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On some geometric properties of sequence spaces of generalized arithmetic divisor sum function

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Abstract

Recently, some new sequence spaces $\ell_p(\mathfrak{A}^\alpha)$ $(0 < p < \infty)$, $c_0(\mathfrak{A}^\alpha)$, $c(\mathfrak{A}^\alpha)$, and $\ell_\infty(\mathfrak{A}^\alpha)$ have been studied by Yaying et al. (Forum Math., 2024, https://doi.org/10.1515/forum-2023-0138) as matrix domains of $\mathfrak{A}^\alpha = (a^\alpha_{n,v})$, where

$$a^\alpha_{\mathfrak{m},v} = \begin{cases} \dfrac{v^\alpha}{\rho^{(\alpha)}(\mathfrak{m})} & , \quad v \mid \mathfrak{m}, \\ 0 & , \quad v \nmid \mathfrak{m}, \end{cases}$$

and $\rho^{(\alpha)}(\mathfrak{m})$:= sum of the α^{th} power of the positive divisors of $\mathfrak{m} \in \mathbb{N}$. They obtained their duals, matrix transformations and associated compact matrix operators for these matrix classes.

This article deals with some geometric properties of these sequence spaces.

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1 Introduction

We recall some known arithmetic functions [1, 16]:

$$d(\mathfrak{m}) = \sum_{\mathfrak{v}|\mathfrak{m}} 1, \text{(Divisor function)}$$

$$\rho(\mathfrak{m}) = \sum_{\mathfrak{v}|\mathfrak{m}} \mathfrak{v}, \text{(Divisor sum function)}$$

$$\rho^{(\alpha)}(\mathfrak{m}) = \sum_{\mathfrak{v}|\mathfrak{m}} v^\alpha, \text{(Divisor sum function of order } \alpha)$$

$$\mu(\mathfrak{m}) = \begin{cases} 1 & , \quad \mathfrak{m} = 1, \\ (-1)^{\mathfrak{v}} & , \quad \mathfrak{m} = p_1 p_2 \cdots p_v, \\ 0 & , \quad p^2|\mathfrak{m}, \text{ for any prime } p, \end{cases} \qquad \text{(Möbius function)}$$

$$\varphi(\mathfrak{m}) = \mathfrak{m}\sum_{\mathfrak{v}|\mathfrak{m}} \frac{\mu(\mathfrak{v})}{\mathfrak{v}}, \text{ (Euler's totient function)}$$

$$J_r(\mathfrak{m}) = \mathfrak{m}^r\sum_{\mathfrak{v}|\mathfrak{m}} \frac{\mu(\mathfrak{v})}{\mathfrak{v}^r}, \text{ (Jordan's totient function)},$$

where $\mathfrak{m} \in \mathbb{N}$ and $p_{\mathfrak{v}}$ denote successive prime numbers.

Lemma 1.1 [16] *For any* $\mathfrak{m} \in N$, $f(\mathfrak{m}) = \sum_{\nu|\mathfrak{m}} g(\nu)$ *iff* $g(\mathfrak{m}) = \sum_{\mathfrak{v}|\mathfrak{m}} \mu(\nu) g\left(\frac{\mathfrak{m}}{\mathfrak{v}}\right) = \sum_{\mathfrak{v}|\mathfrak{m}} \mu\left(\frac{\mathfrak{m}}{\mathfrak{v}}\right) g(\mathfrak{v})$.

We highlight some of the interesting properties of $\rho^{(\alpha)}(\mathfrak{m})$ (see [1]):

(a) $\rho^{(\alpha)}(\mathfrak{m}\mathfrak{n}) = \rho^{(\alpha)}(\mathfrak{m})\rho^{(\alpha)}(\mathfrak{n})$.

(b) By Lemma 1.1,

$$\rho^{(\alpha)}(\mathfrak{m}) = \sum_{\mathfrak{v}|m} \mathfrak{v} \text{ iff } \mathfrak{m}^\alpha = \sum_{\nu|m} \mu\left(\frac{\mathfrak{m}}{\mathfrak{v}}\right)\rho^{(\alpha)}(\mathfrak{v}). \tag{1.1}$$

(c) For any prime p,

$$\rho^{(\alpha)}(p^{\mathfrak{v}}) = \begin{cases} \dfrac{p^{\alpha(\mathfrak{v}+1)}}{p^\alpha - 1} & , \quad \alpha \neq 0, \\ \mathfrak{v}+1 & , \quad \alpha = 0. \end{cases}$$

In general, if $\mathfrak{m} = p_1^{k_1} p_2^{k_2} \cdots p_{\mathfrak{v}}^{k_\nu}$, then

$$\rho^{(\alpha)}(\mathfrak{m}) = \frac{p^{\alpha(k_1+1)}}{p^\alpha - 1} \cdot \frac{p^{\alpha(k_2+1)}}{p^\alpha - 1} \cdots \frac{p^{\alpha(k_\nu+1)}}{p^\alpha - 1}.$$

For $\alpha = 0$, $\rho^{(\alpha)}(\mathfrak{m}) = \rho^{(0)}(\mathfrak{m}) = d(\mathfrak{m})$. For $\alpha = 1$, $\rho^{(\alpha)}(\mathfrak{m}) = \rho^{(1)}(\mathfrak{m}) = \rho(\mathfrak{m})$.

We write ω for the set of all real or complex valued sequeces. We further denote by ℓ_p $(1 \le p < \infty)$ the set of all p-absolutely summable sequences, ℓ_∞ for all bounded sequences, c_0 for all convergent to zero sequences), and c for all convergent sequences [14].

Let $\mathsf{A} = (\mathsf{a}_{rs})$ be an infinite matrix and A_r denotes its r^{th} row. Then, we term the sequence $\mathsf{A}x = \{(\mathsf{A}x)_r\} = \left\{\sum_{s=0}^r \mathsf{a}_{rs} x_s\right\}$ as the A-transform of the sequence $x = (x_s)$. Let X and Y be any two sequence spaces. We say that A defines a matrix mapping from X to Y if for each $x \in X$, $\mathsf{A}x \in Y$. We use the notation (X, Y) to denote the family of all matrix mappings such that $X \to Y$. Further, for any sequence space X, the set X_A that contains all the sequences whose A-transforms belong to X is called as the domain of A in X, i.e., $X_{\mathsf{A}} = \{x \in \omega : \mathsf{A}x \in X\}$. For different matrix domains in classical sequence spaces, one can refer to [2, 8–11, 15].

Recently, Yaying et al. [22] defined the following sequence spaces via $\rho^{(\alpha)}(\mathfrak{n})$:

$$\begin{aligned}
\ell_p(\mathfrak{A}^\alpha) &:= \left\{\mathfrak{x} = (\mathfrak{x}_\nu) \in \omega : \mathfrak{A}^\alpha \mathfrak{x} \in \ell_p\right\}, \\
c_0(\mathfrak{A}^\alpha) &:= \{\mathfrak{x} = (\mathfrak{x}_\nu) \in \omega : \mathfrak{A}^\alpha \mathfrak{x} \in c_0\}, \\
c(\mathfrak{A}^\alpha) &:= \{\mathfrak{x} = (\mathfrak{x}_\nu) \in \omega : \mathfrak{A}^\alpha \mathfrak{x} \in c\}, \\
\ell_\infty(\mathfrak{A}^\alpha) &:= \{\mathfrak{x} = (\mathfrak{x}_\nu) \in \omega : \mathfrak{A}^\alpha \mathfrak{x} \in \ell_\infty\},
\end{aligned}$$

where the matrix $\mathfrak{A}^{\alpha} = (a^{\alpha}_{\mathfrak{n},\mathfrak{v}})_{\mathfrak{n},\mathfrak{v}\in\mathbb{N}}$ is

$$a^{\alpha}_{\mathfrak{n},\nu} = \begin{cases} \dfrac{\mathfrak{v}^{\alpha}}{\rho^{(\alpha)}(\mathfrak{n})} & , \quad \mathfrak{v} \mid \mathfrak{n}, \\ 0 & , \quad \mathfrak{v} \nmid \mathfrak{n}. \end{cases}$$

That is

$$\mathfrak{A}^{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \dfrac{1^{\alpha}}{1^{\alpha}+2^{\alpha}} & \dfrac{2^{\alpha}}{1^{\alpha}+2^{\alpha}} & 0 & 0 & 0 & \cdots \\ \dfrac{1^{\alpha}}{1^{\alpha}+3^{\alpha}} & 0 & \dfrac{3^{\alpha}}{1^{\alpha}+3^{\alpha}} & 0 & 0 & \cdots \\ \dfrac{1^{\alpha}}{1^{\alpha}+2^{\alpha}+4^{\alpha}} & \dfrac{2^{\alpha}}{1^{\alpha}+2^{\alpha}+4^{\alpha}} & 0 & \dfrac{4^{\alpha}}{1^{\alpha}+2^{\alpha}+4^{\alpha}} & 0 & \cdots \\ \dfrac{1^{\alpha}}{1^{\alpha}+5^{\alpha}} & 0 & 0 & 0 & \dfrac{5^{\alpha}}{1^{\alpha}+5^{\alpha}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since $\mathfrak{A}^{\alpha}$ is a triangle, its unique inverse by (1.1) is $(\mathfrak{A}^{\alpha})^{-1} = (a^{-\alpha}_{\mathfrak{n},\mathfrak{v}})$, where

$$a^{-\alpha}_{\mathfrak{n},\mathfrak{v}} = \begin{cases} \dfrac{\mu\left(\frac{\mathfrak{n}}{\mathfrak{v}}\right)\rho^{(\alpha)}(\mathfrak{v})}{\mathfrak{n}^{\alpha}} & , \quad \mathfrak{v} \mid \mathfrak{n}, \\ 0 & , \quad \mathfrak{v} \nmid \mathfrak{n}. \end{cases}$$

$\mathfrak{A}^{\alpha}$-transform of a sequence $\mathfrak{x} = (\mathfrak{x}_{\mathfrak{v}})$ is given by $\eta = (\eta_{\mathfrak{n}})$

$$\eta_{\mathfrak{n}} = (\mathfrak{A}^{\alpha}\mathfrak{x})_{\mathfrak{n}} = \sum_{\mathfrak{v}\mid\mathfrak{n}} \frac{\mathfrak{v}^{\alpha}}{\rho^{(\alpha)}(\mathfrak{n})}\mathfrak{x}_{\nu} \ (\mathfrak{n} \in \mathbb{N}). \tag{1.2}$$

The relation (1.2) is represented by

$$\mathfrak{x}_{\mathfrak{n}} = \left((\mathfrak{A}^{\alpha})^{-1}\eta\right)_{\mathfrak{n}} = \sum_{\mathfrak{v}\mid\mathfrak{n}} \frac{\mu\left(\frac{\mathfrak{n}}{\mathfrak{v}}\right)\rho^{(\alpha)}(\mathfrak{v})}{\mathfrak{n}^{\alpha}}\eta_{\mathfrak{v}} \ (\mathfrak{n} \in \mathbb{N}).$$

The readers are suggested to consult the papers [18–20] for more insights into sequence spaces that are constructed by using arithmetic functions. Clearly $X(\mathfrak{A}^{\alpha}) = X_{\mathfrak{A}^{\alpha}}$, where $X = \ell_p, c_0, c$, or ℓ_{∞}.

Remark 1.2 For $\alpha = 1$, $\ell_p(\mathfrak{A}^{\alpha})$, $c_0(\mathfrak{A}^{\alpha})$, $c(\mathfrak{A}^{\alpha})$ and $\ell_{\infty}(\mathfrak{A}^{\alpha})$ reduce to the spaces defined in [21].

Theorem 1.3 *We have*

(1) $c_0(\mathfrak{A}^{\alpha})$, $c(\mathfrak{A}^{\alpha})$, $\ell_{\infty}(\mathfrak{A}^{\alpha})$ *are BK-spaces with the norm*

$$\|\mathfrak{x}\|_{\ell_{\infty}(\mathfrak{A}^{\alpha})} = \|\mathfrak{A}^{\alpha}\mathfrak{x}\|_{\ell_{\infty}} = \sup_{\mathfrak{n}\in\mathbb{N}} \left| \sum_{\mathfrak{v}\mid\mathfrak{n}} \frac{\mathfrak{v}^{\alpha}}{\rho^{(\alpha)}(\mathfrak{n})}\mathfrak{x}_{\mathfrak{v}} \right|.$$

(2) $\ell_p(\mathfrak{A}^\alpha)(1 \le p < \infty)$ *is a BK-space with the norm*

$$\|\mathfrak{x}\|_{\ell_p(\mathfrak{A}^\alpha)} = \|\mathfrak{A}^\alpha \mathfrak{x}\|_{\ell_p} = \left[\sum_{\mathfrak{n}=0}^{\infty}\left|\sum_{\mathfrak{v}|\mathfrak{n}} \frac{\nu^\alpha}{\rho^{(\alpha)}(\mathfrak{n})}\mathfrak{x}_\mathfrak{v}\right|^p\right]^{1/p} < \infty.$$

In this paper, we study some geometric properties of these sequence spaces.

2 Geometric properties

We recall some geometric properties to study in our case. For Banach spaces λ and μ, let $L : \lambda \to \mu$ be a linear operator. We denote $B(\lambda, \mu)$ and $C(\lambda, \mu)$ for the spaces of bounded linear operators and compact linear operators, respectively.

L is weakly compact [13, Definition 3.5.1] if $L(Q)$ is a relatively weakly compact subset of μ whenever Q is a bounded subset of λ.

Approximation property [13, Definition 3.4.26] is possesed by λ if the set of finite rank members of $B(\mu, \lambda)$ is dense in $C(\mu, \lambda)$ for any μ.

A Banach space is said to have the approximation property (AP), if every compact operator is a limit of finite-rank operators.

The approximation property is satisfied by the space ℓ_p $(1 \le p < \infty)$ (see [13]).

The Dunford–Pettis property (in short, D-P property) is possessed by λ if every continuous weakly compact operator $L : \lambda \to \mu$ transforms weakly compact sets in λ into a compact sets in μ (such operators are called completely continuous).

Theorem 2.1 [17] *Let $L_0 \in B(\nu, \ell_\infty)$. Then, the operator L_0 may be extended to $L \in B(\lambda, \ell_\infty)$ with $\|L_0\| = \|L\|$, where ν is a linear subspace of λ. In this case, ℓ_∞ is said to have Hahn–Banach extension property.*

Let

$$S_\lambda = \{s \in \lambda : \|s\| = 1\}.$$

A normed space λ is said to be rotund (or strictly convex) [13, Definition 5.1.1] if for any $s_1, s_2 \in S_\lambda$ $(s_1 \neq s_2)$ and $0 < \alpha < 1$,

$$\|\alpha s_1 + (1-\alpha)s_2\| < 1.$$

A normed space λ is rotund [13] iff

$$\left\|\frac{s_1 + s_2}{2}\right\| < 1$$

for any $s_1, s_2 \in S_\lambda$ $(s_1 \neq s_2)$.

Proposition 2.2 [13, *Proposition* 5.1.9] *Any normed space that is isometrically isomorphic to a rotund space is also rotund.*

Let X be a Banach space.

If every bounded sequence (ξ_r) in X has a subsequence (χ_r) such that the sequence $\{t_k(\chi)\}$ converges in the norm, then X has the Banach–Saks property [15], where

$$\{t_i(\chi)\} = \frac{1}{i+1}(\chi_0 + \chi_1 + \cdots)\ (i \in \mathbb{N}).$$

If any weakly null sequence (ξ_r) in X has a subsequence (χ_r) such that $\{t_i(\chi)\}$ is strongly convergent to zero, then X has the weak Banach–Saks property.

The following coefficient is provided by Garcia-Falset [4],

$$R(X) = \sup\{\lim_{r\to\infty} \inf \|\xi_r - \xi\| : (\xi_r) \subset D(X), \xi_r \to \xi(w),\ \xi \in D(X)\},$$

where $D(X)$ represents $X's$ unit ball.

Remark 2.3 X has weak fixed point characteristics when $R(X) < 2$ [5].

For $1 < p < \infty$, the property $(BS)_p$, also known as Banach–Saks type p, is that if a subsequence (ξ_{k_l}) of every weakly null sequence (ξ_k) satisfies

$$\left\| \sum_{l=0}^{u} \xi_{k_l} \right\| < Q.(u+1)^{\frac{1}{p}}$$

for each $Q > 0$ and for all $u \in \mathbb{N}$ ([12]).

The Gurarii's modulus of convexity (see [6, 7]) is defined by

$$\beta_X(\epsilon) = \inf \left\{ 1 - \inf_{0\le\delta\le1} ||\delta x + (1-\delta)y||; x, y \in S_X, ||x-y|| = \epsilon \right\},$$

where $0 \le \epsilon \le 2$, and S_X denotes the unit sphere in X.

Most recently such properties are studied in [3].

3 Main results

Here we study such geometric properties for our sequence spaces.

Theorem 3.1 *The approximation property is possessed by the space $\ell_p(\mathfrak{A}^\alpha)$ for $1 \le p < \infty$.*

Proof Let $L \in C(\lambda, \ell_p(\mathfrak{A}^\alpha))$ for any Banach space λ. It follows that for each bounded sequence $s = (s_n) \in \lambda$, the sequence (Ls_n) has a convergent sub-sequence $\left(Ls_{n_v}\right)$ in $\ell_p(\mathfrak{A}^\alpha)$, i.e.,

$$\|Ls_{n_u} - Ls_{n_v}\|^p_{\ell_p(\mathfrak{A}^\alpha)} = \left\|L\left(s_{n_u} - s_{n_v}\right)\right\|^p_{\ell_p(\mathfrak{A}^\alpha)} = \left\|(\mathfrak{A}^\alpha L)\left(s_{n_u} - s_{n_v}\right)\right\|^p_{\ell_p} \to 0$$

as $u, v \to \infty$. Then, $\mathfrak{A}^\alpha L \in C(\lambda, \ell_p)$. Since ℓ_p possesses the approximation property, there exists a sequence $T_n \in B(\lambda, \ell_p)$ of finite rank operators such that

$$\|\mathfrak{A}^\alpha L - T_n\| \to 0.$$

Consequently, the sequence $\left((\mathfrak{A}^{\alpha})^{-1}T_n\right) \in B(\lambda, \ell_p(\mathfrak{A}^{\alpha}))$ is the required sequence of finite rank. Also

$$\begin{aligned}
\left\|L - (\mathfrak{A}^{\alpha})^{-1}T_n\right\| &= \sup_{\|s\|=1} \left\|\left(L - (\mathfrak{A}^{\alpha})^{-1}T_n\right)s\right\|^p_{\ell_p(\mathfrak{A}^{\alpha})} \\
&= \sup_{\|s\|=1} \left\|Ls - \left((\mathfrak{A}^{\alpha})^{-1}T_n\right)s\right\|^p_{\ell_p(\mathfrak{A}^{\alpha})} \\
&= \sup_{\|s\|=1} \|\mathfrak{A}^{\alpha}Ls - T_n s\|^p_{\ell_p} \\
&= \sup_{\|s\|=1} \|(\mathfrak{A}^{\alpha}L - T_n)s\|^p_{\ell_p} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

This completes the proof. □

Theorem 3.2 *The D-P property is possessed by the space $\ell_1(\mathfrak{A}^{\alpha})$.*

Proof Suppose that $L: \ell_1(\mathfrak{A}^{\alpha}) \to \lambda$ is a weakly compact operator. Then, $L\{\mathfrak{A}^{\alpha}\}^{-1} : \ell_1 \to \lambda$ is a bounded linear operator. Let $B \subset \ell_1$ be bounded. Then, it follows that $\{\mathfrak{A}^{\alpha}\}^{-1}B \subset \ell_1(\mathfrak{A}^{\alpha})$ is bounded. It follows that the set

$$L\left(\{\mathfrak{A}^{\alpha}\}^{-1}B\right) = \left(L\{\mathfrak{A}^{\alpha}\}^{-1}\right)B$$

is relatively weakly compact in λ, since L is weakly compact. Therefore, $L\{\mathfrak{A}^{\alpha}\}^{-1}: \ell_1 \to \lambda$ is a weakly compact operator. Now, the operator $L\{\mathfrak{A}^{\alpha}\}^{-1}$ is completely continuous, since the space ℓ_1 has the D-P property. Suppose that Q is a weakly compact subset of $\ell_1(\mathfrak{A}^{\alpha})$. Then, $\mathfrak{A}^{\alpha}Q$ is a weakly compact subset of ℓ_1. Therefore, $L\{\mathfrak{A}^{\alpha}\}^{-1}(\mathfrak{A}^{\alpha})(Q) = L(Q)$ is a compact set in μ, since $L\{\mathfrak{A}^{\alpha}\}^{-1}$ is completely continuous. Hence, L is completely continuous as required. □

Theorem 3.3 *The space $\ell_\infty(\mathfrak{A}^{\alpha})$ has the Hahn–Banach extension property.*

Proof Let ν be a linear subspace of a Banach space λ and $L_0 \in B(\nu, \ell_\infty(\mathfrak{A}^{\alpha}))$. Then, $\mathfrak{A}^{\alpha}L_0 \in B(\nu, \ell_\infty)$. Then the operator $\mathfrak{A}^{\alpha}L_0$ can be extended to $T \in B(\lambda, \ell_\infty)$ with $\|\mathfrak{A}^{\alpha}L_0\| = \|T\|$, since by Theorem 2.1 ℓ_∞ has the Hahn–Banach extension property. Choose the operator $L = \{\mathfrak{A}^{\alpha}\}^{-1}T$. Then, $L \in B(\lambda, \ell_\infty(\mathfrak{A}^{\alpha}))$. Also, we observe that

$$Ls = \left(\{\mathfrak{A}^{\alpha}\}^{-1}T\right)s = \{\mathfrak{A}^{\alpha}\}^{-1}(Ts) = \{\mathfrak{A}^{\alpha}\}^{-1}\left((\mathfrak{A}^{\alpha}L_0)s\right) = L_0 s.$$

for any $s \in \nu$. Additionally

$$\|L\| = \left\|\{\mathfrak{A}^{\alpha}\}^{-1}T\right\| = \left\|\{\mathfrak{A}^{\alpha}\}^{-1}(\mathfrak{A}^{\alpha}L_0)\right\| = \|L_0\|,$$

as desired. □

Theorem 3.4 *The space $\ell_p(\mathfrak{A}^{\alpha})$ $(1 < p < \infty)$ is rotund.*

Proof Since ℓ_p $(1 < p < \infty)$ is a rotund, using Proposition 2.2 we get the result. □

Theorem 3.5 *The spaces $\ell_1(\mathfrak{A}^\alpha)$ and $\ell_\infty(\mathfrak{A}^\alpha)$ are not rotund.*

Proof Choose $a_\nu, b_\nu \in \ell_1(\mathfrak{A}^\alpha)$ given by

$$a_\nu = \begin{cases} \dfrac{\mu(\nu) + (1^\alpha + 2^\alpha)\mu\left(\frac{\nu}{2}\right)}{\nu^\alpha} &, \quad \nu \text{ is even} \\ \dfrac{\mu(\nu)}{\nu^\alpha} &, \quad \nu \text{ is odd,} \end{cases} \quad \text{and}$$

$$b_\nu = \begin{cases} \dfrac{\mu(\nu) - (1^\alpha + 2^\alpha)\mu\left(\frac{\nu}{2}\right)}{\nu^\alpha} &, \quad \nu \text{ is even} \\ \dfrac{\mu(\nu)}{\nu^\alpha} &, \quad \nu \text{ is odd,} \end{cases}$$

for all $\nu \in \mathbb{N}$. Then, $\mathfrak{A}^\alpha a = (1,1,0,0,\ldots) \in \ell_p$ and $\mathfrak{A}^\alpha b = (1,-1,0,0,\ldots) \in \ell_p$. It follows that $\|a\|_{\ell_1(\mathfrak{A}^\alpha)} = 1$ and $\|b\|_{\ell_1(\mathfrak{A}^\alpha)} = 1$. That is $a, b \in S_{\ell_1(\mathfrak{A}^\alpha)}$.

Let $s = \frac{a+b}{2}$. Then, $\mathfrak{A}^\alpha s = \{\dfrac{\mu(\nu)}{\nu^\alpha}\}$. Thus,

$$\|s\|_{\ell_1(\mathfrak{A}^\alpha)} = \|\mathfrak{A}^\alpha s\|_{\ell_1} = 1.$$

Hence, we see that

$$\|s\|_{\ell_1(\mathfrak{A}^\alpha)} \not< 1.$$

Therefore, the space $\ell_1(\mathfrak{A}^\alpha)$ is not rotund. Similarly, non-rotundness of $\ell_\infty(\mathfrak{A}^\alpha)$ can be proved. □

Theorem 3.6 *The space $\ell_p(\mathfrak{A}^\alpha)$ $(1 < p < \infty)$ has the property $(BS)_p$.*

Proof For a positive number sequence (ϵ_r) such that $\sum_{r=1}^{\infty} \epsilon_r \leq \dfrac{1}{2}$ and a weakly null sequence $(\xi_r) \in B(\ell_p(\mathfrak{A}^\alpha))$. Put $\chi_0 = \xi_0 = 0$ and $\chi_1 = \xi_{r_1} = \xi_1$. Therefore, there exists $\nu_1 \in \mathbb{N}$ such that

$$\left\| \sum_{k=\nu_1+1}^{\infty} \chi_1(k) e^{(k)} \right\|_{\ell_p(\mathfrak{A}^\alpha)} < \epsilon_1.$$

There is an $r_2 \in \mathbb{N}$ such that

$$\left\| \sum_{k=0}^{\nu_1} \xi_r(k) e^{(k)} \right\|_{\ell_p(\mathfrak{A}^\alpha)} < \epsilon_1,$$

when $r \geq r_2$, since (ξ_r) is a weakly null sequence, then $\xi_r \to 0$ coordinatewise. Set $\chi_2 = \xi_{r_2}$. Therefore there exists an $r_2 > r_1$ such that

$$\left\| \sum_{k=\nu_2+1}^{\infty} \chi_2(k) e^{(k)} \right\|_{\ell_p(\mathfrak{A}^\alpha)} < \epsilon_2.$$

By using $\xi_r \to 0$ coordinatewise, there exists $r_3 > r_2$ such that

$$\left\| \sum_{k=0}^{\nu_2} \xi_r(k) e^{(k)} \right\|_{\ell_p(\mathfrak{A}^\alpha)} < \epsilon_2,$$

when $r \geq r_3$.

By following this procedure, two increasing subsequences (ν_k) and (r_k) can be obtained such that

$$\left\| \sum_{k=0}^{\nu_\alpha} \xi_r(k) e^{(k)} \right\|_{\ell_p(\mathfrak{A}^\alpha)} < \epsilon_\alpha,$$

for each $r \geq r_{\alpha+1}$ and

$$\left\| \sum_{k=\nu_\alpha+1}^{\infty} \chi_j(k) e^{(k)} \right\|_{\ell_p(\mathfrak{A}^\alpha)} < \epsilon_\alpha,$$

where $\chi_\alpha = \xi_{r_\alpha}$. Thus

$$\begin{aligned}
&\left\| \sum_{\alpha=0}^{r} \chi_\alpha \right\|_{\ell_p(\mathfrak{A}^\alpha)} \\
&\quad = \left\| \sum_{\alpha=0}^{r} \left(\sum_{k=0}^{\nu_{\alpha-1}} \chi_\alpha(k) e^{(k)} + \sum_{k=\nu_{\alpha-1}+1}^{\nu_j} \chi_\alpha(k) e^{(k)} + \sum_{k=\nu_\alpha+1}^{\infty} \chi_\alpha(k) e^{(k)} \right) \right\|_{\ell_p(\mathfrak{A}^\alpha)} \\
&\quad \leq \left\| \sum_{\alpha=0}^{r} \left(\sum_{k=\nu_{\alpha-1}+1}^{\nu_\alpha} \chi_\alpha(k) e^{(k)} \right) \right\|_{\ell_p(\mathfrak{A}^\alpha)} + 2 \sum_{\alpha=0}^{r} \epsilon_\alpha.
\end{aligned}$$

However, we see that $\|\xi\|_{\ell_p(\mathfrak{A}^\alpha)} \leq 1$. Thus, we have

$$\left\| \sum_{\alpha=0}^{r} \left(\sum_{k=\nu_{\alpha-1}+1}^{\nu_\alpha} \chi_\alpha(k) e^{(k)} \right) \right\|_{\ell_p(\mathfrak{A}^\alpha)}^p \leq (r+1).$$

So, we have

$$\left\| \sum_{\alpha=0}^{r} \sum_{k=\nu_{\alpha-1}+1}^{\nu_\alpha} \chi_\alpha(k) e^{(k)} \right\|_{\ell_p(\mathfrak{A}^\alpha)}^p \leq (r+1)^{\frac{1}{p}}.$$

By using $1 \leq (r+1)^{\frac{1}{p}}$ for all $r \in \mathbb{N}$ and $1 < p < \infty$, we have

$$\left\| \sum_{\alpha=0}^{r} \chi_\alpha \right\|_{\ell_p(\mathfrak{A}^\alpha)} \leq (r+1)^{\frac{1}{p}} + 1 \leq 2(r+1)^{\frac{1}{p}}.$$

Therefore, $\ell_p(\mathfrak{A}^\alpha)$ has Banach–Saks type p. □

Remark 3.7 The space $\ell_p(\mathfrak{A}^\alpha)$ is linearly isomorphic to ℓ_p and $R(\ell_p(\mathfrak{A}^\alpha)) = R(\ell_p) = 2^{\frac{1}{p}}$.

Theorem 3.8 *The space $\ell_p(\mathfrak{A}^\alpha)$ $(1 < p < \infty)$ has weak fixed-point property.*

Proof The proof is straightforward and follows from Remark 2.3 and 3.7. □

Theorem 3.9 *The Gurarii's modulus of convexity for $\ell_p(\mathfrak{A}^\alpha)$ $(p \geq 1)$ is*

$$\beta_{\ell_p(\mathfrak{A}^\alpha)}(\delta) \leq 1 - \left(1 - \left(\frac{\delta}{2}\right)^p\right)^{1/p},$$

where $0 \leq \delta \leq 2$.

Proof Let $\mathfrak{x} \in \ell_p(\mathfrak{A}^\alpha)$. Then

$$\|\mathfrak{x}\|_{\ell_p(\mathfrak{A}^\alpha)} = \|\mathfrak{A}^\alpha \mathfrak{x}\|_{\ell_p} = \left[\sum_{\mathfrak{n}=0}^{\infty} \left|\sum_{\mathfrak{v}|\mathfrak{n}} \frac{\nu^\alpha}{\rho^{(\alpha)}(\mathfrak{n})} \mathfrak{x}_{\mathfrak{v}}\right|^p\right]^{1/p}.$$

For $0 \leq \delta \leq 2$, define

$$x = \left(\{\mathfrak{A}^\alpha\}^{-1}\left(1 - \left(\frac{\delta}{2}\right)^p\right)^{1/p}, \{\mathfrak{A}^\alpha\}^{-1}\left(\frac{\delta}{2}\right), 0, 0, 0, \ldots\right)$$

and

$$y = \left(\{\mathfrak{A}^\alpha\}^{-1}\left(1 - \left(\frac{\delta}{2}\right)^p\right)^{1/p}, \{\mathfrak{A}^\alpha\}^{-1}\left(-\frac{\delta}{2}\right), 0, 0, 0, \ldots\right).$$

Then, $\|\mathfrak{A}^\alpha x\|_{\ell_p} = \|x\|_{\ell_p(\mathfrak{A}^\alpha)} = 1$ and $\left\|\mathfrak{A}^\alpha y\right\|_{\ell_p} = \|y\|_{\ell_p(\mathfrak{A}^\alpha)} = 1$. That is, $x, y \in S(\ell_p(\mathfrak{A}^\alpha))$ and $\left\|\mathfrak{A}^\alpha x - \mathfrak{A}^\alpha y\right\|_{\ell_p} = \|x - y\|_{\ell_p(\mathfrak{A}^\alpha)} = \delta$. For $0 \leq \delta \leq 1$,

$$\begin{aligned}&\|\alpha x + (1-\alpha)y\|^p_{\ell_p(\mathfrak{A}^\alpha)} = \|\alpha \mathfrak{A}^\alpha x + (1-\alpha)\mathfrak{A}^\alpha y\|^p_{\ell_p} \\ &= 1 - \left(\frac{\delta}{2}\right)^p + [2\alpha - 1]\left(\frac{\delta}{2}\right)^p.\end{aligned}$$

Hence

$$\inf_{0 \leq \delta \leq 1} \|\alpha x + (1-\alpha)y\|^p_{\ell_p(\mathfrak{A}^\alpha)} = 1 - \left(\frac{\delta}{2}\right)^p.$$

That is, for $p \geq 1$,

$$\beta_{\ell_p(\mathfrak{A}^\alpha)}(\delta) \leq 1 - \left(1 - \left(\frac{\delta}{2}\right)^p\right)^{1/p}.$$

Hence proved. □

Corollary 3.10 *(i) If $\delta = 2$, then $\beta_{\ell_p(\mathfrak{A}^\alpha)}(\delta) = 1$. So, $\ell_p(\mathfrak{A}^\alpha)$ is strictly convex. (ii) If $0 < \delta \leq 2$, then $0 < \beta_{\ell_p(\mathfrak{A}^\alpha)}(\delta) \leq 1$. So, $\ell_p(\mathfrak{A}^\alpha)$ is uniformly convex.*

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Data availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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