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Li-Yau type estimation of a semilinear parabolic system along geometric flow

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Abstract

This article provides a Li–Yau-type gradient estimate for a semilinear weighted parabolic system of semilinear equations along an abstract geometric flow on a smooth measure space. A Harnack-type inequality on the system is also derived at the end.

Keywords: Gradient estimate; Weighted Laplacian; Parabolic system; Geometric flow; Li-Yau type estimation

1 Introduction

In the field of modern geometric analysis, a challenging problem is to determine the intrinsic qualities of a heat type equation on an evolving manifold. Gradient estimation is a standard technique to understand the local and global behavior of positive solutions to the heat type equation. Heat type equations are very much well known in mathematics and physics. This type of study becomes more interesting when different curvature restrictions were introduced. This estimation was popularized after the work of Li and Yau [14], where they studied the equation

$$(\Delta - q(x) - \partial_t)u(x,t) = 0,$$

and stated a bound for the quantity $\frac{\|\nabla u\|}{u}$, where Δ is the Laplace–Beltrami operator and ∇ is the gradient operator. Today this estimation is known as Li–Yau-type estimation. Next, P. Souplet and Q. S. Zhang [21] established an elliptic type gradient estimate for bounded solutions of the heat equation for complete noncompact manifold, by adding a logarithmic correction term. This is called the Souplet–Zhang-type gradient estimate. In [8, 9], Hamilton developed a Harnack estimate on Riemannian manifold with weakly positive Ricci tensor, which was used in solving the Poincaré conjecture. In recent days, Hui et al. [12] studied Hamilton–Souplet–Zhang-type estimation along general geometric flow. In [10] Hui et al. studied weighted elliptic equations on weighted Riemannian manifold not evolving along any geometric flow. Next, for system of equations on Riemannian manifolds, we can start with Shen and Ding's [20] study on the system

$$\begin{cases} u_t = \Delta u^m + k_1(t)f_1(v), \\ v_t = \Delta v^n + k_2(t)f_2(u), \end{cases}$$

with nonlinear boundary conditions. They proved that the above system blows up in finite time using differential Sobolev inequality. Wu and Yang [24] established the global existence and finite time blow up of the solution of the semilinear system

$$\begin{cases} u_t = \Delta u + e^{\alpha t} v^p, \\ v_t = \Delta v + e^{\beta t} u^q. \end{cases} \tag{1}$$

There are numerous applications of the relevant equations and inequalities. One can see [7, 13, 16–18, 27] and the references therein for applications.

Motivated by the works of Wu [23], we consider a closed n-dimensional weighted Riemannian manifold with Riemannian metric g denoted by $(M^n, g, e^{-\phi}d\mu)$, also known as smooth measure space, where $e^{-\phi}d\mu$ is the weighted volume form and ϕ is a twice differentiable function on M. Let the Riemannian metric $g(t)$ be evolving along the geometric flow

$$\frac{\partial}{\partial t} g_{ij} = 2S_{ij}, \tag{2}$$

where $\mathcal{S}(e_i, e_j) := S_{ij}(t)$ is a smooth symmetric 2-tensor on $(M, g(t))$. We denote $S = tr(S_{ij}) = g^{ij}S_{ij}$. Some important of geometric flows are the Ricci flow [8] when $S_{ij} = -Ric_{ij}$, where Ric is the Ricci tensor, Yamabe flow [6] when $S_{ij} = -\frac{1}{2}Rg_{ij}$, where R is the scalar curvature, Ricci–Bourguignon flow [5] when $S_{ij} = -Ric_{ij} + \rho R g_{ij}$, where ρ is constant. For any twice differentiable function ϕ on M and any smooth function f, the weighted Laplacian operator is defined by

$$\Delta_\phi f = \Delta f - \nabla\phi\nabla f,$$

where Δ is the Laplace–Beltrami operator.

Differential Harnack estimations on system (1) have been studied by Wu [23] on hyperbolic spaces. We have already studied the Hamilton and Souplet–Zhang-type estimation for positive solution [11] for positive solutions of the following system of weighted semilinear heat type equations

$$\begin{cases} \Delta_\phi f - f_t = -e^{\lambda_1 t} h^p, \\ \Delta_\phi h - h_t = -e^{\lambda_2 t} f^q, \end{cases} \tag{3}$$

where $p, q, \lambda_1, \lambda_2$ are positive constants and f, h are smooth functions on M. In this article, we consider the system (3) along the geometric flow (2), and we confined ourselves to Li–Yau-type gradient estimate of (3) along (2). Our results are the generalization of the results Wu [23].

2 Preliminaries

This section contains some basic results and evolution formulas related to the gradient estimation.

Lemma 1 [2] *The weighted Bochner formula for any smooth function u is given by*

$$\frac{1}{2}\Delta_\phi\|\nabla u\|^2 = \|Hess\ u\|^2 + \langle\nabla\Delta_\phi u, \nabla u\rangle + Ric_\phi(\nabla u, \nabla u),$$

where $Ric_\phi := Ric + Hess\ \phi$, *is called the Bakry–Émery–Ricci tensor and Hess is the Hessian operator. For* $m > n$, *the* $(m-n)$*-Bakry–Émery–Ricci tensor* [3] *is given by*

$$Ric_\phi^{m-n} := Ric + Hess\ \phi - \frac{\nabla\phi\otimes\nabla\phi}{m-n}.$$

Lemma 2 [2] *If a Riemannian manifold M evolves by the geometric flow* (2), *then for any smooth function u, the expression* $\|\nabla u\|^2$ *evolves by*

$$\frac{\partial}{\partial t}\|\nabla u\|^2 = -2\mathcal{S}(\nabla u, \nabla u) + 2\langle\nabla u, \nabla u_t\rangle,$$

and the expression $\Delta_\phi u$ *evolves by*

$$\frac{\partial}{\partial t}(\Delta_\phi u) = \Delta_\phi u_t - 2S^{ij}\nabla_i\nabla_j u - \langle 2div\ S_{ij} - \nabla S, \nabla u\rangle + 2\mathcal{S}(\nabla\phi, \nabla u) - \langle\nabla u, \nabla\phi_t\rangle,$$

where div S_{ij} *denotes the divergence of* S_{ij}.

Lemma 3 [2] *For any smooth function f and* $m > n$, *we have*

$$\|Hess\ f\|^2 \geq \frac{(\Delta_\phi f)^2}{m} - \frac{1}{m-n}\langle\nabla f, \nabla\phi\rangle^2. \tag{4}$$

Lemma 4 (Young's inequality) [26] *If a, b are nonnegative real numbers and* $p > 1$, $q > 1$ *are real numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{5}$$

For any $\alpha > 0$, we see that

$$ab \leq \frac{\alpha^p a^p}{p} + \frac{b^q}{\alpha^q q}. \tag{6}$$

The above inequality is a generalized version of Young's inequality. For convenience, we categorize both (5) and (6) as Young's inequality.

Let $T > 0$ be any real number. For any two points $x, y \in M$ and for any $t \in [0, T]$, the quantity $d(x, y, t)$ denotes the geodesic distance between x and y under the metric $g(t)$. For any fixed $x_0 \in M$ and $R > 0$, we define a compact set $Q_{2R,T} = \{(x, t) : d(x, x_0, t) \leq 2R, 0 \leq t \leq T\} \subset M^n \times (-\infty, +\infty)$.

Let $\psi : [0,\infty) \to [0,1]$ be a C^2-cut off function given by

$$\psi(s) = \begin{cases} 1, & s \in [0,1], \\ 0, & s \in [2,\infty), \end{cases} \tag{7}$$

satisfying $\psi(s) \in [0,1]$, $-c_0 \le \psi'(s) \le 0$, $\psi''(s) \ge -c_1$ and $\frac{\|\psi''(s)\|^2}{\psi(s)} \le c_1$, where c_1 is a constant. For $R > 1$, we define

$$\eta(x,t) = \psi\left(\frac{r(x,t)}{R}\right), \tag{8}$$

where $r(x,t) = d(x,x_0,t)$. Since ψ is Lipschitz, so by Calabi's argument [4], we can assume that ψ is everywhere smooth and hence we can use maximum principle to find our estimation. Using generalized Laplacian comparison theorem [15, 19, 25], we get

(i) $\Delta_\phi r(x,t) \le (m-1)\sqrt{k_1}\coth(\sqrt{k_1}r(x,t))$,
(ii) $\Delta_\phi \eta \ge -\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) - \frac{c_1}{R^2}$,
(iii) $\frac{\|\nabla\eta\|^2}{\nabla\eta} \le \frac{c_1}{R^2}$.

3 Li–Yau-type gradient estimation

In this section, we provide a detailed derivation of the Li–Yau-type estimation of the system (3) along the flow (2). At the end, a Harnack-type inequality is also derived.

Fix x_0 in M and let $T > 0$ be any real number. Throughout the paper, we consider $(f,h) = (e^u, e^v)$ as a positive solution to the system (3) with the restrictions

$$\tilde{\kappa}_1 \le u \le \kappa_1,$$
$$\tilde{\kappa}_2 \le v \le \kappa_2,$$

for some positive constants κ_1, κ_2, $\tilde{\kappa}_1$, $\tilde{\kappa}_2$. We define some nonnegative constants

$$\sup_{Q_{2R,T}} \|\nabla\phi\| = m_1, \qquad \sup_{Q_{2R,T}} \|\nabla\phi_t\| = \gamma_1,$$
$$\sup_{M\times[0,T]} \|\nabla\phi\| = M_1, \qquad \sup_{M\times[0,T]} \|\nabla\phi_t\| = \Gamma_1,$$

Putting $f = e^u$, $h = e^v$ in (3), we have

$$\begin{cases} \Delta_\phi u - u_t = -\|\nabla u\|^2 - e^{\lambda_1 t + vp - u} \\ \Delta_\phi v - v_t = -\|\nabla v\|^2 - e^{\lambda_2 t + uq - v}. \end{cases} \tag{9}$$

Let $\bar{u} = -e^{\lambda_1 t + vp - u}$ and $\bar{v} = -e^{\lambda_2 t + uq - v}$, hence system (9) reduces to

$$\begin{cases} (\Delta_\phi - \partial_t)u = -\|\nabla u\|^2 + \bar{u} \\ (\Delta_\phi - \partial_t)v = -\|\nabla v\|^2 + \bar{v}. \end{cases} \tag{10}$$

Lemma 5 *Let (u,v) be a solution to the equation (10). If there exist positive constants k_1, k_2, k_3, k_4 such that*

$$Ric_\phi^{m-n} \ge -(m-1)k_1 g, \; -k_2 g \le \mathcal{S} \le k_3 g, \; \|\nabla\mathcal{S}\| \le k_4$$

on $Q_{2R,T}$, then for any $\epsilon \in (0, \frac{1}{\lambda})$, the function $F_1 := t(\|\nabla u\|^2 - \lambda(u_t + \bar{u}))$ satisfies

$$\begin{aligned}(\Delta_\phi - \partial_t)F_1 \geq{}& 2t(1-\lambda\epsilon)\frac{(\Delta_\phi u)^2}{m} - \frac{\lambda t k_2}{2\epsilon}\|\nabla u\|^2 - 2\lambda t k_2\epsilon\|\nabla\phi\|^2 - 2\nabla F_1\nabla u - \frac{F_1}{t} \\ &- 2t(1-\lambda\epsilon)(m-1)k_1\|\nabla u\|^2 - 2t(\lambda-1)k_3\|\nabla u\|^2 - \frac{n\lambda t}{2\epsilon}(k_2+k_3)^2 \\ &- 3\lambda t\sqrt{n}k_4\|\nabla u\|^2 + \mathcal{H}\end{aligned} \tag{11}$$

and the function $F_2 := t(\|\nabla v\|^2 - \lambda(v_t + \bar{v}))$ satisfies

$$\begin{aligned}(\Delta_\phi - \partial_t)F_2 \geq{}& 2t(1-\lambda\epsilon)\frac{(\Delta_\phi v)^2}{m} - \frac{\lambda t k_2}{2\epsilon}\|\nabla v\|^2 - 2\lambda t k_2\epsilon\|\nabla\phi\|^2 - 2\nabla F_2\nabla v - \frac{F_2}{t} \\ &- 2t(1-\lambda\epsilon)(m-1)k_1\|\nabla v\|^2 - 2t(\lambda-1)k_3\|\nabla v\|^2 - \frac{n\lambda t}{2\epsilon}(k_2+k_3)^2 \\ &- 3\lambda t\sqrt{n}k_4\|\nabla v\|^2 + \mathcal{K}\end{aligned} \tag{12}$$

where $\mathcal{H} = -2t(\lambda-1)\nabla\bar{u}\nabla u - \lambda t\nabla u\nabla\phi_t - \lambda t\Delta_\phi\bar{u}$ and $\mathcal{K} = -2t(\lambda-1)\nabla\bar{v}\nabla v - \lambda t\nabla v\nabla\phi_t - \lambda t\Delta_\phi\bar{v}$.

Proof Given that $F_1 = t(\|\nabla u\|^2 - \lambda(u_t + \bar{u}))$. Using Lemma 1 we have

$$\Delta_\phi F_1 = 2t\|\text{Hess } u\|^2 + 2t\langle\nabla\Delta_\phi u, \nabla u\rangle + 2tRic_\phi(\nabla u, \nabla u) - \lambda t\Delta_\phi u_t - \lambda t\Delta_\phi\bar{u}. \tag{13}$$

From $\frac{F_1}{t} = \|\nabla u\|^2 - \lambda(u_t + \bar{u})$, we get

$$\Delta_\phi u = -\frac{F_1}{t} - (\lambda-1)(u_t + \bar{u}), \tag{14}$$

$$\nabla\Delta_\phi u = -\frac{\nabla F_1}{t} - (\lambda-1)(\nabla u_t + \nabla\bar{u}). \tag{15}$$

Using (14) and (15) in (13) we deduce

$$\begin{aligned}\Delta_\phi F_1 ={}& 2t\|\text{Hess } u\|^2 - 2\nabla F_1\nabla u - 2t(\lambda-1)(\nabla u_t + \nabla\bar{u})\nabla u + 2tRic_\phi(\nabla u, \nabla u) \\ &-\lambda t\Delta_\phi u_t - \lambda t\Delta_\phi\bar{u}.\end{aligned} \tag{16}$$

From (14), we get $\partial_t(\Delta_\phi u) = \dfrac{F_1}{t^2} - \dfrac{\partial_t F_1}{t} - (\lambda-1)(u_{tt} + \bar{u}_t)$. Thus (16) reduces to

$$\begin{aligned}\Delta_\phi F_1 ={}& 2t\|\text{Hess } u\|^2 - 2\nabla F_1\nabla u - 2t(\lambda-1)(\nabla u_t + \nabla\bar{u})\nabla u + 2tRic_\phi(\nabla u, \nabla u) \\ &-\frac{\lambda F_1}{t} + \lambda\partial_t F_1 + \lambda(\lambda-1)t(u_{tt} + \bar{u}_t) - \lambda t\langle 2\text{div } S_{ij} - \nabla S, \nabla u\rangle \\ &-2\lambda t\langle\mathcal{S}, \text{Hess } u\rangle + 2\lambda t\mathcal{S}(\nabla\phi, \nabla u) - \lambda t\langle\nabla u, \nabla\phi_t\rangle - \lambda t\Delta_\phi\bar{u}.\end{aligned} \tag{17}$$

By Lemma 2, the evolution of F_1 is given by

$$\begin{aligned}\partial_t F_1 &= \frac{F_1}{t} + t(\partial_t\|\nabla u\|^2 - \lambda(u_{tt} + \bar{u}_t)) \\ &= \frac{F_1}{t} + t(-2\mathcal{S}(\nabla u, \nabla u) + 2\langle\nabla u, \nabla u_t\rangle) - \lambda t(u_{tt} + \bar{u}_t).\end{aligned} \tag{18}$$

Combining (17) and (18), we have

$$(\Delta_\phi - \partial_t)F_1 = 2t\|\text{Hess } u\|^2 + 2tRic_\phi(\nabla u, \nabla u) - 2\nabla F_1 \nabla u - \frac{F_1}{t} + 2\lambda t\mathcal{S}(\nabla\phi, \nabla u)$$
$$-2t(\lambda-1)\mathcal{S}(\nabla u, \nabla u) - \lambda t\langle 2\text{div } S_{ij} - \nabla S, \nabla u\rangle - 2\lambda t\langle \mathcal{S}, \text{Hess } u\rangle$$
$$-2\lambda t\langle \text{div } S_{ij} - \frac{1}{2}\nabla S, \nabla u\rangle + \mathcal{H}, \tag{19}$$

where $\mathcal{H} = -2t(\lambda-1)\nabla\bar{u}\nabla u - \lambda t\nabla u\nabla\phi_t - \lambda t\Delta_\phi\bar{u}$.

Since $-(k_2+k_3)g_{ij} \le S_{ij} \le (k_2+k_3)g_{ij}$ implies $\|S\|^2 \le n(k_2+k_3)^2$, hence for any $\epsilon \in (0, \frac{1}{\lambda})$ using Young's inequality we get

$$\langle \mathcal{S}, \text{Hess } u\rangle \le \epsilon\|\text{Hess } u\|^2 + \frac{n}{4\epsilon}(k_2+k_3)^2, \tag{20}$$

$$2\lambda t\mathcal{S}(\nabla\phi, \nabla u) \ge -\frac{\lambda t k_2}{2\epsilon}\|\nabla u\|^2 - 2\lambda t k_2\epsilon\|\nabla\phi\|^2. \tag{21}$$

In similar way we find

$$\|\text{div}S_{ij} - \frac{1}{2}\nabla S\| \le \frac{3}{2}\sqrt{n}k_4, \tag{22}$$

and using Lemma 3 we get

$$\|\text{Hess } u\|^2 \ge \frac{(\Delta_\phi u)^2}{m} - \frac{1}{m-n}\langle\nabla u, \nabla\phi\rangle^2. \tag{23}$$

Using (20) to (23) in (19) we have (11).

Due to the symmetry in the system of equations (10), replacing F_1 by F_2, u with v and $\mathcal{H}$ by $\mathcal{K}$ in (11) we obtain (12). □

Theorem 1 *If k_1, k_2, k_3, k_4 are positive constants such that*

$$Ric_\phi^{m-n} \ge -(m-1)k_1 g,\ -k_2 g \le \mathcal{S} \le k_3 g,\ \|\nabla\mathcal{S}\| \le k_4$$

on $Q_{2R,T}$ and (f,h) is a positive solution to the system (3) along the flow (2), then for any for any $\lambda > 1$ and $\epsilon \in (0, \frac{1}{\lambda})$ we have

$$\frac{\|\nabla f\|^2}{f^2} - \lambda\left(\frac{f_t}{f} - e^{\lambda_1 t}h^p\right) \le \frac{4m\lambda^2}{3t(1-\lambda\epsilon)} + \frac{2m\lambda^2}{3(1-\lambda\epsilon)}(2\Omega + (\bar{u}^* + \bar{v}^*)) + D_1, \tag{24}$$

$$\frac{\|\nabla h\|^2}{h^2} - \lambda\left(\frac{h_t}{h} - e^{\lambda_2 t}f^q\right) \le \frac{4m\lambda^2}{3t(1-\lambda\epsilon)} + \frac{2m\lambda^2}{3(1-\lambda\epsilon)}\left(\frac{11}{4}\Omega + (\bar{u}^* + \frac{7}{4}\bar{v}^*)\right) + \tilde{D}_1, \tag{25}$$

where

$$D_1 = \frac{4}{3}\sqrt{\frac{m\lambda^2}{2(1-\lambda\epsilon)}}(E_2 + \tilde{E}_2) + \frac{m\lambda^2 p\bar{u}_*}{2(1-\lambda\epsilon)},$$

$$\tilde{D}_1 = \sqrt{\frac{m\lambda^2}{2(1-\lambda\epsilon)}}\left(\frac{4}{3}\sqrt{E_2} + \frac{7}{3}\sqrt{\tilde{E}_2}\right) + \frac{m\lambda^2 p}{2(1-\lambda\epsilon)}\left(\bar{u}_* + \frac{1}{4}\bar{v}_*\right),$$

$$\begin{aligned}
E_2 &= 2\lambda k_2\epsilon m_1^2 + \frac{n\lambda}{2\epsilon}(k_2+k_3)^2 + \frac{1}{4}\lambda^2\gamma_1^2 + \frac{m\lambda^2 E_0^2}{8(1-\lambda\epsilon)(\lambda-1)^2},\\
\tilde{E}_2 &= 2\lambda k_2\epsilon m_1^2 + \frac{n\lambda}{2\epsilon}(k_2+k_3)^2 + \frac{1}{4}\lambda^2\gamma_1^2 + \frac{m\lambda^2 \tilde{E}_0^2}{8(1-\lambda\epsilon)(\lambda-1)^2},\\
E_0 &= \frac{\lambda\eta k_2}{2\epsilon} + 2\eta(1-\lambda\epsilon)(m-1)k_1 + 2\eta k_3(\lambda-1) + 3\eta\lambda\sqrt{n}k_4\\
&\quad +(3\lambda-2)\bar{u}^*\eta + p\eta\bar{u}^*\lambda + \bar{u}^*(n-1)+1,\\
\tilde{E}_0 &= \frac{\lambda\eta k_2}{2\epsilon} + 2\eta(1-\lambda\epsilon)(m-1)k_1 + 2\eta k_3(\lambda-1) + 3\eta\lambda\sqrt{n}k_4\\
&\quad +(3\lambda-2)\bar{v}^*\eta + p\eta\bar{v}^*\lambda + \bar{v}^*(n-1)+1,\\
\bar{u}^* &= -e^{\lambda_1 t + p\kappa_2 - \tilde{\kappa}_1} = -\bar{u}_*,\\
\bar{v}^* &= -e^{\lambda_2 t + q\kappa_1 - \tilde{\kappa}_2} = -\bar{v}_*.
\end{aligned}$$

Proof Let $G_1 = \eta F_1$ and $G_2 = \eta F_2$, where η is defined in (8). Fix $T_1 \in (0,T]$ and assume G_1, G_2 achieve maximum at $(x_0,t_0) \in Q_{2R,T_1}$. If $G_1 \le 0$, $G_2 \le 0$, then the proof is trivial. So assume that $G_1(x_0,t_0) \ge 0$, $G_2(x_0,t_0) \ge 0$. Thus at (x_0,t_0) we have

$$\nabla G_1 = 0,\ \Delta G_1 \le 0,\ \partial_t G_1 \ge 0, \tag{26}$$

$$\nabla G_2 = 0,\ \Delta G_2 \le 0,\ \partial_t G_2 \ge 0. \tag{27}$$

Therefore,

$$\nabla F_1 = -\frac{F_1}{\eta}\nabla\eta, \tag{28}$$

$$\nabla F_2 = -\frac{F_2}{\eta}\nabla\eta, \tag{29}$$

and

$$0 \ge (\Delta_\phi - \partial_t)G_1 = F_1(\Delta_\phi - \partial_t)\eta + \eta(\Delta_\phi - \partial_t)F_1 + 2\langle\nabla\eta,\nabla F_1\rangle, \tag{30}$$

$$0 \ge (\Delta_\phi - \partial_t)G_2 = F_2(\Delta_\phi - \partial_t)\eta + \eta(\Delta_\phi - \partial_t)F_2 + 2\langle\nabla\eta,\nabla F_2\rangle. \tag{31}$$

By [22], there is a constant c_2 such that

$$-F_1\eta_t \ge -c_2k_2F_1, \tag{32}$$

$$-F_2\eta_t \ge -c_2k_2F_2. \tag{33}$$

Using (28), (32) and generalized Laplacian comparison theorem in (30) we get

$$0 \ge -\left(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) + \frac{3c_1}{R^2} + c_2k_2\right)F_1 + \eta(\Delta_\phi - \partial_t)F_1. \tag{34}$$

Similarly, using (29), (33) and generalized Laplacian comparison theorem in (31) we have

$$0 \ge -\left(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) + \frac{3c_1}{R^2} + c_2k_2\right)F_2 + \eta(\Delta_\phi - \partial_t)F_2. \tag{35}$$

Following the same techniques as in [2], we set

$$\xi_1 = \frac{\|\nabla u\|^2}{F_1}\Big\|_{(x_0,t_0)} \geq 0, \tag{36}$$

$$\xi_2 = \frac{\|\nabla v\|^2}{F_2}\Big\|_{(x_0,t_0)} \geq 0. \tag{37}$$

We now consider (34) and (36). Then, at (x_0, t_0), we have

$$\|\nabla u\| = \sqrt{\xi_1 F_1}, \tag{38}$$

$$(\xi_1 - \frac{t_0\xi_1 - 1}{\lambda t_0})F_1 = \|\nabla u\|^2 - (u_t + \bar{u}), \tag{39}$$

$$\eta\langle \nabla u, \nabla F_1\rangle \leq \frac{\sqrt{c_1}}{R}\eta^{\frac{1}{2}}F_1\|\nabla u\|, \tag{40}$$

$$3\lambda\sqrt{n}k_4\|\nabla u\| \leq 2k_4\|\nabla u\|^2 + \frac{9}{8}n\lambda^2 k_4. \tag{41}$$

Using (11) of Lemma 5 we get

$$\begin{aligned} 0 \geq{}& -\Omega F - 1 + 2\eta t_0(1-\lambda\epsilon)\frac{1}{m}\left(\xi_1 - \frac{t_0\xi_1 - 1}{t_0\lambda}\right)^2 F_1^2 - \frac{\lambda\eta t_0 k_2}{2\epsilon}\|\nabla u\|^2 \\ &-2\eta\lambda t_0 k_2\epsilon\|\nabla\phi\|^2 - 2\eta t_0(1-\lambda\epsilon)(m-1)k_1\|\nabla u\|^2 - 2\eta\nabla F_1\nabla u - \frac{\eta F_1}{t_0} \\ &-2\eta t_0(\lambda-1)k_3\|\nabla u\|^2 - \frac{n\lambda t_0}{2\epsilon}\eta(k_2+k_3)^2 - 3\eta\lambda t_0\sqrt{n}k_4\|\nabla u\|^2 + \eta\mathcal{H}(t_0), \end{aligned} \tag{42}$$

where $\Omega = \frac{c_0}{R}(m-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2k_2$, $\mathcal{H}(t_0) = -2t_0(\lambda-1)\nabla\bar{u}\nabla u - \lambda t_0\nabla u\nabla\phi_t - \lambda t_0\Delta_\phi\bar{u}$.

Multiplying (42) with ηt_0 we get

$$\begin{aligned} 0 \geq{}& -\Omega t_0 G_1 + 2t_0^2\frac{1-\lambda\epsilon}{m}\left(\xi_1 - \frac{t_0\xi_1 - 1}{\lambda t_0}\right)^2 G_1^2 - \frac{\lambda\eta t_0^2 k_2}{2\epsilon}\xi_1 G_1 - 2\lambda t_0^2\eta^2 k_2\epsilon m_1^2 \\ &-2\eta t_0^2(1-\lambda\epsilon)(m-1)k_1\xi_1 G_1 - 2\eta^2 t_0\nabla F_1\nabla u - \eta G_1 - 2\eta t_0^2 k_3(\lambda-1)\xi_1 G_1 \\ &-\frac{n\lambda t_0^2}{2\epsilon}\eta^2(k_2+k_3)^2 - 3\eta t_0^2\lambda\sqrt{n}k_4\xi_1 G_1 + \eta^2 t_0\mathcal{H}(t_0). \end{aligned} \tag{43}$$

We can find

$$\begin{aligned} 2\eta^2 t_0\langle\nabla F_1, \nabla u\rangle &\leq 2\eta t_0\frac{\sqrt{c_1}}{R}\eta^{\frac{1}{2}}F_1\|\nabla u\| \\ &= \frac{2t_0\sqrt{c_1}}{R}G_1^{\frac{3}{2}}\xi_1^{\frac{1}{2}}. \end{aligned} \tag{44}$$

We now find a bound for $\eta^2 t_0\mathcal{H}$. Given that $\bar{u} = -e^{\lambda_1 t + pv - u}$. Thus

$$\nabla\bar{u} = -e^{\lambda_1 t + pv - u}(p\nabla v - \nabla u),$$

$$\Delta_\phi\bar{u} = -\bar{u}(p^2\|\nabla v\|^2 + \|\nabla u\|^2 - 2p\langle\nabla v, \nabla u\rangle + p\Delta_\phi v - \Delta_\phi u).$$

Hence

$$-\lambda\eta^2 t_0^2\Delta_\phi\bar{u} = \bar{u}\lambda t_0^2\left\{p^2\eta\xi_2 G_2 + \eta\xi_1 G_1 - p\eta\xi_1^{\frac{1}{2}}\xi_2^{\frac{1}{2}}G_1^{\frac{3}{2}}G_2^{\frac{3}{2}} + p\eta^2\Delta_\phi v - \eta^2\Delta_\phi u\right\}, \tag{45}$$

$$2(\lambda-1)t_0^2\eta^2 - \bar{u}(p\langle\nabla v,\nabla u\rangle - \|\nabla u\|^2) \geq -2(\lambda-1)t_0^2\bar{u}\left\{p\eta G_1^{\frac{1}{2}}G_2^{\frac{1}{2}}\xi_1^{\frac{1}{2}}\xi_2^{\frac{1}{2}} - \eta G_1\xi_1\right\}, \tag{46}$$

$$\begin{aligned}-\lambda t_0^2\eta^2\langle\nabla u,\nabla\phi_t\rangle &\geq -\lambda t_0^2\eta^2\|\nabla u\|\|\nabla\phi_t\|\\ &\geq -\lambda t_0^2\eta^{\frac{3}{2}}G_1^{\frac{1}{2}}\xi_1^{\frac{1}{2}}\gamma_1.\end{aligned} \tag{47}$$

Combining the above three equations we get a lower bound for $\eta^2 t_0\mathcal{H}$ given by

$$\begin{aligned}\eta^2 t_0\mathcal{H} \geq{}& -2t_0^2(\lambda-1)\bar{u}p\eta G_1^{\frac{1}{2}}G_2^{\frac{1}{2}}\xi_1^{\frac{1}{2}}\xi_2^{\frac{1}{2}} + 2(\lambda-1)t_0^2\bar{u}\eta G_1\xi_1 - \lambda t_0^2\eta^{\frac{3}{2}}G_1^{\frac{1}{2}}\xi_1^{\frac{1}{2}}\gamma_1\\ &+\bar{u}\lambda t_0^2\left\{p\eta\xi_2 G_2 + \eta\xi_1 G_1 - p\eta\xi_1^{\frac{1}{2}}\xi_2^{\frac{1}{2}}G_1^{\frac{1}{2}}G_2^{\frac{1}{2}} + p\eta^2\Delta_\phi v - \eta^2\Delta_\phi u\right\}\\ \geq{}& -(3\lambda-2)p\eta\bar{u}t_0^2\xi_1^{\frac{1}{2}}\xi_2^{\frac{1}{2}}G_1^{\frac{1}{2}}G_2^{\frac{1}{2}} + (3\lambda-2)\bar{u}\eta t_0^2 G_1\xi_1 + \lambda t_0^2\eta^{\frac{3}{2}}G_1^{\frac{1}{2}}\xi_1^{\frac{1}{2}}\gamma_1\\ &+p\eta\bar{u}\lambda t_0^2 G_2\xi_2 + p\bar{u}\lambda t_0^2\eta^2\Delta_\phi v - \bar{u}\lambda t_0^2\eta^2\Delta_\phi u.\end{aligned} \tag{48}$$

From the definition of ξ_1 and ξ_2, we get

$$\begin{cases}-\Delta_\phi u = \left(\xi_1 - \frac{t_0\xi_1-1}{\lambda t_0}\right)F_1,\\ -\Delta_\phi v = \left(\xi_2 - \frac{t_0\xi_2-1}{\lambda t_0}\right)F_2,\end{cases} \tag{49}$$

or equivalently

$$\begin{cases}\eta^2\Delta_\phi u = -\frac{1}{\lambda t_0}G_1 - \frac{\lambda-1}{\lambda}\xi_1 G_1,\\ \eta^2\Delta_\phi v = -\frac{1}{\lambda t_0}G_2 - \frac{\lambda-1}{\lambda}\xi_2 G_2.\end{cases} \tag{50}$$

Using (44), (48), (49), (50) in (43) we get

$$\begin{aligned}0 \geq{}& -\Omega t_0 G_1 + 2t_0^2\frac{1-\lambda\epsilon}{m}\left(\xi_1 - \frac{t_0\xi_1-1}{\lambda t_0}\right)^2 G_1^2 - \lambda\frac{\eta t_0^2 k_2}{2\epsilon}\xi_1 G_1 - 2\lambda t_0^2\eta^2 k_2\epsilon m_1^2\\ &-2\eta t_0^2(1-\lambda\epsilon)(m-1)k_1\xi_1 G_1 - 2\eta t_0\frac{\sqrt{c_1}}{R}G_1^{\frac{3}{2}}\xi_1^{\frac{1}{2}} - \eta G_1 - 2\eta t_0^2 k_3(\lambda-1)G_1\xi_1\\ &-\frac{n\lambda t_0^2}{2\epsilon}\eta^2(k_2+k_3)^2 - 3\eta t_0^2\lambda\sqrt{n}k_4 G_1\xi_1 - (3\lambda-2)p\eta\bar{u}t_0^2 G_1^{\frac{1}{2}}G_2^{\frac{1}{2}}\xi_1^{\frac{1}{2}}\xi_2^{\frac{1}{2}}\\ &+(3\lambda-2)\bar{u}\eta t_0^2\xi_1 G_1 + \lambda t_0^2\eta^{\frac{3}{2}}G_1^{\frac{1}{2}}\xi_1^{\frac{1}{2}}\gamma_1 + p\eta\bar{u}\lambda t_0^2 G_2\xi_2\\ &+p\bar{u}\lambda t_0^2\left(-\frac{1}{\lambda t_0}G_2 - \frac{\lambda-1}{\lambda}\xi_2 G_2\right) - \bar{u}\lambda t_0^2\left(-\frac{1}{\lambda t_0}G_1 - \frac{\lambda-1}{\lambda}\xi_1 G_1\right).\end{aligned} \tag{51}$$

By Young's inequality we have

$$2\eta t_0\frac{\sqrt{c_1}}{R}G_1^{\frac{1}{2}}\xi_1^{\frac{1}{2}}G_1 \leq \frac{4(1-\lambda\epsilon)(\lambda-1)}{m\lambda^2}t_0^2\xi_1 G_1^2 + \frac{t_0 m\lambda^2 c_1}{4R^2(1-\lambda\epsilon)(\lambda-1)}G_1. \tag{52}$$

Let

$$E_1 = \Omega t_0 + 1 + \bar{u}^* t_0, \text{ where } \bar{u}^* = -e^{\lambda_1 t_0 + p\kappa_2 - \tilde{\kappa}_1},$$
$$E_2 = 2\lambda k_2 \epsilon m_1^2 + \frac{n\lambda}{2\epsilon}(k_2 + k_3)^2,$$
$$E_0 = \frac{\lambda \eta k_2}{2\epsilon} + 2\eta(1-\lambda\epsilon)(m-1)k_1 + 2\eta k_3(\lambda - 1) + 3\eta\lambda\sqrt{n}k_4 + \bar{u}^*(3\lambda - 2)\eta$$
$$+ p\eta\bar{u}^*\lambda + \bar{u}^*(\lambda - 1)$$

and $\tilde{E} = p\bar{u}^*(\eta\lambda - \lambda + 1)$.

By Young's inequality we have

$$\lambda t_0^2 \eta^{\frac{3}{2}} G_1^{\frac{1}{2}} \xi_1^{\frac{1}{2}} \gamma_1 \le \frac{(t_0\lambda\gamma_1)^2}{4} + t_0^2 G_1 \xi_1. \tag{53}$$

Using (52) and (53) in (51) we get

$$0 \ge \frac{2(1-\lambda\epsilon)}{m\lambda^2} G_1^2 + \frac{2t_0^2(1-\lambda\epsilon)(\lambda-1)^2}{m\lambda^2} \xi_1^2 G_1^2 - \frac{t_0 m \lambda^2 c_1}{4R^2(1-\lambda\epsilon)(\lambda-1)} G_1 - E_1 G_1$$
$$- t_0^2 E_2 - E_0 t_0^2 \xi_1 G_1 + \tilde{E}\xi_2 G_2 + p t_0 \bar{u}^* G_2 - (3\lambda-2)p\bar{u}^* t_0^2 G_1^{\frac{1}{2}} G_2^{\frac{1}{2}} \xi_1^{\frac{1}{2}} \xi_2^{\frac{1}{2}}. \tag{54}$$

By Young's inequality, we have

$$((3\lambda-2)p\bar{u}^* t_0^2 G_1^{\frac{1}{2}} \xi_1^{\frac{1}{2}}) G_2^{\frac{1}{2}} \xi_2^{\frac{1}{2}} \le \frac{(3\lambda-2)^2 p^2 (\bar{u}^*)^2 t_0^2}{4\tilde{E}} G_1 \xi_1 + t_0^2 G_2 \xi_2 \tilde{E}. \tag{55}$$

Using (55) in (54) and updating E_0, E_1 and E_2 we obtain

$$0 \ge \frac{2(1-\lambda\epsilon)}{m\lambda^2} G_1^2 + \frac{2t_0^2(1-\lambda\epsilon)(\lambda-1)^2}{m\lambda^2} \xi_1^2 G_1^2 - E_1 G_1 - t_0^2 E_2$$
$$- E_0 t_0^2 \xi_1 G_1 + p t_0 \bar{u}^* G_2. \tag{56}$$

Applying Young's inequality on the term $E_0 t_0^2 \xi_1 G_1$ we find

$$E_0 t_0^2 \xi_1 G_1 \le \frac{E_0^2 t_0^2 m \lambda^2}{8(1-\lambda\epsilon)(\lambda-1)^2} + \xi_1^2 G_1^2 \frac{2t_0^2(1-\lambda\epsilon)(\lambda-1)^2}{m\lambda^2}. \tag{57}$$

Using (57) in (56) we infer

$$0 \ge \frac{2(1-\lambda\epsilon)}{m\lambda^2} G_1^2 - E_1 G_1 - t_0^2 E_2 + p t_0 \bar{u}^* G_2. \tag{58}$$

Similarly, using (35) and (37), we can deduce

$$0 \ge \frac{2(1-\lambda\epsilon)}{m\lambda^2} G_2^2 - \tilde{E}_1 G_2 - t_0^2 \tilde{E}_2 + q t_0 \bar{v}^* G_1, \tag{59}$$

or equivalently

$$0 \ge \frac{2(1-\lambda\epsilon)}{m\lambda^2} G_1^2 - E_1 G_1 - \left(t_0^2 E_2 + p t_0 \bar{u}_* G_2\right), \tag{60}$$

$$0 \geq \frac{2(1-\lambda\epsilon)}{m\lambda^2} G_2^2 - \tilde{E}_1 G_2 - \left(t_0^2 \tilde{E}_2 + q t_0 \bar{v}_* G_1\right), \tag{61}$$

where the terms E_1, $\tilde{E}_1$, $E_2 \tilde{E}_2$, E_0, $\tilde{E}_0$ are defined as follows

$$\begin{aligned}
E_1 &= \Omega t_0 + 1 - \bar{u}^* t_0 \\
\tilde{E}_1 &= \Omega t_0 + 1 - \bar{v}^* t_0, \\
E_2 &= 2\lambda k_2 \epsilon m_1^2 + \frac{n\lambda}{2\epsilon}(k_2+k_3)^2 + \frac{1}{4}\lambda^2 \gamma_1^2 + \frac{m\lambda^2 E_0^2}{8(1-\lambda\epsilon)(\lambda-1)^2}, \\
\tilde{E}_2 &= 2\lambda k_2 \epsilon m_1^2 + \frac{n\lambda}{2\epsilon}(k_2+k_3)^2 + \frac{1}{4}\lambda^2 \gamma_1^2 + \frac{m\lambda^2 \tilde{E}_0^2}{8(1-\lambda\epsilon)(\lambda-1)^2}, \\
E_0 &= \frac{\lambda\eta k_2}{2\epsilon} + 2\eta(1-\lambda\epsilon)(m-1)k_1 + 2\eta k_3(\lambda-1) + 3\eta\lambda\sqrt{n}k_4 \\
&\quad + (3\lambda-2)\bar{u}^*\eta + p\eta\bar{u}^*\lambda + \bar{u}^*(n-1) + 1, \\
\tilde{E}_0 &= \frac{\lambda\eta k_2}{2\epsilon} + 2\eta(1-\lambda\epsilon)(m-1)k_1 + 2\eta k_3(\lambda-1) + 3\eta\lambda\sqrt{n}k_4 \\
&\quad + (3\lambda-2)\bar{v}^*\eta + p\eta\bar{v}^*\lambda + \bar{v}^*(n-1) + 1.
\end{aligned}$$

For any $a > 0$ and $b, c \geq 0$ the equation $ax^2 - bx - c \leq 0$ implies $x \leq \frac{b}{a} + \sqrt{\frac{c}{a}}$. Hence from (60) and (61), we get

$$G_1 \leq \frac{m\lambda^2 E_1}{2(1-\lambda\epsilon)} + \sqrt{\frac{m\lambda^2}{2(1-\lambda\epsilon)}(t_0^2 E_2 + p t_0 \bar{u}_* G_2)}, \tag{62}$$

$$G_2 \leq \frac{m\lambda^2 \tilde{E}_1}{2(1-\lambda\epsilon)} + \sqrt{\frac{m\lambda^2}{2(1-\lambda\epsilon)}(t_0^2 \tilde{E}_2 + p t_0 \bar{v}_* G_1)}. \tag{63}$$

Using an elementary inequality $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for nonnegative x, y, in (62) and (63) we obtain

$$G_1 \leq \frac{m\lambda^2 E_1}{2(1-\lambda\epsilon)} + \sqrt{\frac{m\lambda^2}{2(1-\lambda\epsilon)} t_0^2 E_2} + \sqrt{\frac{m\lambda^2}{2(1-\lambda\epsilon)} p t_0 \bar{u}_* G_2}, \tag{64}$$

$$G_2 \leq \frac{m\lambda^2 \tilde{E}_1}{2(1-\lambda\epsilon)} + \sqrt{\frac{m\lambda^2}{2(1-\lambda\epsilon)} t_0^2 \tilde{E}_2} + \sqrt{\frac{m\lambda^2}{2(1-\lambda\epsilon)} q t_0 \bar{v}_* G_1}. \tag{65}$$

Applying Young's inequality in (64) and (65), we get

$$\begin{aligned}
G_1 &\leq \frac{m\lambda^2 E_1}{2(1-\lambda\epsilon)} + \sqrt{\frac{m\lambda^2 t_0^2 E_2}{2(1-\lambda\epsilon)}} + \frac{1}{4}\left(\frac{m\lambda^2 p t_0 \bar{u}_*}{2(1-\lambda\epsilon)}\right) + G_2 \\
&\leq \frac{m\lambda^2 E_1}{2(1-\lambda\epsilon)} + \sqrt{\frac{m\lambda^2 t_0^2 E_2}{2(1-\lambda\epsilon)}} + \frac{1}{4}\left(\frac{m\lambda^2 p t_0 \bar{u}_*}{2(1-\lambda\epsilon)}\right) + \frac{m\lambda^2 \tilde{E}_1}{2(1-\lambda\epsilon)} \\
&\quad + \sqrt{\frac{m\lambda^2 t_0^2 \tilde{E}_2}{2(1-\lambda\epsilon)}} + \sqrt{\frac{m\lambda^2 p t_0 \bar{v}_*}{2(1-\lambda\epsilon)}}\sqrt{G_1} \\
&\leq \frac{2m\lambda^2}{3(1-\lambda\epsilon)}(E_1 + \tilde{E}_1) + \frac{4}{3}\sqrt{\frac{m\lambda^2 t_0^2}{2(1-\lambda\epsilon)}}(\sqrt{E_2} + \sqrt{\tilde{E}_2})
\end{aligned}$$

$$+\frac{m\lambda^2 p t_0 \bar{u}_*}{2(1-\lambda\epsilon)}. \tag{66}$$

Again, using Young's inequality in (65) and using (66) we have

$$\begin{aligned} G_2 \leq{}& \frac{m\lambda^2 \tilde{E}_1}{2(1-\lambda\epsilon)} + \sqrt{\frac{m\lambda^2 t_0^2 \tilde{E}_2}{2(1-\lambda\epsilon)}} + \frac{1}{4}\left(\frac{m\lambda^2 p t_0 \bar{v}_*}{2(1-\lambda\epsilon)}\right) + \frac{2m\lambda^2}{3(1-\lambda\epsilon)}(E_1+\tilde{E}_1) \\ &+ \frac{4}{3}\sqrt{\frac{m\lambda^2 t_0^2}{2(1-\lambda\epsilon)}}(\sqrt{E_2}+\sqrt{\tilde{E}_2}) + \frac{m\lambda^2 p t_0 \bar{u}_*}{2(1-\lambda\epsilon)}. \end{aligned} \tag{67}$$

Setting

$$\begin{aligned} D_1 &= \frac{4}{3}\sqrt{\frac{m\lambda^2}{2(1-\lambda\epsilon)}}(E_2+\tilde{E}_2) + \frac{m\lambda^2 p \bar{u}_*}{2(1-\lambda\epsilon)}, \\ \tilde{D}_1 &= \sqrt{\frac{m\lambda^2}{2(1-\lambda\epsilon)}}\left(\frac{4}{3}\sqrt{E_2}+\frac{7}{3}\sqrt{\tilde{E}_2}\right) + \frac{m\lambda^2 p}{2(1-\lambda\epsilon)}\left(\bar{u}_* + \frac{1}{4}\bar{v}_*\right), \end{aligned}$$

(66) and (67) reduces to

$$\begin{cases} G_1 \leq \frac{2m\lambda^2}{3(1-\lambda\epsilon)}(E_1+\tilde{E}_1) + t_0 D_1, \\ G_2 \leq \frac{2m\lambda^2}{3(1-\lambda\epsilon)}(E_1+\frac{7}{4}\tilde{E}_1) + t_0 \tilde{D}_1. \end{cases} \tag{68}$$

We have $\eta = 1$ whenever $d(x, x_0, T_1) \leq R$. To obtain the result on F_1, F_2, we put $\eta = 1$ and thus

$$\begin{aligned} \|\nabla u\|^2 - \lambda(u_t+\bar{u})\Big\|_{(x,T_1)} &= \frac{F_1(x,T_1)}{T_1} \leq \frac{G_1(x_0,t_0)}{T_1} \leq \frac{2m\lambda^2}{3T_1(1-\lambda\epsilon)}(E_1+\tilde{E}_1) + D_1, \\ \|\nabla v\|^2 - \lambda(v_t+\bar{v})\Big\|_{(x,T_1)} &= \frac{F_2(x,T_1)}{T_1} \leq \frac{G_2(x_0,t_0)}{T_1} \leq \frac{2m\lambda^2}{3T_1(1-\lambda\epsilon)}(E_1+\frac{7}{4}\tilde{E}_1) + \tilde{D}_1. \end{aligned}$$

The rest of the proof is clear as T_1 is chosen arbitrarily. □

The above theorem gives the local Li–Yau-type gradient estimation. The following Corollary gives the global Li–Yau-type gradient estimation.

Corollary 1 *If k_1, k_2, k_3, k_4 are positive constants such that*

$$Ric_{\phi}^{m-n} \geq -(m-1)k_1 g,\ -k_2 g \leq \mathcal{S} \leq k_3 g,\ \|\nabla \mathcal{S}\| \leq k_4$$

on $M \times [0,T]$ and (f,h) is a positive solution to the system (3) along the flow (2), then for any $\lambda > 1$ and $\epsilon \in (0, \frac{1}{\lambda})$ we have

$$\frac{\|\nabla f\|^2}{f^2} - \lambda\left(\frac{f_t}{f} - e^{\lambda_1 t}h^p\right) \leq \frac{4m\lambda^2}{3t(1-\lambda\epsilon)} + \frac{2m\lambda^2}{3(1-\lambda\epsilon)}(2c_2k_2 + \bar{u}^* + \bar{v}^*) + \hat{D}_1, \tag{69}$$

$$\frac{\|\nabla h\|^2}{h^2} - \lambda\left(\frac{h_t}{h} - e^{\lambda_2 t}f^q\right) \leq \frac{4m\lambda^2}{3t(1-\lambda\epsilon)} + \frac{2m\lambda^2}{3(1-\lambda\epsilon)}\left(\frac{11}{4}c_2k_2 + \bar{u}^* + \frac{7}{4}\bar{v}^*\right) + \hat{\tilde{D}}_1, \tag{70}$$

where

$$\hat{D}_1 = \frac{4}{3}\sqrt{\frac{m\lambda^2}{2(1-\lambda\epsilon)}}(\hat{E}_2 + \hat{\hat{E}}_2) + \frac{m\lambda^2 p\bar{u}_*}{2(1-\lambda\epsilon)},$$

$$\hat{\hat{D}}_1 = \sqrt{\frac{m\lambda^2}{2(1-\lambda\epsilon)}}\left(\frac{4}{3}\sqrt{\hat{E}_2} + \frac{7}{3}\sqrt{\hat{\hat{E}}_2}\right) + \frac{m\lambda^2 p}{2(1-\lambda\epsilon)}\left(\bar{u}_* + \frac{1}{4}\bar{v}_*\right),$$

$$\hat{E}_2 = 2\lambda k_2\epsilon M_1^2 + \frac{n\lambda}{2\epsilon}(k_2+k_3)^2 + \frac{1}{4}\lambda^2\Gamma_1^2 + \frac{m\lambda^2 E_0^2}{8(1-\lambda\epsilon)(\lambda-1)^2},$$

$$\hat{\hat{E}}_2 = 2\lambda k_2\epsilon M_1^2 + \frac{n\lambda}{2\epsilon}(k_2+k_3)^2 + \frac{1}{4}\lambda^2\Gamma_1^2 + \frac{m\lambda^2 \tilde{E}_0^2}{8(1-\lambda\epsilon)(\lambda-1)^2},$$

$$E_0 = \frac{\lambda\eta k_2}{2\epsilon} + 2\eta(1-\lambda\epsilon)(m-1)k_1 + 2\eta k_3(\lambda-1) + 3\eta\lambda\sqrt{n}k_4$$
$$+(3\lambda-2)\bar{u}^*\eta + p\eta\bar{u}^*\lambda + \bar{u}^*(n-1) + 1,$$

$$\tilde{E}_0 = \frac{\lambda\eta k_2}{2\epsilon} + 2\eta(1-\lambda\epsilon)(m-1)k_1 + 2\eta k_3(\lambda-1) + 3\eta\lambda\sqrt{n}k_4$$
$$+(3\lambda-2)\bar{v}^*\eta + p\eta\bar{v}^*\lambda + \bar{v}^*(n-1) + 1,$$

$$\bar{u}^* = -e^{\lambda_1 t + p\kappa_2 - \tilde{\kappa}_1} = -\bar{u}_*,$$

$$\bar{v}^* = -e^{\lambda_2 t + q\kappa_1 - \tilde{\kappa}_2} = -\bar{v}_*.$$

Proof Taking $R \to \infty$ in (24), (25) and using the global bounds of $\|\nabla\phi\|$, $\|\nabla\phi_t\|$ we have (69) and (70), respectively. □

Theorem 2 (Harnack-type inequality) *If k_1, k_2, k_3, k_4 are positive constants such that*

$$Ric_\phi^{m-n} \geq -(m-1)k_1 g,\ -k_2 g \leq \mathcal{S} \leq k_3 g,\ \|\nabla\mathcal{S}\| \leq k_4$$

on M and (f,h) is a positive solution to (3) along (2), then we have the Harnack-type inequality given by

$$\frac{f(y_1,s_1)}{f(y_2,s_2)} \leq \left(\frac{s_2}{s_1}\right)^{\frac{4m\lambda}{3(1-\lambda\epsilon)}} \exp\left\{\mathcal{C}[(y_1,s_1),(y_2,s_2)] + \int_{s_1}^{s_2} Q dt\right\}, \tag{71}$$

$$\frac{h(y_1,s_1)}{h(y_2,s_2)} \leq \left(\frac{s_2}{s_1}\right)^{\frac{11m\lambda^2}{6(1-\lambda\epsilon)}} \exp\left\{\mathcal{C}[(y_1,s_1),(y_2,s_2)] + \int_{s_1}^{s_2} \tilde{Q} dt\right\}, \tag{72}$$

where

$$Q = \bar{u}^* + \frac{2m\lambda}{3(1-\lambda\epsilon)}(2c_2k_2 + \bar{u}^* + \bar{v}^*) + \frac{D_1}{\lambda},$$

$$\tilde{Q} = \bar{v}^* + \frac{2m\lambda}{3(1-\lambda\epsilon)}\left(\frac{11}{4}c_2k_2 + \bar{u}^* + \frac{7}{4}\bar{v}^*\right) + \frac{\tilde{D}_1}{\lambda}$$

and $\mathcal{C}[(y_1,s_1),(y_2,s_2)] = \frac{\lambda}{4}\sup_\nu \int_{s_1}^{s_2} \|\nu'(t)\|^2 dt$*, the supremum is taken over all possible curves joining (y_1,s_1), (y_2,s_2) over M.*

Proof Let $(y_1,s_1),(y_2,s_2)\in M\times(0,T]$ be two points such that $s_1<s_2$. Choose a geodesic path $\nu:[s_1,s_2]\to M$ satisfying $\nu(s_1)=y_1$, $\nu(s_2)=y_2$. Hence for $f=e^u$, $h=e^v$, we have from Corollary 1

$$\begin{aligned} u(y_1,s_1)-u(y_2,s_2) &= -\int_{s_1}^{s_2}\frac{d}{dt}u(\nu(t),t)dt \\ &= -\int_{s_1}^{s_2}\partial_t u dt-\int_{s_1}^{s_2}\langle\nabla u,\nu'(t)\rangle dt \\ &\leq -\int_{s_1}^{s_2}\left(\frac{\|\nabla u\|^2}{\lambda}+\langle\nabla u,\nu'(t)\rangle\right)dt \\ &\quad +\frac{4m\lambda}{3(1-\lambda\epsilon)}\ln(\frac{s_2}{s_1})+\int_{s_1}^{s_2}Qdt, \end{aligned} \tag{73}$$

and

$$\begin{aligned} v(y_1,s_1)-v(y_2,s_2) &= -\int_{s_1}^{s_2}\frac{d}{dt}v(\nu(t),t)dt \\ &= -\int_{s_1}^{s_2}\partial_t v dt-\int_{s_1}^{s_2}\langle\nabla v,\nu'(t)\rangle dt \\ &\leq -\int_{s_1}^{s_2}\left(\frac{\|\nabla v\|^2}{\lambda}dt+\langle\nabla v,\nu'(t)\rangle\right)dt \\ &\quad +\frac{4m\lambda}{3(1-\lambda\epsilon)}\ln(\frac{s_2}{s_1})+\int_{s_1}^{s_2}\tilde{Q}dt. \end{aligned} \tag{74}$$

We know that $-ax^2-bx\leq\frac{b^2}{4a}$. Setting $x=\nabla u$, $a=\frac{1}{\lambda}$, $b=\nu'(t)$ we deduce

$$-\frac{\|\nabla u\|^2}{\lambda}-\langle\nu'(t),\nabla u\rangle\leq\frac{\lambda\|\nu'(t)\|^2}{4}. \tag{75}$$

Similarly, putting $x=\nabla v$ $a=\frac{1}{\lambda}$, $b=\nu'(t)$ we get

$$-\frac{\|\nabla v\|^2}{\lambda}-\langle\nu'(t),\nabla v\rangle\leq\frac{\lambda\|\nu'(t)\|^2}{4}. \tag{76}$$

Combining (75) and (73) we get

$$u(y_1,s_1)-u(y_2,s_2)\leq\frac{\lambda}{4}\int_{s_1}^{s_2}\|\nu'(t)\|^2dt+\frac{4m\lambda}{3(1-\lambda\epsilon)}\ln(\frac{s_2}{s_1})+\int_{s_1}^{s_2}Qdt.$$

Taking supremum on the right-hand side of the above equation over all possible curves ν, joining (y_1,s_1), (y_2,s_2) we find

$$u(y_1,s_1)-u(y_2,s_2)\leq\mathcal{C}[(y_1,s_1),(y_2,s_2)]+\frac{4m\lambda}{3(1-\lambda\epsilon)}\ln(\frac{s_2}{s_1})+\int_{s_1}^{s_2}Qdt.$$

Taking exponent on both sides by putting $u=\ln f$, $v=\ln h$ we get (71). In similar way, using (76) and (74) we get

$$v(y_1,s_1)-v(y_2,s_2)\leq\frac{\lambda}{4}\int_{s_1}^{s_2}\|\nu'(t)\|^2dt+\frac{4m\lambda}{3(1-\lambda\epsilon)}\ln(\frac{s_2}{s_1})+\int_{s_1}^{s_2}\tilde{Q}dt.$$

Again, taking supremum on the right-hand side just like before, we derive

$$v(y_1,s_1) - v(y_2,s_2) \le \mathcal{C}[(y_1,s_1),(y_2,s_2)] + \frac{4m\lambda}{3(1-\lambda\epsilon)} \ln(\frac{s_2}{s_1}) + \int_{s_1}^{s_2} \tilde{Q}dt.$$

Taking exponent on both sides after putting $u = \ln f$, $v = \ln h$, we get (72). This completes the proof. □

4 Concluding remark

In this paper, we have presented a detailed work on finding certain bounds for the quantities $\frac{\|\nabla f\|^2}{f^2} - \lambda\left(\frac{f_t}{f} - e^{\lambda_1 t} h^p\right)$ and $\frac{\|\nabla h\|^2}{h^2} - \lambda\left(\frac{h_t}{h} - e^{\lambda_2 t} f^q\right)$ on a smooth measure space $(M^n, g, e^{-\phi}d\mu)$, evolving along the geometric flow (2), where p, q, λ_1, λ_2 are positive constants, $\lambda > 1$ is a real number and (f,h) is a positive solution to the system (3) along (2). We have also derived a Harnack-type inequality given by (71) and (72), which provides information about the amount of heat located in two different places of the manifold in two different time. As future work, we suggest to extend this method of deriving gradient estimates for single as well as for system of heat type equations to space-times. As future work, one can extend these results to heat equations on Finsler manifold (see [1]).

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Declarations

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The authors declare no competing interests.

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