b}_{nn}}{a_{nn}} \lambda_n X_n + \sum_{\nu=0}^{n-1} (\hat{b}_{n\nu} \lambda_\nu a'_{\nu\nu} + \hat{b}_{n, \nu+1} \lambda_{\nu+1} \hat{a}'_{\nu+1, \nu}) X_\nu + \sum_{\nu=2}^n \hat{b}_{n\nu} \lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\ &= \frac{\hat{b}_{nn}}{a_{nn}} \lambda_n X_n + \sum_{\nu=0}^{n-1} (\hat{b}_{n\nu} \lambda_\nu a'_{\nu\nu} + \hat{b}_{n, \nu+1} \lambda_{\nu+1} a'_{\nu\nu} - \hat{b}_{n, \nu+1} \lambda_{\nu+1} a'_{\nu\nu} + \hat{b}_{n, \nu+1} \lambda_{\nu+1} \hat{a}'_{\nu+1, \nu}) X_\nu \\ &\quad + \sum_{\nu=2}^n \hat{b}_{n\nu} \lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\ &= \frac{\hat{b}_{nn}}{a_{nn}} \lambda_n X_n + \sum_{\nu=0}^{n-1} \frac{\Delta_\nu (\hat{b}_{n\nu} \lambda_\nu)}{a_{\nu\nu}} X_\nu + \sum_{\nu=0}^{n-1} \hat{b}_{n, \nu+1} \lambda_{\nu+1} (a'_{\nu\nu} + \hat{a}'_{\nu+1, \nu}) X_\nu + \sum_{\nu=2}^n \hat{b}_{n\nu} \lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i. \end{aligned} \quad (9)$$

Using the fact that

$$a'_{\nu\nu} + \hat{a}'_{\nu+1, \nu} = \frac{1}{a_{\nu\nu}} \left(\frac{a_{\nu\nu} - a_{\nu+1, \nu}}{a_{\nu+1, \nu+1}} \right), \quad (10)$$

and substituting (10) into (9), we have the following:

$$\begin{aligned} Y_n &= \frac{\hat{b}_{nn}}{a_{nn}} \lambda_n X_n + \sum_{\nu=0}^{n-1} \frac{\Delta_\nu (\hat{b}_{n\nu} \lambda_\nu)}{a_{\nu\nu}} X_\nu + \sum_{\nu=0}^{n-1} \hat{b}_{n, \nu+1} \lambda_{\nu+1} \left(\frac{a_{\nu\nu} - a_{\nu+1, \nu}}{a_{\nu\nu} a_{\nu+1, \nu+1}} \right) X_\nu + \sum_{\nu=2}^n \hat{b}_{n\nu} \lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\ &= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \text{ say.} \end{aligned} \quad (11)$$

By Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{s-1} |T_{ni}|^s < \infty, \quad i = 1, 2, 3, 4. \tag{12}$$

Using (i),

$$\begin{aligned} J_1 &:= \sum_{n=1}^{\infty} n^{s-1} |T_{n1}|^s = \sum_{n=1}^{\infty} n^{s-1} \left| \frac{b_{nn}\lambda_n}{a_{nn}} X_n \right|^s \\ &= O(1) \sum_{n=1}^{\infty} n^{s-1} (n^{1/s-1/k})^s |X_n|^s \\ &= O(1) \sum_{n=1}^{\infty} n^{k-1} |X_n|^k (n^{s-s/k-k+1} |X_n|^{s-k}). \end{aligned} \tag{13}$$

But $n^{s-s/k-k+1} |X_n|^{s-k} = (n^{1-1/k} |X_n|)^{s-k} = O((n|X_n|)^{s-k}) = O(1)$, from (ii) of Theorem 1. Since $\sum a_n$ is summable $|A|_k, J_1 = O(1)$.

Using (i), (iv), (v), (ii), and Hölder's inequality,

$$\begin{aligned} J_2 &:= \sum_{n=1}^{\infty} n^{s-1} |T_{n2}|^s = \sum_{n=1}^{\infty} n^{s-1} \left| \sum_{\nu=0}^{n-1} \frac{\Delta_{\nu}(\hat{b}_{n\nu}\lambda_{\nu})}{a_{\nu\nu}} X_{\nu} \right|^s \\ &\leq \sum_{n=1}^{\infty} n^{s-1} \left\{ \sum_{\nu=0}^{n-1} \nu^{1/s-1/k} |b_{\nu\nu}\lambda_{\nu}|^{-1} |\Delta_{\nu}(\hat{b}_{n\nu}\lambda_{\nu})| |X_{\nu}| \right\}^s \\ &= O(1) \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{\nu=0}^{n-1} \nu^{1-s/k} |b_{\nu\nu}\lambda_{\nu}|^{-s} |\Delta_{\nu}(\hat{b}_{n\nu}\lambda_{\nu})| |X_{\nu}|^s \right) \times \left(\sum_{\nu=0}^{n-1} |\Delta_{\nu}(\hat{b}_{n\nu}\lambda_{\nu})| \right)^{s-1} \\ &= O(1) \sum_{n=1}^{\infty} (n |b_{nn}\lambda_n|)^{s-1} \sum_{\nu=0}^{n-1} \nu^{1-s/k} |b_{\nu\nu}\lambda_{\nu}|^{-s} |\Delta_{\nu}(\hat{b}_{n\nu}\lambda_{\nu})| |X_{\nu}|^s \\ &= O(1) \sum_{\nu=1}^{\infty} \nu^{1-s/k} |b_{\nu\nu}\lambda_{\nu}|^{-s} |X_{\nu}|^s \sum_{n=\nu+1}^{\infty} (n |b_{nn}\lambda_n|)^{s-1} |\Delta_{\nu}(\hat{b}_{n\nu}\lambda_{\nu})| \\ &= O(1) \sum_{\nu=1}^{\infty} \nu^{1-s/k} |b_{\nu\nu}\lambda_{\nu}|^{-s} |X_{\nu}|^s \nu^{s-1} |b_{\nu\nu}\lambda_{\nu}|^s = O(1) \sum_{\nu=1}^{\infty} \nu^{s-s/k} |X_{\nu}|^s \\ &= O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} |X_{\nu}|^k (\nu^{s-s/k-k+1} |X_{\nu}|^{s-k}) = O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} |X_{\nu}|^k = O(1). \end{aligned} \tag{14}$$

Using (iii), (vi), (vii), (ii), and Hölder's inequality,

$$\begin{aligned}
J_3 &:= \sum_{n=1}^{\infty} n^{s-1} |T_{n3}|^s = \sum_{n=1}^{\infty} n^{s-1} \left| \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1} \lambda_{\nu+1} \left(\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu} a_{\nu+1,\nu+1}} \right) X_{\nu} \right|^s \\
&\leq \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{\nu=0}^{n-1} |\hat{b}_{n,\nu+1} \lambda_{\nu+1}| \left| \frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu} a_{\nu+1,\nu+1}} \right| |X_{\nu}| \right)^s \\
&= O(1) \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{\nu=0}^{n-1} |\hat{b}_{n,\nu+1} \lambda_{\nu+1}| |X_{\nu}| \right)^s \\
&= O(1) \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{\nu=0}^{n-1} \left(\frac{|b_{\nu\nu} \lambda_{\nu+1}|}{|b_{\nu\nu} \lambda_{\nu+1}|} \right) |\hat{b}_{n,\nu+1} \lambda_{\nu+1}| |X_{\nu}| \right)^s \\
&= O(1) \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{\nu=0}^{n-1} |b_{\nu\nu} \lambda_{\nu+1}|^{1-s} |\hat{b}_{n,\nu+1} \lambda_{\nu+1}| |X_{\nu}|^s \right) \times \left(\sum_{\nu=0}^{n-1} |b_{\nu\nu} \lambda_{\nu+1}| |\hat{b}_{n,\nu+1} \lambda_{\nu+1}| \right)^{s-1} \\
&= O(1) \sum_{n=1}^{\infty} (n |b_{nn} \lambda_{n+1}|)^{s-1} \sum_{\nu=0}^{n-1} |b_{\nu\nu} \lambda_{\nu+1}|^{1-s} |\hat{b}_{n,\nu+1} \lambda_{\nu+1}| |X_{\nu}|^s \\
&= O(1) \sum_{\nu=0}^{\infty} |b_{\nu\nu} \lambda_{\nu+1}|^{1-s} |X_{\nu}|^s \sum_{n=\nu+1}^{\infty} (n |b_{nn} \lambda_{n+1}|)^{s-1} |\hat{b}_{n,\nu+1} \lambda_{\nu+1}| \\
&= O(1) \sum_{\nu=0}^{\infty} |b_{\nu\nu} \lambda_{n+1}|^{1-s} |X_{\nu}|^s \nu^{s-1} |b_{\nu\nu} \lambda_{\nu+1}|^{s-1} = O(1) \sum_{\nu=0}^{\infty} \nu^{s-1} |X_{\nu}|^s \\
&= O(1) \sum_{\nu=0}^{\infty} \nu^{k-1} |X_{\nu}|^k (\nu |X_{\nu}|)^{s-k} = O(1) \sum_{\nu=0}^{\infty} \nu^{k-1} |X_{\nu}|^k = O(1).
\end{aligned} \tag{15}$$

From (viii),

$$\sum_{n=1}^{\infty} n^{s-1} |T_{n4}|^s = \sum_{n=1}^{\infty} n^{s-1} \left| \sum_{\nu=2}^n \hat{b}_{n\nu} \lambda_{\nu} \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \right|^s = O(1). \tag{16}$$

□

We now state sufficient conditions, when $k = s$.

COROLLARY 1. *Let $\{\lambda_n\}$ be a sequence of constants, A and B triangles satisfying*

- (i) $|b_{nm}|/|a_{nm}| = O(1/|\lambda_n|)$,
- (ii) $|a_{nm} - a_{n+1,n}| = O(|a_{nm} a_{n+1,n+1}|)$,
- (iii) $\sum_{\nu=0}^{n-1} |\Delta_{\nu}(\hat{b}_{n\nu} \lambda_{\nu})| = O(|b_{nn} \lambda_n|)$,
- (iv) $\sum_{n=\nu+1}^{\infty} (n |b_{nn} \lambda_n|)^{k-1} |\Delta_{\nu}(\hat{b}_{n\nu} \lambda_{\nu})| = O(\nu^{k-1} |b_{\nu\nu} \lambda_{\nu}|^k)$,
- (v) $\sum_{\nu=0}^{n-1} |b_{\nu\nu} \lambda_{\nu+1}| |\hat{b}_{n,\nu+1} \lambda_{\nu+1}| = O(|b_{nn} \lambda_{n+1}|)$,
- (vi) $\sum_{n=\nu+1}^{\infty} (n |b_{nn} \lambda_{n+1}|)^{k-1} |\hat{b}_{n,\nu+1} \lambda_{\nu+1}| = O((\nu |b_{\nu\nu} \lambda_{\nu+1}|)^{k-1})$,
- (vii) $\sum_{n=1}^{\infty} n^{k-1} \left| \sum_{\nu=2}^n \hat{b}_{n\nu} \lambda_{\nu} \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \right|^k = O(1)$,

then $\lambda \in (|A|_k, |B|_k)$.

A weighted mean matrix is a lower triangular matrix with entries p_k/P_n , $0 \leq k \leq n$, where $P_n = \sum_{k=0}^n p_k$.

COROLLARY 2. *Let $1 < k \leq s < \infty$. Let $\{\lambda_n\}$ be a sequence of constants, let B be a triangle such that B , and let $\{p_n\}$ satisfy*

- (i) $b_{\nu\nu}\lambda_\nu = O((p_\nu/P_\nu)\nu^{1/s-1/k})$,
 - (ii) $(n|X_n|)^{s-k} = O(1)$,
 - (iii) $\sum_{\nu=1}^{n-1} |\Delta_\nu(\widehat{b}_{n\nu}\lambda_\nu)| = O(|b_{nn}\lambda_n|)$,
 - (iv) $\sum_{n=\nu+1}^\infty (n|b_{nn}\lambda_n|)^{s-1} |\Delta_\nu(\widehat{b}_{n\nu}\lambda_\nu)| = O(\nu^{s-1}|b_{\nu\nu}\lambda_\nu|^s)$,
 - (v) $\sum_{\nu=1}^{n-1} |b_{\nu\nu}\lambda_\nu| |\widehat{b}_{n,\nu}\lambda_\nu| = O(|b_{nn}\lambda_n|)$,
 - (vi) $\sum_{n=\nu+1}^\infty (n|b_{nn}\lambda_n|)^{s-1} |\widehat{b}_{n,\nu}\lambda_\nu| = O((\nu|b_{\nu\nu}\lambda_\nu|)^{s-1})$,
- then $\lambda \in (|\overline{N}, p_n|_k, |B|_s)$.

Proof. Conditions (i), (ii), (iii)–(vii) of Theorem 1 reduce to conditions (i)–(vi), respectively, of Corollary 1.

With $A = (\overline{N}, p_n)$,

$$a_{nm} - a_{n+1,n} = \frac{p_n}{P_n} - \frac{p_n}{P_{n+1}} = \frac{p_n p_{n+1}}{P_n P_{n+1}} = a_{nn} a_{n+1,n+1}, \tag{17}$$

and condition (ii) of Theorem 1 is automatically satisfied.

A matrix A is said to be factorable if $a_{nk} = b_n c_k$ for each n and k .

Since A is a weighted mean matrix, \widehat{A} is a factorable triangle and it is easy to show that its inverse is bidiagonal. Therefore condition (viii) of Theorem 1 is trivially satisfied. \square

We now turn our attention to obtaining necessary conditions.

THEOREM 2. *Let $1 < k \leq s < \infty$, and let A and B be two lower triangular matrices with A satisfying*

$$\sum_{n=\nu+1}^\infty n^{k-1} |\Delta_\nu \widehat{a}_{n\nu}|^k = O(|a_{\nu\nu}|^k). \tag{18}$$

Then necessary conditions for $\lambda \in (|A|_k, |B|_s)$ are

- (i) $|b_{\nu\nu}\lambda_\nu| = O(|a_{\nu\nu}| \nu^{1/s-1/k})$,
- (ii) $\sum_{n=\nu+1}^\infty n^{s-1} |\Delta_\nu \widehat{b}_{n\nu}\lambda_\nu|^s = O(|a_{\nu\nu}|^s \nu^{s-s/k})$,
- (iii) $\sum_{n=\nu+1}^\infty n^{s-1} |\widehat{b}_{n,\nu+1}\lambda_{\nu+1}|^s = O(\sum_{n=\nu+1}^\infty n^{k-1} |\widehat{a}_{n,\nu+1}|^k)^{s/k}$.

Proof. Define

$$\begin{aligned} A^* &= \{ \{a_i\} : \sum a_i \text{ is summable } |A|_k \}, \\ B^* &= \{ \{b_i\} : \sum b_i \lambda_i \text{ is summable } |B|_s \}. \end{aligned} \tag{19}$$

With Y_n and X_n as defined by (6) and (7), the spaces A^* and B^* are BK-spaces, with norms given by

$$\begin{aligned}\|a\|_1 &= \left\{ |X_0|^k + \sum_{n=1}^{\infty} n^{k-1} |X_n|^k \right\}^{1/k}, \\ \|a\|_2 &= \left\{ |Y_0|^s + \sum_{n=1}^{\infty} n^{s-1} |Y_n|^s \right\}^{1/s},\end{aligned}\tag{20}$$

respectively.

From the hypothesis of the theorem, $\|a\|_1 < \infty$ implies that $\|a\|_2 < \infty$. The inclusion map $i: A^* \rightarrow B^*$ defined by $i(x) = x$ is continuous, since A^* and B^* are BK-spaces. Applying the Banach-Steinhaus theorem, there exists a constant $K > 0$ such that

$$\|a\|_2 \leq K \|a\|_1.\tag{21}$$

Let e_n denote the n th coordinate vector. From (6) and (7), with $\{a_n\}$ defined by $a_n = e_n - e_{n+1}$, $n = \nu$, $a_n = 0$ otherwise, we have

$$\begin{aligned}X_n &= \begin{cases} 0, & n < \nu, \\ \hat{a}_{n\nu}, & n = \nu, \\ \Delta_\nu \hat{a}_{n\nu}, & n > \nu, \end{cases} \\ Y_n &= \begin{cases} 0, & n < \nu, \\ \hat{b}_{n\nu} \lambda_\nu, & n = \nu, \\ \Delta_\nu (\hat{b}_{n\nu} \lambda_\nu), & n > \nu. \end{cases}\end{aligned}\tag{22}$$

From (20),

$$\begin{aligned}\|a\|_1 &= \left\{ \nu^{k-1} |a_{\nu\nu}|^k + \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu \hat{a}_{n\nu}|^k \right\}^{1/k}, \\ \|a\|_2 &= \left\{ \nu^{s-1} |b_{\nu\nu} \lambda_\nu|^s + \sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta_\nu (\hat{b}_{n\nu} \lambda_\nu)|^s \right\}^{1/s},\end{aligned}\tag{23}$$

recalling that $\hat{b}_{\nu\nu} = \bar{b}_{\nu\nu} = b_{\nu\nu}$.

From (21), using (18), we obtain

$$\begin{aligned}& \nu^{s-1} |b_{\nu\nu} \lambda_\nu|^s + \sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta_\nu (\hat{b}_{n\nu} \lambda_\nu)|^s \\ & \leq K^s \left(\nu^{k-1} |a_{\nu\nu}|^k + \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu \hat{a}_{n\nu}|^k \right)^{s/k} \leq K^s \left(\nu^{k-1} |a_{\nu\nu}|^k + O(1) |a_{\nu\nu}|^k \right)^{s/k} \\ & = O(|a_{\nu\nu}|^k (\nu^{k-1} + 1))^{s/k} = O(\nu^{k-1} |a_{\nu\nu}|^k)^{s/k}.\end{aligned}\tag{24}$$

The above inequality will be true if and only if each term on the left-hand side is $O(\nu^{k-1} |a_{\nu\nu}|^k)^{s/k}$. Using the first term,

$$\nu^{s-1} |b_{\nu\nu}\lambda_\nu|^s = O(\nu^{k-1} |a_{\nu\nu}|^k)^{s/k}, \tag{25}$$

which implies that $|b_{\nu\nu}\lambda_\nu| = O(|a_{\nu\nu}| \nu^{1/s-1/k})$, and (i) is necessary.

Using the second term, we obtain

$$\sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta_\nu(\widehat{b}_{n\nu}\lambda_\nu)|^s = O(\nu^{k-1} |a_{\nu\nu}|^k)^{s/k} = O(\nu^{s-s/k} |a_{\nu\nu}|^s), \tag{26}$$

which is condition (ii).

If we now define $a_n = e_{n+1}$ for $n = \nu$, $a_n = 0$ otherwise, then from (6) and (7), we obtain

$$X_n = \begin{cases} 0, & n \leq \nu, \\ \widehat{a}_{n,\nu+1}, & n > \nu, \end{cases} \tag{27}$$

$$Y_n = \begin{cases} 0, & n \leq \nu, \\ \widehat{b}_{n,\nu+1}\lambda_{\nu+1}, & n > \nu. \end{cases}$$

The corresponding norms are

$$\|a\|_1 = \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} |\widehat{a}_{n,\nu+1}|^k \right\}^{1/k}, \tag{28}$$

$$\|a\|_2 = \left\{ \sum_{n=\nu+1}^{\infty} n^{s-1} |\widehat{b}_{n,\nu+1}\lambda_{\nu+1}|^s \right\}^{1/s}.$$

Applying (21),

$$\sum_{n=\nu+1}^{\infty} n^{s-1} |\widehat{b}_{n,\nu+1}\lambda_{\nu+1}|^s \leq K^s \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} |\widehat{a}_{n,\nu+1}|^k \right\}^{s/k}, \tag{29}$$

which implies condition (iii). □

COROLLARY 3. *Let A and B be two lower triangular matrices with A satisfying*

$$\sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu \widehat{a}_{n\nu}|^k = O(|a_{\nu\nu}|^k). \tag{30}$$

Then necessary conditions for $\lambda \in (|A|_k, |B|_k)$ are

- (i) $|b_{\nu\nu}\lambda_\nu| = O(|a_{\nu\nu}|)$,
- (ii) $(\sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu \widehat{b}_{n\nu}\lambda_\nu|^k)^{1/k} = O(|a_{\nu\nu}| \nu^{1-1/k})$,
- (iii) $\sum_{n=\nu+1}^{\infty} n^{k-1} |\widehat{b}_{n,\nu+1}\lambda_{\nu+1}|^k = O(\sum_{n=\nu+1}^{\infty} n^{k-1} |\widehat{a}_{n,\nu+1}|^k)$.

COROLLARY 4. Let $1 < k \leq s < \infty$. Let B be a lower triangular matrix, $\{p_n\}$ is a sequence satisfying

$$\sum_{n=\nu+1}^{\infty} n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k = O\left(\frac{1}{P_\nu^k} \right). \quad (31)$$

Then necessary conditions for $\lambda \in (|\overline{N}, p_n|_k, |B|_s)$ are

- (i) $P_\nu |b_{\nu\nu} \lambda_\nu| / p_\nu = O(\nu^{1/s-1/k})$,
- (ii) $\sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta_\nu(\widehat{b}_{n\nu} \lambda_\nu)|^s = O(\nu^{s-s/k} (p_\nu/P_\nu)^s)$,
- (iii) $\sum_{n=\nu+1}^{\infty} n^{s-1} |\widehat{b}_{n,\nu+1} \lambda_{\nu+1}|^s = O(1)$.

Proof. With $A = (\overline{N}, p_n)$, (18) becomes (31), and conditions (i)–(iii) of Theorem 2 become conditions (i)–(iii) of Corollary 4, respectively. \square

Every summability factor theorem becomes an inclusion theorem by setting each $\lambda_n = 1$.

COROLLARY 5 (see [3]). Let $1 < k \leq s < \infty$. Let A and B be triangles satisfying

- (i) $|b_{nn}|/|a_{nn}| = O(\nu^{1/s-1/k})$,
- (ii) $(n|X_n|)^{s-k} = O(1)$,
- (iii) $|a_{nn} - a_{n+1,n}| = O(|a_{nn} a_{n+1,n+1}|)$,
- (iv) $\sum_{\nu=0}^{n-1} |\Delta_\nu(\widehat{b}_{n\nu})| = O(|b_{nn}|)$,
- (v) $\sum_{n=\nu+1}^{\infty} (n|b_{nn}|)^{s-1} |\Delta_\nu(\widehat{b}_{n\nu})| = O(\nu^{s-1} |b_{\nu\nu}|^s)$,
- (vi) $\sum_{\nu=0}^{n-1} |b_{\nu\nu}| |\widehat{b}_{n,\nu+1}| = O(|b_{nn}|)$,
- (vii) $\sum_{n=\nu+1}^{\infty} (n|b_{nn}|)^{s-1} |\widehat{b}_{n,\nu+1}| = O((\nu|b_{\nu\nu}|)^{s-1})$,
- (viii) $\sum_{n=1}^{\infty} n^{s-1} |\sum_{\nu=2}^n \widehat{b}_{n\nu} \sum_{i=0}^{\nu-2} \widehat{a}'_{\nu i} X_i|^s = O(1)$.

Then if $\sum a_n$ is summable $|A|_k$, it is summable $|B|_s$.

COROLLARY 6 (see [4]). Let be $\{p_n\}$ a sequence of positive constants, B is a triangle satisfying

- (i) $P_n |b_{nn}| = O((p_n) \nu^{1/s-1/k})$,
- (ii) $(n|X_n|)^{s-k} = O(1)$,
- (iii) $\sum_{\nu=0}^{n-1} |\Delta_\nu(\widehat{b}_{n\nu})| = O(|b_{nn}|)$,
- (iv) $\sum_{n=\nu+1}^{\infty} (n|b_{nn}|)^{s-1} |\Delta_\nu(\widehat{b}_{n\nu})| = O(\nu^{s-1} |b_{\nu\nu}|^s)$,
- (v) $\sum_{\nu=0}^{n-1} |b_{\nu\nu} \widehat{b}_{n,\nu+1}| = O(|b_{nn}|)$,
- (vi) $\sum_{n=\nu+1}^{\infty} (n|b_{nn}|)^{s-1} |\widehat{b}_{n,\nu+1}| = O((\nu|b_{\nu\nu}|)^{s-1})$.

Then if $\sum a_n$ is summable $|\overline{N}, p_n|_k$, it is summable $|B|_s$.

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