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On new extended cone b-metric-like spaces over a real Banach algebra

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Abstract

In this article, we introduce the structure of a new extended cone b-metric-like space over a Banach algebra. In this generalized space, we define the notion of generalized Reich-type mappings and prove a fixed point result. We also prove a fixed point result using $\alpha^* - \psi$ multivalued contraction and provide some of its consequences. Finally, we furnish with applications to establish the validity of our results.

Keywords: Cone metric space; Reich-type mappings; $\alpha^* - \psi$ multivalued contraction

1 Introduction

In 2007, Huang and Zhang [1] reinitiated the concept of a cone metric space over a Banach space as the generalization of metric spaces. They used an ordered Banach space $\mathbb{A}$ instead of $\mathbb{R}$ as the range set of metric d, that is, they used $d : X \times X \to \mathbb{A}$. They proved some fixed point results, including the Banach Contraction principle. After that, many researchers published many articles involving a cone b-metric. For more details, see [2, 3].

Later in 2016, Huang and Radenović [4] extended the idea of a cone metric space over Banach algebras to cone b-metric spaces over Banach algebras. They proved Banach and Kannan-type theorems for such spaces.

Recently, Du [5] noted that fixed point theorems in generalized cone metric spaces and in usual metric spaces are equivalent. In particular, the author proved that the celebrated fixed-point theorems of Banach, Kannan and Chatterjea in both topological vector spaces and cone metric can be derived easily from the usual metric space set-up, by a simple manipulation, namely, using a scalarization function.

Very recently, Liu and Xu [6] introduced a cone metric space over a Banach algebra and defined generalized Lipschitz mappings where the contractive coefficient is a vector instead of the usual real constant. They proved the existence of fixed points in such settings under the conditions that the underlying cones are normal cones. Furthermore, they gave an example to explain that the fixed point theorems in cone metric spaces over Banach algebras are not equivalent to those in metric spaces. Subsequently, Liu and Xu [6] and Huang and Radenović [4] omitted the normality of cones by using c-sequences.

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In 1993, Czerwik [7] introduced the notion of a b-metric space by replacing the triangular property of a metric space. Later in 2017, Kamran et al. [8] further extended the concept of b-metric spaces by introducing extended b-metric spaces. They introduced a function $\theta : X \times X \to [1,\infty)$ instead of b in the triangular inequality. They established a Banach-like contraction and proved some fixed-point results in such spaces. This shows that the class of such spaces is much more larger than the class of b-metric spaces and the class of metric spaces.

In 2017, Fernandez et al. [9] proposed cone b-metric-like spaces over Banach algebra, as a generalization of b-metric-like spaces and investigated the fixed point of generalized contractions and expansive mapping in the setting of cone b-metric-like spaces over Banach algebras with a nonnormal cone.

In 1971, Reich [10] introduced a new type of contraction which we call Reich contraction. It generalizes the two eminent contractions (i.e., Banach contraction and Kannan contraction). On the other hand, Samet et al. [11] in 2012 initiated the idea of α-admissible mappings in metric spaces. Recently, in 2015 and 2017, Malhotra et al. [12] used the idea of α-admissibility in cone metric spaces by using Banach algebras and proved Banach and Kannan type theorems. Later in 2017, Hussain et al. [13] used the concept of α-admissible mapping in cone b-metric spaces over Banach algebras and proved the Banach type results for such spaces.

Recently, Ullah et al. [14] presented the definition of an extended cone b-metric space over a Banach algebra and then proved some related fixed point results. Inspired by the concept of a new extended b-metric space and a cone b-metric-like space, we will here define a generalized space called a new extended cone b-metric-like space which generalizes many spaces.

Nadler [15] extended the Banach contraction principle from single-valued to multivalued contraction maps, considering the metric defined on closed and bounded subsets of a nonempty set M. Moreover, by changing the wide structure of underlying space, the fixed-point theory is widespread by introducing the notions of a b-metric space [16], a b-metric-like space [17], a partial metric space [18], a quasi b-metric space [19, 20], a cone rectangular metric space [21], new extended b-metric space [22], cone metric spaces [23, 24], cone b-metric spaces [25], extended b-metric spaces [26]. Some useful applications related to partial differential equations and fractional derivatives are presented in [27],[28] and [29]. In 2012, Samet et al. [30] introduced the notion of $\alpha - \psi$-contractive type mappings. Recently, Asl et al. [31] introduced the notion of $\alpha^* - \psi$-contractive mappings to extend the notion $\alpha - \psi-$ contractive mappings. Mehmood and Ahmad [32] generalized the result of [31] and obtained some fixed point results in cone metric spaces.

In this work, we initiate the setting of a new extended cone b-metric-like space over a Banach algebra and proved fixed point results for single valued mappings. We also obtain some fixed points results involving multi-valued mappings generalizing the result of Mehmood and Ahmad [32] in the context of new extended cone b-metric-like space over a Banach algebra. We also present some illustrated examples and applications.

2 Preliminaries

We need the following definitions and results, in the sequel.

Let $\mathbb{A}$ be a real Banach algebra. That is, $\mathbb{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties: $(\forall x, y, z \in \mathbb{A}$ and $\alpha \in \mathbb{R})$

(1) $(xy)z = x(yz)$,
(2) $x(y + z) = xy + yz$ and $(x + y)z = xz + yz$,
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$,
(4) $\|xy\| \leq \|x\|\|y\|$.

Here, we will assume that a Banach algebra has a unit e such that $ex = xe = e$, for unit e and for all $x \in \mathbb{A}$. An element $x \in \mathbb{A}$ is said to be invertible if there is an inverse element $y \in \mathbb{A}$ such that $xy = yx = e$.

We have the following proposition.

Proposition 2.1 [24] *Let $\mathbb{A}$ be a real Banach algebra with unit e and $x \in \mathbb{A}$. If the spectral radius verifies*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. Actually, we have

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Let $\mathbb{A}$ be a real Banach algebra and θ be its zero element. A subset $\mathbb{K}$ of $\mathbb{A}$ is called a cone if

(1) $\mathbb{K}$ is non-empty closed;
(2) $\alpha\mathbb{K} + \beta\mathbb{K} \subset \mathbb{K}$, for all non-negative real numbers α, β;
(3) $\mathbb{K}^2 = \mathbb{K} \cap \mathbb{K} \subset \mathbb{K}$;
(4) $\mathbb{K} \cap (-\mathbb{K}) = \theta$.

If the interior of $\mathbb{K}$, denoted by $int\mathbb{K}$, is nonempty, then the cone $\mathbb{K}$ is called a solid cone.

For a given cone $\mathbb{K} \subset \mathbb{A}$, we can define a partial ordering $\preceq$ with respect to $\mathbb{K}$ by $x \preceq y$ if and only if $y - x \in \mathbb{K}$. Here, $x < y$ will stand for $y - x \in int\mathbb{K}$.

We define another partial order $\ll$ on $\mathbb{A}$ by $x \ll y$ iff $x - y \in int\mathbb{K}$.

The cone $\mathbb{K}$ is called normal if there is a number $M > 0$ such that for all $x, y \in \mathbb{K}$,

$\theta \leq x \leq y \Longrightarrow \|x\| \leq M\|y\|$. The least positive number satisfying the above is called the normal constant of $\mathbb{K}$.

A complex number $\lambda \in \mathbb{C}$ is said to be spectral value of $x \in \mathbb{A}$, if $x - \lambda e$ is non-invertible in $\mathbb{A}$. The set of spectral values of $x \in \mathbb{A}$, denoted by $\sigma(x)$, is called the spectrum of x. The spectral radius $r(x)$ is defined as $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$.

Lemma 2.2 [33] *Let $\mathbb{A}$ be a real Banach algebra and $x, y \in \mathbb{A}$. If x and y commute, then*
1. $r(x + y) = r(x) + r(y)$,
2. $r(xy) = r(x)r(y)$.

Definition 2.3 [24] Let $\mathbb{A}$ be a real Banach algebra with a solid cone $\mathbb{K}$. A c-sequence is a sequence $\{x_i\}$ in $\mathbb{K}$ such that for every $c \in \mathbb{A}$ with $c \gg \theta$, there exists $n \in \mathbb{N}$ such that $x_i \ll c$, for all $i > n$.

Lemma 2.4 [4] *Let $\alpha, \beta \in \mathbb{K}$ be any two arbitrary vectors and $\{a_n\}$, $\{b_n\}$ be two c-sequences in a solid cone $\mathbb{K}$ of a Banach algebra $\mathbb{A}$. Then, $\{\alpha a_n + \beta b_n\}$ is a c-sequence.*

Definition 2.5 Let X be a nonempty set and $\mathbb{A}$ be a Banach algebra. Suppose that a mapping $d : X \times X \to \mathbb{A}$ satisfies for all $x, y, z \in X$:

(1) $\theta \preceq d(x,y)$ and $d(x,y) = \theta$ iff $x = y$

(2) $d(x,y) = d(y,x)$,

(3) $d(x,z) \preceq d(x,y) + d(y,z)$.

Then, d is called a cone metric on X and (X,d) is called cone metric space over Banach algebra.

The notion of a b-metric space was considered by Bakhtin and Czerwik [7] as a generalization of a metric space.

Definition 2.6 Let X be a nonempty set and let $b \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is called a b-metric if the following conditions are satisfied for all $x, y, z \in X$:

(1): $0 \leq d(x,y)$ and $d(x,y) = 0$ iff $x = y$,

(2): $d(x,y) = d(y,x)$,

(3): $d(x,z) \leq b[d(x,y) + d(y,z)]$.

Then d is called a b-metric on X and (X,d) is called a b-metric space.

Definition 2.7 [4] Let X be a nonempty set, $b \geq 1$ and $\mathbb{A}$ be a Banach algebra. Suppose that a mapping $d : X \times X \to \mathbb{A}$ satisfies for all $x, y, z \in X$:

(1): $\theta \preceq d(x,y)$ and $d(x,y) = \theta$ iff $x = y$;

(2): $d(x,y) = d(y,x)$;

(3): $d(x,z) \preceq b[d(x,y) + d(y,z)]$.

Then, d is called a cone b-metric on X and (X,d) is called a cone b-metric space over a Banach algebra.

Remark 2.8 If $b = 1$, then we say that d is a cone metric over a Banach algebra $\mathbb{A}$. So, we can say that a cone b-metric is the generalization of a cone metric.

Example 2.9 Consider the Banach algebra $\mathbb{A} = C[0,1]$ with unit element $e(t) = 1$ and supremum norm, where multiplication is defined pointwise as $(xy)(t) = x(t)y(t)$. Let $X = \mathbb{R}$ and $\mathbb{K} = \{x \in \mathbb{A} : x(t) > 0; \forall t \in [0,1]\}$. Define $d : X \times X \to \mathbb{A}$ by

$$d(x,y)(t) = \|x - y\|_{\infty}^{p} e^{t}$$

for all $x, y \in X$, where $p > 1$ is a constant then (X,d) is a cone b-metric space over Banach algebra with coefficient $s = 2^{p-1}$, but it is not a cone metric space over Banach algebra.

Definition 2.10 [22] Let X be a nonempty set, θ be a function defined as $\theta : X \times X \times X \to [1,\infty)$. A new extended b-metric is a function $d : X \times X \to [0,\infty)$ such that for all $x, y, z \in X$:

(E1) $d(x,y) \geq 0$,

(E2) $d(x,y) = 0 \Leftrightarrow x = y$,

(E3) $d(x,y) = d(y,x)$,

(E4) $d(x,z) \leq \theta(x,y,z)[d(x,y) + d(y,z)]$.

The pair (X,d) is called a new extended b-metric space.

Recently, Ullah at el. [14] introduced an extended cone b-metric space over a real Banach algebra as follows:

Definition 2.11 Let X be a nonempty set, $\mathbb{A}$ be a Banach algebra and $\phi : X \times X \to [1,\infty)$ be a given function. Suppose that a mapping $d : X \times X \to \mathbb{A}$ satisfies for all $x, y, z \in X$:

(1) $\theta \preceq d(x,y)$ and $d(x,y) = \theta$ iff $x = y$,

(2) $d(x,y) = d(y,x)$,

(3) $d(x,z) \preceq \phi(x,z)[d(x,y) + d(y,z)]$.

Then d is called an extended cone b-metric on X and (X,d) is called an extended cone b-metric space over Banach algebra.

We recall the definition of generalized α-admissible, α-regular, and generalized Reich-type mappings in the setting of extended cone b-metric spaces over a Banach algebra.

Definition 2.12 [14] Let (X,d) be an extended cone b-metric space over $\mathbb{A}$ with $\mathbb{K}$ an underlying solid cone. Given $\alpha : X \times X \to [0,\infty)$. Let $T : X \to X$ be a mapping.

(1) T is said to be a generalized α-admissible mapping if for $x, y \in X$, $\alpha(x,y) \geq \phi(x,y)$, we have $\alpha(Tx, Ty) \geq \phi(Tx, Ty)$.

(2) (X,d) is said to be α-regular if for any sequence $\{x_n\} \in X$ with $\alpha(x_n, x_{n+1}) \geq \phi(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x$, we have $\alpha(x_n, x) \geq \phi(x_n, x)$.

Definition 2.13 [14] Let $\mathbb{A}$ be a Banach algebra with underlying solid cone $\mathbb{K}$ and (X,d) be an extended cone b-metric space over $\mathbb{A}$. Given $\phi : X \times X \to [1,\infty)$ and $\alpha : X \times X \to [0,\infty)$. Then, the self map T on X is called a generalized Reich-type contraction if there exist $k_1, k_2, k_3 \in K$ such that $\forall x, y \in X$ with $\alpha(x,y) \geq \phi(x,y)$, we have

(1) $2\phi(x,y)r(k_1) + (\phi(x,y) + 1)r(k_2 + k_3) < 2$,

(2) $d(Tx, Ty) \preceq k_1 d(x,y) + k_2 d(x, Tx) + k_3 d(y, Ty)$.

3 Main results

Here, we generalize the extended cone b-metric space to a new extended cone b-metric-like space over a real Banach algebra $\mathbb{A}$. We also extend the generalized Reich-type contraction theorem in this space.

First, we define a new extended cone b-metric space.

Definition 3.1 Let X be a nonempty set, $\mathbb{A}$ be a Banach algebra and $\phi : X \times X \times X \to [1,\infty)$ be a mapping. Suppose that a mapping $d : X \times X \to \mathbb{A}$ satisfies, forall $x, y, z \in X$:

(1): $\theta \preceq d(x,y)$ and $d(x,y) = \theta$ iff $x = y$,

(2): $d(x,y) = d(y,x)$,

(3): $d(x,z) \preceq \phi(x,y,z)[d(x,y) + d(y,z)]$.

Then d is called a new extended cone b-metric on X and (X,d) is called a new extended cone b-metric space over a Banach algebra.

Example 3.2 Let $\mathbb{A} = C[0,1]$ be a usual unital Banach algebra with the supremum norm. Let $\mathbb{K} = \{f \in C[0,1] : f(t) \geq 0, \forall t \in [0,1]\}$ and $X = \mathbb{R}$. Define $d : X \times X \to \mathbb{A}$ by

$$d(x,y)(t) = (1 + |x| + |y| + |z|)|x - y|^p e^t,$$

for any $x, y, z \in X$ and for all $t \in [0,1]$. Conditions (1) and (2) are clearly satisfied. For (3) take x, y, z as arbitrary.

$$\begin{aligned} d(x,z)(t) &= ((1+|x|+|y|+|z|)|x-z|e^t \\ &\leq (1+|x|+|y|+|z|)(|x-y|+|y-z|)e^t \\ &\leq (1+|x|+|y|+|z|)(d(x,y)(t)+d(y,z)(t))e^t \\ &\leq (1+|x|+|y|+|z|)(d(x,y)+d(y,z))(t), \forall t \in [0,1] \end{aligned}$$

Therefore $d(x,z) \preceq (1+|x|+|y|+|z|)(d(x,y)+d(y,z))$. Since x, y, z are chosen arbitrarily, thus d is a new extended cone$b$-metric space over Banach Algebra with $\phi(x,y,z) = 1+|x|+|y|+|z|$, forall $x, y, z \in X$.

Definition 3.3 Let X be a nonempty set, $\mathbb{A}$ be a Banach algebra and $\phi : X \times X \times X \to [1,\infty)$ be a mapping. Suppose that a mapping $d : X \times X \to \mathbb{A}$ satisfies, for all $x, y, z \in X$:

(1) $\theta \preceq d(x,y)$ and $d(x,y) = \theta \implies x = y$,

(2) $d(x,y) = d(y,x)$,

(3) $d(x,z) \preceq \phi(x,y,z)[d(x,y)+d(y,z)]$.

Then, d is called a new extended cone b-metric-like on X and (X,d) is called a new extended cone b-metric-like space over a Banach algebra.

Example 3.4 Let $\mathbb{A}$ be a real Banach algebra, $X = \{0\} \cup \mathbb{N}$ and p be a positive even integer. Define a mapping $\phi : X \times X \times X \to [1,+\infty)$ by

$$\phi(x,y,z) = \begin{cases} |x|+|y|+|z|+1, & \text{if } x \neq y; \\ 1, & \text{if } x = y. \end{cases}$$

for all $x, y, z \in X$. Define $d : X \times X \to \mathbb{A}$ by

$$d(x,y)(t) = |x+y|^p e^t$$

for all $t \in [0,1]$. Then, (X,d) is a new extended cone b-metric-like space over Banach algebra. Conditions (1) and (2) are clearly satisfied. Now, to prove (3), we take $x \in X$ as arbitrary and see that

(i) If $x = y$, then (3) is clear.

(ii) If $x \neq y$, $x = z$, then

$$\begin{aligned} \phi(x,y,z)[d(x,z)+d(z,y)](t) &= (|x|+|y|+|z|+1)[|x+z|^p+|z+y|^p]e^t. \\ &\geq (|x|+|y|+|z|+1)|x+y|^p e^t \\ &\geq |x+y|^p e^t \\ &= d(x,y)(t). \end{aligned}$$

(iii) If $x \neq y$, $y \neq z$, $z \neq x$, then

$$\phi(x,y,z)[d(x,z)+d(z,y)](t) = (|x|+|y|+|z|+1)[|x+z|^p+|z+y|^p]e^t.$$

$$\begin{aligned}
&\geq \frac{(|x|+|y|+|z|+1)}{2}|x+z+z+y|^p e^t \\
&= \frac{(|x|+|y|+|z|+1)}{2}|x+2z+y|^p e^t \\
&\geq |x+2z+y|^p e^t \\
&\geq |x+y|^p e^t \\
&= d(x,y)(t).
\end{aligned}$$

Example 3.5 Consider the real Banach algebra $\mathbb{A}=\mathbb{R}^2$ with solid cone $\mathbb{K}=\{(a,b)\in\mathbb{R}^2 : ab\geq 0\}$. Let $\phi : X\times X\times X\to[1,\infty)$ be defined as $\phi(x,y,z)=x+y+z+1$ for $X=\{1,2,3\}$. Also, we define $d : X\times X\to\mathbb{A}$ by

$d(1,2)=d(2,1)=(80,80)$,

$d(1,3)=d(3,1)=(1000,1000)$,

$d(2,3)=d(3,2)=(600,600)$,

$d(1,1)=d(2,2)=d(3,3)=(0,0)=\theta$.

Clearly, the first two conditions are satisfied. For the third condition, let us consider

$$\begin{aligned}
&\phi(1,3,2)[d(1,3)+d(3,2)]-d(1,2) \\
&\quad=7[(1000,1000)+(600,600)]-(80,80)=(11120,11120)\in\mathbb{K}, \\
&\phi(1,2,3)[d(1,2)+d(2,3)]-d(1,2) \\
&\quad=7[(80,80)+(600,600)]-(1000,1000)=(3760,3760)\in\mathbb{K}, \\
&\phi(2,1,3)[d(2,1)+d(1,3)]-d(1,2) \\
&\quad=7[(80,80)+(1100,1000)]-(600,600)=(1660,1660)\in\mathbb{K}.
\end{aligned}$$

Therefore, for all $x,y,z\in X$, we have

$$d(x,z)\preceq\phi(x,y,z)[d(x,y)+d(y,z)].$$

Hence, (X,d) is a new extended cone b-metric -like space over $\mathbb{A}$.

Example 3.6 Consider the Banach Algebra $\mathbb{A}=C^1_{\mathbb{R}}[0,1]$ and cone $\mathbb{K}=\{x\in\mathbb{A} : x(t)\geq 0,\forall t\in[0,1]\}$. Let $\phi : X\times X\times X\to[1,\infty)$ be defined as $\phi(x,y,z)=x+y+z+1$ for $X=\{1,2,3\}$. Also we define $d : X\times X\to\mathbb{A}$ by;

$d(1,1)(t)=d(2,2)(t)=d(3,3)(t)=0$, $d(1,2)(t)=d(2,1)(t)=10e^t$, $d(2,3)(t)=d(3,2)(t)=40e^t$, $d(1,3)(t)=d(3,1)(t)=80e^t$,

for all $t\in[0,1]$ then (X,d) is new extended cone b-metric-like space over a Banach algebra but it is not a usual cone metric-like-space over Banach algebra. Since

$$d(1,2)+d(2,3)=10e^t+40e^t<80e^t=d(1,3).$$

Remark 3.7 (1) A new extended con b-metric-like space generalizes a new extended cone b-metric space since a new extended cone b-metric-like space on X satisfies all of the conditions of a new extended cone b-metric space, except that $d((x,x)$ need not be θ for $x\in X$.

(2) If $\phi(x,y,z) = \phi(x,z)$ then a new extended cone b-metric-like space over Banach algebra reduces to an extended con b-metric-like space over Banach algebra,

(3) If $\phi(x,z) = b \geq 1$, then an extended cone b-metric-like space over a Banach algebra reduces to a cone b-metric-like space over a Banach algebra.

Remark 3.8 In general, a new extended cone b-metric-like is not necessarily a continuous function, but here we will consider it as a continuous function.

Definition 3.9 Let (X,d) be a new extended cone b-metric-like over a Banach algebra $\mathbb{A}$ and $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is said to be: (i) a convergent sequence which converges to $x \in X$ if and only if

$$\lim_{n\to\infty} d(x_n,x) = d(x,x)$$

(ii) a θ-Cauchy sequence if $d(x_n,x_m)$ is a c-sequence that is for every $c \in \int \mathbb{K}$ (i.e. $\theta \ll c$), there exists a natural number N such that $d(x_n,x_m) \ll c$ for all $n,m \geq N$.

(iii) A θ-complete if every θ-Cauchy sequence in X is convergent to a point $x \in X$, that is

$$\lim_{n,m\to\infty} d(x_n,x_m) = \lim_{n\to\infty} d(x_n,x) = d(x,x) = \theta$$

Remark 3.10 (1) If $\{x_n\}$ converges to $x \in X$, then $d(x_n,x)$ and $d(x_n,x_{n+m})$ are c-sequences for any $m \in \mathbb{N}$.

(2) If $\|x_n\| \to 0$ as $n \to \infty$, then for any $c \gg \theta$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $x_n \ll c$.

We now define generalized α-admissible mapping and α-regular space in terms of a new extended cone b-metric-like spaces over Banach algebras.

Definition 3.11 Let (X,d) be a new extended cone b-metric-like space over $\mathbb{A}$ with $\mathbb{K}$ an underlying solid cone. Let $\alpha : X \times X \to [0,\infty)$ and $T : X \to X$ b mappings, then

(1) T is said to be generalized α-admissible mapping if for $x,y,z \in X$, $\alpha(x,y) \geq \phi(x,y,z)$ implies that $\alpha(Tx,Ty) \geq \phi(Tx,Ty,Tz)$.

(2) (X,d) is said to be α-regular if any sequence $\{x_n\} \in X$ with $\alpha(x_n,x_{n+1}) \geq \phi(x_n,x_{n+1},x_{n+2})$ for all $n \in \mathbb{N}$ and $x_n \to x$ implies that $\alpha(x_n,x) \geq \phi(x_n,x,x_{n+1})$.

We now define a generalized Reich-type contraction by using the new extended cone b-metric-like spaces over Banach algebras.

Definition 3.12 Let $\mathbb{A}$ be a Banach algebra with underlying solid cone $\mathbb{K}$ and (X,d) a new extended cone b-metric-like space over $\mathbb{A}$, and $\phi : X \times X \to [1,\infty)$ and $\alpha : X \times X \to [0,\infty)$ be mappings. Then, the self map T on X is called a generalized Reich-type contraction if there exists $k_1,k_2,k_3 \in K$ such that $\forall x,y \in X$ with $\alpha(x,y) \geq \phi(x,y,z)$:

(1). $2\phi(x,y)r(k_1) + (\phi(x,y)+1)r(k_2+k_3) < 2$ and for each $x_0 \in X$ with $x_n = T^n x_0$,

$$\lim_{n\to\infty} \phi(x_n,x_{n+1},x_{n+2}) < \frac{1}{\|q\|},$$

where $q = (2e-k)^{-1}(2k_1+k)$ for $k = k_2+k_3$.

(2) $d(Tx,Ty) \preceq k_1 d(x,y) + k_2 d(x,Tx) + k_3 d(y,Ty)$.

We now prove our main result.

Theorem 3.13 *Let (X,d) be a complete new extended cone b-metric-like space over a Banach algebra $\mathbb{A}$. Given $\alpha : X \times X \to [0,\infty)$ and an underlying solid cone $\mathbb{K}$. Let $T : X \to X$ be a generalized Reich-type contraction with vectors $k_1, k_2, k_3 \in \mathbb{K}$. Suppose that:*
(1) *T is generalised α-admissible;*
(2) *There exists an element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \phi(x_0, Tx_0, T^2x_0)$;*
(3) *(X,d) is regular or T is continuous.*
Then, T has a fixed point $x \in X$.

Proof Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \phi(x_0, Tx_0, T^2x_0)$.
Define

$$x_n = T^n(x_0).$$

We have

$$\alpha(x_0, x_1) \geq \phi(x_0, x_1, x_2).$$

Since T is generalized α-admissible, one writes

$$\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq \phi(x_1, x_2, x_3).$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq \phi(x_n, x_{n+1}, x_{n+2}).$$

Consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \preceq k_1 d(x_{n-1}, x_n) + k_2 d(x_{n-1}, Tx_{n-1}) + k_3 d(x_n, Tx_n).$$

That is,

$$(e - k_3) d(x_n, x_{n+1}) \preceq (k_1 + k_2) d(x_{n-1}, x_n). \tag{1}$$

Similarly,

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \preceq k_1 d(x_n, x_{n-1}) + k_2 d(x_n, Tx_n) + k_3 d(x_{n-1}, Tx_{n-1}).$$

Hence,

$$(e - k_2) d(x_{n+1}, x_n) \preceq (k_1 + k_3) d(x_{n-1}, x_n). \tag{2}$$

Adding (1) to (2), we get

$$(2e - k_2 - k_3) d(x_n, x_{n+1}) \preceq (2k_1 + k_2 + k_3) d(x_{n-1}, x_n).$$

Put $k = k_2 + k_3$. We have

$$(2e-k)d(x_n,x_{n+1}) \preceq (2k_1+k)d(x_{n-1},x_n). \tag{3}$$

Note that

$$2r(k) \leq [\phi(x_n,x_{n+1},x_{n+2})+1]r(k) \leq 2r(k_1)+[d(x_n,x_{n+1})+1]r(k) < 2,$$

then

$$r(k) < 1 < 2 \implies r(k) < 2.$$

Now, by Proposition 2.1, $2e-k$ is invertible and $(2e-k)^{-1} = \sum_{n=0}^{\infty} = \frac{k^n}{2^{n+1}}$, $r(2e-k)^{-1} < \frac{1}{2-r(k)}$. Hence, (3) becomes

$$d(x_n,x_{n+1}) \preceq qd(x_{n-1},x_n),$$

where, $q = (2e-k)^{-1}(2k_1+k)$. Thus, for all $n \in \mathbb{N}$, we have,

$$\begin{aligned} d(x_n,x_{n+1}) &\preceq qd(x_{n-1},x_n) \\ &\preceq q^2 d(x_{n-2},x_{n-1}) \\ &\quad .... \\ &\preceq q^n d(x_0,x_1). \end{aligned} \tag{4}$$

For $m > n$, we have

$$\begin{aligned} d(x_n,x_m) &\preceq \phi(x_n,x_{n+1},x_m)[d(x_n,x_{n+1})+d(x_{n+1},x_m)] \\ &\preceq \phi(x_n,x_{n+1},x_m)[d(x_n,x_{n+1})+\phi(x_{n+1},x_{n+2},x_m)[d(x_{n+1},x_{n+2})+d(x_{n+2},x_m)]] \\ &\preceq \phi(x_n,x_{n+1},x_m)d(x_n,x_{n+1})+\phi(x_n,x_{n+1},x_m)\phi(x_{n+1},x_{n+2},x_m)d(x_{n+1},x_{n+2}) \\ &\quad +\cdots+\phi(x_n,x_{n+1},x_m)\phi(x_{n+1},x_{n+2},x_m)...\phi(x_{m-2},x_{m-1},x_m)d(x_{m-1},x_m) \\ &\preceq \phi(x_n,x_{n+1},x_m)q^n d(x_0,x_1)+\phi(x_n,x_{n+1},x_m)\phi(x_{n+1},x_{n+2},x_m)q^{n+1}d(x_0,x_1) \\ &\quad +\cdots+\phi(x_n,x_{n+1},x_m)\phi(x_{n+1},x_{n+2},x_m)...\phi(x_{m-2},x_{m-1},x_m)q^{m-1}d(x_0,x_1) \\ &\preceq \phi(x_1,x_2,x_m)\phi(x_2,x_3,x_m)...\phi(x_n,x_{n+1},x_m)q^n d(x_0,x_1) \\ &\quad +\phi(x_1,x_2,x_m)\phi(x_2,x_3,x_m)...\phi(x_n,x_{n+1},x_m)\phi(x_{n+1},x_{n+2},x_m)q^{n+1}d(x_0,x_1) \\ &\quad +\cdots+\phi(x_1,x_2,x_m)\phi(x_2,x_3,x_m)...\phi(x_{m-2},x_{m-1},x_m)q^{m-1}d(x_0,x_1) \\ &\preceq d(x_0,x_1)[\phi(x_1,x_2,x_m)\phi(x_2,x_3,x_m)...\phi(x_n,x_{n+1},x_m)q^n \\ &\quad +\phi(x_1,x_2,x_m)\phi(x_2,x_3,x_m)...\phi(x_n,x_{n+1},x_m)\phi(x_{n+1},x_{n+2},x_m)q^{n+1} \\ &\quad +\cdots+\phi(x_1,x_2,x_m)\phi(x_2,x_3,x_m)...\phi(x_{m-2},x_{m-1},x_m)q^{m-1}] \\ &= d(x_0,x_1)[q^n\prod_{i=1}^{n}\phi(x_i,x_{i+1},x_m)+q^{n+1}\prod_{i=1}^{n+1}\phi(x_i,x_{i+1},x_m) \end{aligned}$$

$$+ \cdots + q^{m-1} \prod_{i=1}^{m-1} \phi(x_{i-1}, x_i, x_m)].$$

Since $\lim_{n,m\to\infty} \|q\| \phi(x_n, x_{n+1}, x_m) < 1$, the series $\sum_{n=1}^{\infty} q^n \Pi_{i=1}^{n} \phi(x_i, x_{i+1}, x_m)$ converges absolutely by ratio test.

Let $S = \sum_{n=1}^{\infty} q^n \Pi_{i=1}^{n} \phi(x_i, x_{i+1}, x_m)$ and $S_n = \sum_{j=1}^{n} q^j \Pi_{i=1}^{n} \phi(x_i, x_{i+1}, x_m)$.

Since $\mathbb{A}$ is a Banach algebra and the series S_n is absolutely convergent, it is convergent in $\mathbb{A}$. Thus,

$$\begin{aligned} S_{m-1} - S_n = q^n \prod_{i=1}^{n} \phi(x_i, x_{i+1}, x_m) + q^{n+1} \prod_{i=1}^{n+1} \phi(x_i, x_{i+1}, x_m) + \cdots \\ + q^{m-1} \prod_{i=1}^{m-1} \phi(x_{i-1}, x_i, x_m) \to \theta \end{aligned}$$

as $n, m \to \infty$, so is $d(x_0, x_1)[S_{m-1} - S_n]$.

For $m > n$, we have

$$d(x_n, x_m) \preceq d(x_0, x_1)[S_{m-1} - S_n].$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. We show that x is fixed point of T.

Suppose that T is continuous, then $x_{n+1} = Tx_n \to Tx$ as $n \to \infty$. Since limit of a sequence is unique, we have $Tx = x$.

If (X, d) is α-regular, then

$$\alpha(x_n, x) \geq \phi(x_n, x, x_{n+1}).$$

Now,

$$\begin{aligned} d(x, Tx) &\preceq \phi(x, x_{n+1}, Tx)[d(x, Tx_n) + d(Tx_n, Tx)] \\ &\preceq \phi(x, x_{n+1}, Tx)d(x, Tx_n) + \phi(x, x_{n+1}, Tx)[k_1 d(x_n, x) + k_2 d(x_n, Tx_n) + k_3 d(x, Tx)] \\ &\preceq \phi(x, x_{n+1}, Tx)d(x, Tx_n) + \phi(x, x_{n+1}, Tx)k_1 d(x_n, x) + \phi(x, x_{n+1}, Tx)k_2 d(x_n, Tx_n) \\ &\quad + \phi(x, x_{n+1}, Tx)k_3 d(x, Tx) \\ &\preceq \phi(x, x_{n+1}, Tx)d(x, x_{n+1}) + \phi(x, x_{n+1}, Tx)k_1 d(x_n, x) + \phi(x, x_{n+1}, Tx)k_3 d(x, Tx) \\ &\quad + \phi(x, x_{n+1}, Tx)k_2 \phi(x_n, x_{n+1}, x)[d(x_n, x) + d(x, x_{n+1})] \\ &\preceq [\phi(x, x_{n+1}, Tx)(e + \phi(x_n, x_{n+1}, x)k_2]d(x, x_{n+1}) + \phi(x, x_{n+1}, Tx)k_3 d(x, Tx) \\ &\quad + \phi(x, x_{n+1}, Tx)[k_1 + \phi(x_n, x_{n+1}, x)k_2]d(x_n, x), \end{aligned}$$

which further implies that

$$\begin{aligned} [e - k_3 \phi(x, x_{n+1}, Tx)]d(x, Tx) &\preceq [\phi(x, x_{n+1}, Tx)(e + \phi(x_n, x_{n+1}, x)k_2]d(x, x_{n+1}) \\ &\quad + \phi(x, x_{n+1}, Tx)[k_1 + \phi(x_n, x_{n+1}, x)k_2]d(x_n, x). \end{aligned} \tag{5}$$

Similarly,

$$\begin{aligned}
d(x,Tx) &\preceq \phi(x,x_{n+1},Tx)[d(x,Tx_n)+d(Tx_n,Tx)] \\
&\preceq \phi(x,x_{n+1},Tx)d(x,Tx_n)+\phi(x,x_{n+1},Tx)[k_1d(x,x_n)+k_2d(x,Tx)+k_3d(x_n,Tx_n)] \\
&\preceq \phi(x,x_{n+1},Tx)d(x,Tx_n)+\phi(x,x_{n+1},Tx)k_1d(x_n,x)+\phi(x,x_{n+1},Tx)k_2d(x,Tx) \\
&\quad +\phi(x,x_{n+1},Tx)k_3d(x_n,Tx_n) \\
&\preceq \phi(x,x_{n+1},Tx)d(x,Tx_n)+\phi(x,x_{n+1},Tx)k_1d(x_n,x)+\phi(x,x_{n+1},Tx)k_2d(x,Tx) \\
&\quad +\phi(x,x_{n+1},x)k_3[d(x_n,x)+d(x,x_{n+1}] \\
&\preceq \phi(x,x_{n+1},Tx)[e+\phi(x,x_{n+1},x)]d(x,x_{n+1})+\phi(x,x_{n+1},Tx)k_2d(x,Tx) \\
&\quad +\phi(x,x_{n+1},Tx)[k_1+\phi(x,x_{n+1},x)k_3]d(x_n,x),
\end{aligned}$$

which implies

$$\begin{aligned}
[e-k_2\phi(x,x_{n+1},Tx)]d(x,Tx) &\preceq [\phi(x,x_{n+1},Tx)(e+\phi(x_n,x_{n+1},x)k_3]d(x,x_{n+1}) \\
&\quad +\phi(x,x_{n+1},Tx)[k_1+\phi(x_n,x_{n+1},x)k_3]d(x_n,x). \qquad (6)
\end{aligned}$$

Adding (5) to (6), we have

$$\begin{aligned}
[2e-k_2\phi(x,x_{n+1},Tx)-\phi(x,Tx,x_{n+1})]d(x,Tx) &\preceq [\phi(x,x_{n+1},Tx)[2e+\phi(x,Tx,x_{n+1}) \\
&\quad +\phi(x_n,x_{n+1},x)k_3]d(x,x_{n+1}) \\
&\quad +\phi(x,x_{n+1},Tx)[2k_1+\phi(x,x_{n+1},Tx)k_2 \\
&\quad +\phi(x_n,x_{n+1},x)k_3]d(x_n,x).
\end{aligned}$$

That is,

$$\begin{aligned}
[2e-k\phi(x,x_{n+1},Tx)]d(x,Tx) &\preceq [\phi(x,x_{n+1},Tx)[2e+\phi(x,Tx,x_{n+1})k]d(x,x_{n+1}) \\
&\quad +\phi(x,x_{n+1},Tx)[2k_1+\phi(x,x_{n+1},Tx)k]d(x_n,x). \qquad (7)
\end{aligned}$$

Also,

$$\begin{aligned}
r[\phi(x,x_{n+1},Tx)k] &= \phi(x,x_{n+1},Tx)r(k) \\
&\leq 2\phi(x,x_{n+1},Tx)]r(k_1)+[\phi(x,x_{n+1},Tx)+1]r(k)<2
\end{aligned}$$

By Proposition 2.1 and (7), $2e-\phi(x,x_{n+1},Tx)k$ is invertible so Eq. (7) implies

$$\begin{aligned}
d(x,Tx) &\preceq [2e-k_2\phi(x,x_{n+1},Tx)]^{-1}[2e-k_2\phi(x,x_{n+1},Tx)(2e+k\phi(x,x_{n+1},Tx)) \\
&\quad d(x,x_{n+1})+\phi(x,x_{n+1},Tx)(2k_1+\phi(x,x_{n+1},Tx)k)d(x_n,x)].
\end{aligned}$$

By using Remark 3.10, the sequences $\{d(x_{n+1},x)\}$ and $\{d(x_n,x)\}$ are c-sequences. Hence, by Lemma 2.4, the sequence $\{\tau_1 d(x_{n+1},x)+\tau_2 d(x_n,x)\}$ is a c-sequence, where

$$\tau_1=[2e-k_2\phi(x,x_{n+1},Tx)]^{-1}\phi(x,x_{n+1},Tx)(2e+k\phi(x,x_{n+1},Tx))$$

and

$$\tau_2 = [2e - k\phi(x, x_{n+1}, Tx)]^{-1}\phi(x, x_{n+1}, Tx)(2k_1 + \phi(x, x_{n+1}, Tx)k)].$$

Therefore, for $c \in int(\mathbb{K})$, there exists $n_0 \in \mathbb{N}$ such that,

$$d(x, Tx) \preceq \tau_1 d(x_{n+1}, x) + \tau_1 d(x_n, x) \ll c.$$

This implies that $d(x, Tx) = \theta$. Hence, $Tx = x$. □

We prove the following result as a consequence of our main theorem.

Theorem 3.14 *Let (X, d) be a new extended cone b-metric-like space over a Banach algebra $\mathbb{A}$ with an associated cone $\mathbb{K}$. Let T be a self map on X such that*

$$d(Tx, Ty) \preceq kd(x, y),$$

where $k \in \mathbb{K}$ such that $r(k) < 1$ and for each $x_0 \in X$, $\lim_{n,m\to\infty} \phi(x_n, x_{n+1}, x_m) < \frac{1}{\|k\|}$, then there exists a unique fixed point $x \in X$. Furthermore, for each $x_0 \in X$, the iterative sequence $\{x_n\}$ defined by $x_n = Tx_{n-1} = T^n x_0$ converges to x.

Proof By taking $k_1 = k$, $k_2 = k_3 = \theta$ and $\alpha(x, y) = \phi(x, y, z)$, all the conditions of Theorem 3.13 are satisfied. Thus, there exists $x \in X$, which is a fixed point of T. T proves uniqueness, let there be $x^* \in X$ such that $Tx^* = x^*$, then we have

$$d(x, x^*) = d(Tx, Tx^*) \preceq kd(x, x^*).$$

Since $r(k) < 1$ so by Proposition 2.1, $e - k$ is invertible, and so $d(x, x^*) = \theta$. □

4 Applications

We will use the following lemma.

Lemma 4.1 *Let ψ be a Lebesgue measurable function defined on $[0, 1]$ with $k \geq 1$. Then, we have*

$$\left|\int_0^1 \psi(s)ds\right|^k \leq \int_0^1 |\psi(s)|^k ds.$$

Example 4.2 Let $\mathbb{A} = X = \mathbb{C}^1_{\mathbb{R}}[0, 1]$ be the space of all real-valued differentiable functions with continuous derivative defined on $[0, 1]$. If $\mathbb{K} = \{f \in \mathbb{A} : f(t) \geq 0, t \in [0, 1]\}$, then $\mathbb{K}$ is a non-normal cone. Let $X = [0, \infty)$ and $\phi : X \times X \times X \to [1, \infty)$ be defined as

$$\phi(x, y, z) = \max|x(t)| + \max|y(t)| + \max|z(t)| + 2^p.$$

Define $d : X \times X \to \mathbb{A}$ by

$$d(x, y)(t) = \|x - y\|^p_\infty e^t,$$

then d is a new extended cone b-metric-like.

Theorem 4.3 *Consider the following nonlinear integral equation*

$$f(t) = \int_0^1 F(t,f(x))ds, \tag{8}$$

where F satisfies the following:

(i) $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous*.

(ii) *There exists a constant $M \in [0, \frac{1}{2})$ such that, for each $f_0 \in X$, we have $M^p < \frac{1}{\lim_{n,m\to\infty} \phi(f_n, f_{n+1}, f_m)}$ and for all $t \in [0,1]$ and $x, y \in \mathbb{R}$,*

$$|F(t,x) - F(t,y)| \le M|x - y|,$$

then the equation (8) *has a unique solution in X.*

Proof Define $F : X \to X$ by

$$F(f(t)) = \int_0^1 F(t,f(s))ds. \tag{9}$$

By using Lemma 4.1, we have

$$\begin{aligned}
d(F(f), F(g))(t) &= e^t \|F(f) - F(g)\|_\infty^p \\
&= e^t \max_{x\in[0,1]} \|F(f)(x) - F(g)(x)\|^p \\
&= e^t \max_{x\in[0,1]} \left| \int_0^1 F(t,f(s))ds - \int_0^1 F(t,g(s))ds \right|^p \\
&= e^t \max_{x\in[0,1]} \left| \int_0^1 (F(t,f(s)) - F(t,g(s)))ds \right|^p \\
&\le e^t \max_{x\in[0,1]} \int_0^1 \left|(F(t,f(s)) - F(t,g(s)))\right|^p ds \\
&\le e^t \int_0^1 (M|f(s) - g(s)|)^p ds \\
&= e^t M^p \int_0^1 |f(s) - g(s)|^p ds \\
&\le e^t M^p \max_{x\in[0,1]} |f(s) - g(s)|^p ds \\
&= M^p d(f,g).
\end{aligned}$$

By taking $k = M^p e$, we have $r(k) < \|M^p e\| = M^p < \frac{1}{\lim_{n,m\to\infty} \phi(f_n, f_{n+1}, f_m)}$. Thus, all the conditions of Theorem 4.3 are satisfied, so there is a unique fixed point of F. □

5 Multivalued fixed-point theorems in new extended cone b-metric-like spaces

In order to extend fixed-point results to multi-valued mappings, we first need to define some concept of distance between two sets. One such notion is the notion of the Hausdorff metric, introduced by Hausdorff. It gives a concept of distance between two sets of a new extended cone b-metric-like space.

We denote by $N(X)(resp.B(X), CB(X))$ the set of nonempty (resp. bounded, sequentially closed and bounded) subset of a new extended cone b-metric-like space.

Let (X,d) be a new extended cone b-metric-like space over a Banach algebra $\mathbb{A}$. For $p \in X$ and $A, B \in CB(X)$, we define

$$s(p) = \{q \in \mathbb{A}; p \preceq q\},$$
$$s(p,B) = \bigcup_{b\in B} s(d(p,b)) = \bigcup_{b\in B}\{x \in \mathbb{A}; d(p,b) \preceq x\},$$

and

$$s(A,B) = \left(\bigcap_{a\in A} s(a,B)\right) \bigcap \left(\bigcap_{b\in B} s(b,A)\right).$$

Definition 5.1 Let Ψ be a family of nondecreasing functions, $\psi : \mathbb{K} \to \mathbb{K}$ such that

(i) $\psi(\theta) = \theta$ and $\theta < \psi(t) < t$ for $t \in \mathbb{K} \setminus \{\theta\}$,

(ii) $t \in \mathrm{Int}\mathbb{K}$ implies $t - \psi(t) \in \mathrm{Int}\mathbb{K}$,

(iii) $\lim_{n\to+\infty} \psi^n(t) = \theta$ for every$t \in \mathbb{K} \setminus \{\theta\}$.

where $\mathbb{K}$ be a solid(normal or non-normal)cone.

Lemma 5.2 *Let (X,d) be a new extended cone b-metric-like space with a cone $\mathbb{K}$. For $x, y \in X$ and $y \in B \subset X$, if $d(x,y) \preceq a$, then $a \in s(x,B)$.*

Lemma 5.3 *Let (X,d) be a new extended cone b-metric like space with a cone $\mathbb{K}$. Then, we have*

(i) Let $p, q \in \mathbb{A}$. If $p \preceq q$, then $s(q) \subset s(p)$;

(ii) Let $x \in X$ and $A \in CL(X)$. If $\theta \in s(x,A)$, then $x \in A$;

(iii) Let $q \in \mathbb{K}$ and let $A, B \in CL(X)$ and $a \in A$. If $q \in s(A,B)$, then $q \in s(a,B)$;

(iv) For all $q \in \mathbb{K}$ and $A, B \in CL(X)$, we have $q \in s(AB)$ if and only if there exist $a \in A$ and $b \in B$ such that $d(a,b) \preceq q$.

(v) Let $k \in \mathbb{K}$ and consider $\gamma \geq 0$, then $\gamma s(k) \subset s(\gamma k)$.

Remark 5.4 Let (X,d) be a new extended b-cone metric-like space. If $\mathbb{A} = \mathbb{R}$ and $\mathbb{K} = [0,+\infty)$, then (X,d) is a metric space. Moreover, for $A, B \in CL(X)$, $H(A,B) = infs(A,B)$ is the Hausdorff distance induced by d. Also, $s(\{x\},\{y\}) = s(d(x,y))$ for all $x, y \in X$.

Definition 5.5 Let X be a nonempty set, $T : X \to C(X)$ a multivalued mapping and $\alpha : X \times X \to [0,\infty)$. The mapping T is called α^*-admissible function if

$$\alpha(x,y) \geq 1 \implies \alpha^*(Tx,Ty) \geq 1$$

where $x, y \in X$.

Definition 5.6 ($\alpha^* - \psi$ Multivalued Contraction) Let (X,d) be a new extended cone b-metric-like space over a Banach algebra$\mathbb{A}$ with $\phi : X \times X \times X \to [1,\infty)$ and $\mathbb{K}$ be the suppressed solid cone. Then the multivalued mapping $T : X \to C(X)$ is said to be $\alpha^* - \psi$

multivalued contraction if there exist $\alpha : X \times X \to [0, +\infty)$ such that

$$\psi d(x,y) \in \alpha^*(Tx, Ty)s(Tx, Ty), \forall x, y \in \mathbb{X}. \tag{10}$$

where $\alpha^*(A,B) = \inf\{\alpha(a,b) : a \in A, b \in B\}$

We now present a fixed-point result using multivalued mappings in a new extended cone b-metric-like space over a Banach algebra.

Theorem 5.7 *Let (X,d) be a complete solid (normal or non-normal) cone metric space with cone $\mathbb{K}$, $\alpha : X \times X \to [0, +\infty)$ be a function, $\psi \in \Psi$ be a strictly increasing map and $T : X \to CB(X)$, F is α^*-admissible and α^*-ψ-contractive multifunction on X. Suppose that there exist $x_0, x_1 \in X$ such that $\alpha(x_0, x_1) \geq 1$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to u$ as $n \to +\infty$ then $\alpha(x_n, u) \geq 1$ for all n. Then, there exists a point x^* in X such that $x^* \in Tx^*$.*

Theorem 5.8 *Let (X,d) be a complete new extended cone b-metric-like space over a Banach algebra $\mathbb{A}$. Let $\alpha : X \times X \to [0,\infty)$ be a function, $\psi \in \Psi$ be a strictly increasing map and $\phi : X \times X \times X \to [1,\infty)$ be such that $\lim_{n,m\to\infty} \|\psi\|\phi(x_n, x_m, x_{n+1}) < 1$. Let the mapping $T : X \to CB(X)$ be an α^*-admissible and α^*-ψ-contractive multi-function on X. Suppose that there exist $x_0, x_1 \in X$ such that $\alpha(x_0, x_1) \geq 1$. Also assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to u$ as $n \to +\infty$ then $\alpha(x_n, u) \geq 1$ for all n. Then, there exists a point x^* in X such that $x^* \in Tx^*$.*

Proof Let $x_0 \in X$ and $x_1 \in Tx_0$ be arbitrary choosen. We may suppose that $x_0 \neq x_1$. Then

$$\psi(d(x_0, x_1)) \in \alpha^*(Tx_0, Tx_1)s(Tx_0, Tx_1).$$

By definition, we have

$$\psi(d(x_0, x_1)) \in \alpha^*(Tx_0, Tx_1)\left(\left(\bigcap_{x\in Tx_0} s(x, Tx_1)\right) \cap \left(\bigcap_{x'\in Tx_1} s\left(x', Tx_0\right)\right)\right)$$

$$\psi(d(x_0, x_1)) \in \alpha^*(Tx_0, Tx_1)\left(\bigcap_{x\in Tx_0} s(x, Tx_1)\right).$$

$$\psi(d(x_0, x_1)) \in \alpha^*(Tx_0, Tx_1)\, s(x, Tx_1) \textit{ for all } x \in Tx_0.$$

Since $x_1 \in Tx_0$, so we have

$$\psi(d(x_0, x_1)) \in \alpha^*(Tx_0, Tx_1)\, s(x_1, Tx_1)$$

$$\psi(d(x_0, x_1)) \in \alpha^*(Tx_0, Tx_1)\, s(x_1, Tx_1) = \alpha^*(Tx_0, Tx_1)\left(\bigcup_{x\in Tx_1} s(d(x_1, x))\right).$$

So there exists some $x_2 \in Tx_1$, such that

$$\psi(d(x_0, x_1)) \in \alpha^*(Tx_0, Tx_1)s(d(x_1, x_2)) = s\left(\alpha^*(Tx_0, Tx_1)d(x_1, x_2)\right)$$

which implies that

$$\alpha^*(Tx_0, Tx_1)d(x_1,x_2) \preceq \psi(d(x_0,x_1)).$$

Hence

$$0 \prec d(x_1,x_2) \preceq \alpha^*(Tx_0, Tx_1)d(x_1,x_2) \preceq \psi(d(x_0,x_1))$$

$x_1 \neq x_2$ and$\alpha(x_1,x_2) \geq 1$. Thus $\alpha^*(Tx_1, Tx_2) \geq$ and $d(x_1,x_2) \leq \psi(d(x_0,x_1))$.
If $x_2 \in Tx_2$, then x_2 is a fixed point of F. Assume that $x_2 \notin Tx_2$. Then

$$\psi(d(x_1,x_2)) \in \alpha^*(Tx_1, Tx_2)s(Tx_1, Tx_2).$$

By definition, we have

$$\psi(d(x_1,x_2)) \in \alpha^*(Tx_1, Tx_2)\left(\left(\bigcap_{x\in Tx_1} s\,(x, Tx_2)\right) \cap \left(\bigcap_{x'\in Tx_2} s\left(x', Tx_1\right)\right)\right)$$

$$\psi(d(x_1,x_2)) \in \alpha^*(Tx_1, Tx_2)\left(\bigcap_{x\in Tx_1} s\,(x, Tx_2)\right).$$

$$\psi(d(x_1,x_2)) \in \alpha^*(Tx_1, Tx_2)\, s\,(x, Tx_2) \textit{ for all } x \in Tx_1.$$

Since $x_2 \in Tx_1$, so we have

$$\psi(d(x_1,x_2)) \in \alpha^*(Tx_1, Tx_2)\, s\,(x_2, Tx_2)$$

$$\psi(d(x_1,x_2)) \in \alpha^*(Tx_1, Tx_2)\, s\,(x_2, Tx_2) = \alpha^*(Tx_1, Tx_2)\left(\bigcup_{x\in Tx_2} s\,(d\,(x_2, x))\right).$$

So there exists some $x_2 \in Tx_2$, such that

$$\psi(d(x_1,x_2)) \in \alpha^*(Tx_1, Tx_2)s\,(d(x_2,x_3)) = s\left(\alpha^*(Tx_1, Tx_2)d(x_2,x_3)\right).$$

That is

$$\alpha^*(Tx_1, Tx_2)d(x_2,x_3) \preceq \psi(d(x_1,x_2)).$$

Hence

$$0 \prec d(x_2,x_3) \preceq \alpha^*(Tx_1, Tx_2)d(x_2,x_3) \preceq \psi(d(x_1,x_2))$$

It is clear that $x_2 \neq x_3$ and $\alpha(x_2,x_3) \geq 1$. Thus $\alpha^*(Tx_2, Tx_3) \geq 1$ and $d(x_2,x_3) < \psi^2(\psi(d(x_0,x_1)))$.
If $x_3 \in Tx_3$, then x_3 is a fixed point of F. Assume that $x_3 \notin Tx_3$.

$$\psi(d(x_2,x_3)) \in \alpha^*(Tx_2, Tx_3)s(Tx_2, Tx_3).$$

By definition, we have

$$\psi(d(x_2,x_3)) \in \alpha^*(Tx_2,Tx_3)\left(\left(\bigcap_{x\in Tx_2} s(x,Tx_3)\right)\cap\left(\bigcap_{x'\in Tx_3} s\left(x',Tx_2\right)\right)\right)$$

$$\psi(d(x_2,x_3)) \in \alpha^*(Tx_2,Tx_3)\left(\bigcap_{x\in Tx_2} s(x,Tx_3)\right).$$

$$\psi(d(x_2,x_3)) \in \alpha^*(Tx_2,Tx_3)\, s(x,Tx_3) \textit{ for all } x \in Tx_2.$$

Since $x_3 \in Tx_2$, so we have

$$\psi(d(x_2,x_3)) \in \alpha^*(Tx_2,Tx_3)\, s(x_3,Tx_3)$$

$$\psi(d(x_2,x_3)) \in \alpha^*(Tx_2,Tx_3)\, s(x_3,Tx_3) = \alpha^*(Tx_2,Tx_3)\left(\bigcup_{x\in Tx_3} s(d(x_3,x))\right).$$

So there exists some $x_3 \in Tx_3$, such that

$$\psi(d(x_2,x_3)) \in \alpha^*(Tx_2,Tx_3)s(d(x_3,x_4)) = s\left(\alpha^*(Tx_2,Tx_3)d(x_3,x_4)\right).$$

That is

$$\alpha^*(Tx_2,Tx_3)d(x_3,x_4) \preceq \psi(d(x_2,x_3)).$$

Hence

$$0 \prec d(x_3,x_4) \preceq \alpha^*(Tx_2,Tx_3)d(x_3,x_4) \preceq \psi(d(x_2,x_3)).$$

It is clear that $x_3 \neq x_4$ and $\alpha(x_3,x_4) \geq 1$. Thus $\alpha^*(Tx_3,Tx_4) \geq 1$ and $d(x_3,x_4) < \psi^3(\psi(d(x_0,x_1)))$. By continuing this process, we obtain a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}, x_n \neq x_{n-1}$, $\alpha(x_n,x_{n+1}) \geq 1$ and $d(x_n,x_{n+1}) \preceq \psi^n(d(x_0,x_1))$ for all n.

For $m > n$, we have

$$\begin{aligned}
d(x_n,x_m) &\preceq \phi(x_n,x_{n+1},x_m)[d(x_n,x_{n+1}) + d(x_{n+1},x_m)]\\
&\preceq \phi(x_n,x_{n+1},x_m)[d(x_n,x_{n+1}) + \phi(x_{n+1},x_{n+2},x_m)[d(x_{n+1},x_{n+2}) + d(x_{n+2},x_m)]]\\
&\preceq \phi(x_n,x_{n+1},x_m)d(x_n,x_{n+1}) + \phi(x_n,x_{n+1},x_m)\phi(x_{n+1},x_{n+2},x_m)d(x_{n+1},x_{n+2})\\
&\quad + \cdots + \phi(x_n,x_{n+1},x_m)\phi(x_{n+1},x_{n+2},x_m)...\phi(x_{m-2},x_{m-1},x_m)d(x_{m-1},x_m)\\
&\preceq \phi(x_n,x_{n+1},x_m)\psi^n d(x_0,x_1) + \phi(x_n,x_{n+1},x_m)\phi(x_{n+1},x_{n+2},x_m)\psi^{n+1}d(x_0,x_1)\\
&\quad + \cdots + \phi(x_n,x_{n+1},x_m)\phi(x_{n+1},x_{n+2},x_m)...\phi(x_{m-2},x_{m-1},x_m)\psi^{m-1}d(x_0,x_1)\\
&\preceq \phi(x_1,x_2,x_m)\phi(x_2,x_3,x_m)...\phi(x_n,x_{n+1},x_m)k\psi^n d(x_0,x_1)\\
&\quad +\phi(x_1,x_2,x_m)\phi(x_2,x_3,x_m)...\phi(x_n,x_{n+1},x_m)\phi(x_{n+1},x_{n+2},x_m)\psi^{n+1}d(x_0,x_1)\\
&\quad + \cdots + \phi(x_1,x_2,x_m)\phi(x_2,x_3,x_m)...\phi(x_{m-2},x_{m-1},x_m)\psi^{m-1}d(x_0,x_1)\\
&\preceq d(x_0,x_1)[\phi(x_1,x_2,x_m)\phi(x_2,x_3,x_m)...\phi(x_n,x_{n+1},x_m)\psi^n
\end{aligned}$$

$$
\begin{aligned}
&+\phi(x_1,x_2,x_m)\phi(x_2,x_3,x_m)...\phi(x_n,x_{n+1},x_m)\phi(x_{n+1},x_{n+2},x_m)\psi^{n+1}\\
&+\cdots+\phi(x_1,x_2,x_m)\phi(x_2,x_3,x_m)...\phi(x_{m-2},x_{m-1},x_m)\psi^{m-1}]\\
=\ &d(x_0,x_1)[\psi^n\prod_{i=1}^{n}\phi(x_i,x_{i+1},x_m)+\psi^{n+1}\prod_{i=1}^{n+1}\phi(x_i,x_{i+1},x_m)+...\\
&+\psi^{m-1}\prod_{i=1}^{m-1}\phi(x_{i-1},x_i,x_m)].
\end{aligned}
$$

Since $\lim_{n,m\to\infty}\|\psi\|\phi(x_n,x_{n+1},x_m)<1$, for each $m\in\mathbb{N}$, the series $\sum_{n=1}^{\infty}\psi^n\Pi_{i=1}^{n}\phi(x_i,x_{i+1},x_m)$ converges absolutely by ratio test.

Let $S=\sum_{n=1}^{\infty}\psi^n\Pi_{i=1}^{n}\phi(x_i,x_{i+1},x_m)$ and $S_n=\sum_{j=1}^{n}\psi^j\Pi_{i=1}^{n}\phi(x_i,x_{i+1},x_m)$.

Since $\mathbb{A}$ is a Banach algebra and the series S_n is absolutely convergent, it is convergent in $\mathbb{A}$. Thus,

$$S_{m-1}-S_n=\psi^n\prod_{i=1}^{n}\phi(x_i,x_{i+1},x_m)+\psi^{n+1}\prod_{i=1}^{n+1}\phi(x_i,x_{i+1},x_m)+\cdots+\psi^{m-1}\prod_{i=1}^{m-1}\phi(x_{i-1}x_i,x_m)$$

converges to θ as$n,m\to\infty$, so is $d(x_0,x_1)[S_{m-1}-S_n]$.

Thus, for $m>n$, we have,

$$d(x_n,x_m)\preceq d(x_0,x_1)[S_{m-1}-S_n].$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, for $n\to\infty$, we have, $x_n\to u\in X$. Now, since $\{x_n\}$ is sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$,

Since $\alpha(x_n,u)\geq 1$ for all n and T is α^*-admissible, so $\alpha^*(Tx_n,Tu)\geq 1$ for all n. From (3.1), we have

$$\psi(d(x_n,u))\in\alpha^*(Tx_n,Tu)s(Tx_n,Tu)$$

or all $n\in\mathbb{N}$. By definition we obtain

$$\psi(d(x_n,u))\in\alpha^*(Tx_n,Tu)s(x_{n+1},Tu).$$

Thus, there exists $v_n\in Tu$ such that

$$\psi(d(x_n,u))\in\alpha^*(Tx_n,Tu)s(d(x_{n+1},v_n))=s(\alpha^*(Tx_n,Tu)d(x_{n+1},v_n))$$

which implies that

$$\alpha^*(Tx_n,Tu)d(x_{n+1},v_n)\preceq\psi(d(x_n,u)).$$

Hence

$$0\prec d(x_{n+1},v_n)\preceq\alpha^*(Tx_n,Tu)d(x_{n+1},v_n)\preceq\psi(d(x_n,u))\preceq d(x_n,u).$$

Since $x_n\to u$ as $n\to\infty$, for $c\in int\mathbb{K}$. Also, there exists $N\in\mathbb{N}$ such that $d(x_n,u)\ll\frac{c}{2\phi(u,x_n,x_{n+1})}$ and$d(x_{n+1},x)\ll\frac{c}{2\phi(u,x_n,x_{n+1})}$, for $N(c)=N\leq n$.

Now, by using triangular inequality, we have

$$\begin{aligned}
d(u,x_n) &\preceq \phi(u,x_n,x_{n+1})[d(u,x_{n+1})d(x_{n+1},x_n)] \\
&\preceq \phi(u,x_{n+1},Tx)d(u,x_{n+1}) + \phi(u,x_n,x_{n+1})d(x_{n+1},x_n)] \\
&\preceq \phi(u,x_{n+1},Tx)d(u,x_{n+1}) + \phi(u,x_n,x_{n+1})\alpha(x_n,u)d(x_{n+1},x_n)] \\
&\preceq \phi(u,x_{n+1},Tx)d(u,x_{n+1}) + \phi(u,x_n,x_{n+1})k(x_{n+1},x_n) \\
&\preceq \phi(u,x_{n+1},Tx)d(u,x_{n+1}) + \phi(u,x_n,x_{n+1})d(x_{n+1},x_n) \\
&\ll \phi(u,x_{n+1},Tx)\frac{c}{2\phi(u,x_n,x_{n+1})} + \phi(u,x_n,x_{n+1})\frac{c}{2\phi(u,x_n,x_{n+1})} \\
&\ll c.
\end{aligned}$$

Thus, $\lim_{n\to\infty} x_n = u$. Since Tu is closed, $u \in Tu$. This shows that x is a fixed point of T. This completes the proof. □

The consequences of the above theorem are presented here as follows:

Corollary 5.9 *Let (X,d) be a complete new extended cone b-metric-like solid(normal or non-normal) with cone $\mathbb{K}$. Let $\alpha : X \times X \to [0,+\infty)$ be a function and $\theta : X \times X \times X \to [1,\infty)$ be such that $\lim_{n,m\to\infty} \phi(x_n,x_m,x_{n+1}) < \frac{1}{\|k\|}$ and $T : X \to CB(X)$ is α^*-admissible and there exists a constant $k \in [0,1)$ such that*

$$kd(x,y) \in \alpha^*(Tx,Ty)s(Tx,Ty)$$

for all $x,y \in X$. Suppose that there exist $x_0,x_1 \in X$ such that $\alpha(x_0,x_1) \geq 1$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to u$ as $\to +\infty$, then $\alpha(x_n,u) \geq 1$ for all n. Then, there exists a point x^* in X such that $x^* \in Tx^*$.

Corollary 5.10 *Let (X,d) be a complete new extended cone b-metric-like solid(normal or non-normal)cone $\mathbb{K}$, $\psi \in \Psi$ be a strictly increasing map, $\phi : X \times X \times X \to [1,\infty)$ be such that $\lim_{n,m\to\infty} \phi(x_n,x_m,x_{n+1}) < 1$ and $T : X \to CB(X)$ be multivalued mapping such that*

$$\psi(d(x,y)) \in s(Tx,Ty)$$

for all $x,y \in X$. Then, there exists a point x^* in X such that $x^* \in Tx^*$.

Corollary 5.11 *Let (X,d) be a complete new extended cone b-metric-like solid (normal or non-normal) with cone $\mathbb{K}$, $\phi : X \times X \times X \to [1,\infty)$ be such that $\lim_{n,m\to\infty} \phi(x_n,x_m,x_{n+1}) < \frac{1}{\|k\|}$ and $T : X \to CB(X)$ be a multivalued mapping. If there exists a constant $k \in [0,1)$ such that*

$$kd(x,y) \in s(Tx,Ty)$$

for all $x,y \in X$. Then, there exists a point x^* in X such that $x^* \in Tx^*$.

Corollary 5.12 *Let (X,d) be a complete new extended cone b-metric-like solid(normal or non-normal) cone $\mathbb{K}$ metric space, $\alpha : X \times X \to [0,+\infty)$ be a function, $\psi \in \Psi$ be a strictly increasing map and $T : X \to CB(X)$ is α^*-admissible such that*

$$\alpha^*(Tx,Ty)H(Tx,Ty) \leq \psi(d(x,y))$$

for all $x,y \in X$. Suppose that there exist $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to u$ asn $\to +\infty$ then $\alpha(x_n,u) \geq 1$ for all n. Then, there exists a point x^* in X such that $x^* \in Tx^*$.

Corollary 5.13 *Let (X,d) be a complete new extended cone b-metric-like solid(normal or non-normal) with cone $\mathbb{K}$, $\alpha : X \times X \to [0,+\infty)$ be a function, $\phi : X \times X \times X \to [1,\infty)$ be such that $\lim_{n,m\to\infty} \phi(x_n,x_m,x_{n+1}) < \frac{1}{\|k\|}$ and $T : X \to CB(X)$ is α^*-admissible. If there exists a constant $k \in [0,1)$ such that*

$$\alpha^*(Tx,Ty)H(Tx,Ty) \leq kd(x,y)$$

for all $x,y \in X$. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to u$ asn $\to +\infty$ then $\alpha(x_n,u) \geq 1$ for all n. Then, there exists a point x^* in X such that $x^* \in Tx^*$.

Corollary 5.14 *Let (X,d) be a complete new extended cone b-metric-like solid (normal or non-normal) with cone $\mathbb{K}$, $\phi : X \times X \times X \to [1,\infty)$ be such that $\lim_{n,m\to\infty} \phi(x_n,x_m,x_{n+1}) < \frac{1}{\|k\|}$ and $T : X \to CB(X)$ be a multivalued mapping. If there exists a constant $k \in [0,1)$ such that*

$$H(Tx,Ty) \leq kd(x,y)$$

for all $x,y \in X$. Then, there exists a point x^* in X such that $x^* \in Tx^*$.

6 Applications

6.1 Non-linear Functional Integral Equations

Let $\mathbb{R}$ be the line and let $J = [0,1] \subset R$. Consider the nonlinear functional integral equation(in shortFIE),

$$x(t) = k(t,x(\mu(t))) + [f(t,x(\theta(t)))](q(t) + \int_0^{\sigma(t)} g(s,x(\eta(s)))ds) \tag{11}$$

for $t \in J$, where $q : J \to R$, μ, θ, σ, η, J and $f,g,k : J \times R \to R$ are continuous.

The FIE(11) and its special case have been studied in the literature very extensively via different fixed point methods for various aspects of the solutions. See Dhage[29], Dhage and O'Regan [30] and the references there in. Here we shall prove the existence of local solutions of FIE(11) by an application of the abstract fixed point theorem of previous section under some suitable conditions different from others.

Let $M(J,R)$ and $B(J,R)$ denote respectively the spaces of all measurable and bounded real-valued functions on J. We shall seek the solution of FIE(11) in the space $BM(J,R)$ of bounded and measurable real-valued functions on J. Define a norm

$$\| x \|_{BM} = \max_{t \in J} |x(t)|. \tag{12}$$

Clearly, $BM(J,R)$ is a Banach algebra with this maximum norm.

By a solution of FIE (11) we mean a function $x \in AC(J,R)$ that satisfies the equation(11), where $AC(J,R)$ is the space of all absolutely continuous real-valued functions on J.

Notice that

$AC(J,R) \subset BM(J,R)$.

Define,

$$T(x(t)) = k(t, x(\mu(t))) + [f(t, x(\theta(t)))](q(t) + \int_0^{\sigma(t)} g(s, x(\eta(s)))ds)$$

Then clearly fixed point of T is the solution of the integral equation(11). Now we apply Theorem 2.12 to prove that T has unique fixed point.

Now consider,

$$\begin{aligned}
&\left|T(x(t)) - T(y(t))\right| \\
&= \left| \begin{array}{c} \left\{ k(t,x(\mu(t))) + [f(t,x(\theta(t)))](q(t) + \int_0^{\sigma(t)} g(s,x(\eta(s)))ds) \right\} \\ - \left\{ k(t,y(\mu(t))) + [f(t,y(\theta(t)))](q(t) + \int_0^{\sigma(t)} g(s,y(\eta(s)))ds) \right\} \end{array} \right| \\
&= \left| \begin{array}{c} k(t,x(\mu(t))) + [f(t,x(\theta(t)))](q(t) + \int_0^{\sigma(t)} g(s,x(\eta(s)))ds) - k(t,y(\mu(t))) \\ -[f(t,y(\theta(t)))](q(t) + \int_0^{\sigma(t)} g(s,y(\eta(s)))ds) \end{array} \right| \\
&= \left| \begin{array}{c} k(t,x(\mu(t))) - k(t,y(\mu(t))) + [f(t,x(\theta(t))) - f(t,y(\theta(t)))] \\ \times \left\{ \int_0^{\sigma(t)} g(s,x(\eta(s)))ds) - \int_0^{\sigma(t)} g(s,y(\eta(s)))ds) \right\} \end{array} \right| \\
&\leq \left|k(t,x(\mu(t))) - k(t,y(\mu(t)))\right| + \left|f(t,x(\theta(t))) - f(t,y(\theta(t)))\right| \\
&\quad \times \left\{ \int_0^{\sigma(t)} \left|g(s,x(\eta(s))) - g(s,y(\eta(s)))\right| ds \right\}
\end{aligned}$$

here assume that

$(H1)$ k and g are Lipschitz with constants L_1 and L_2.

$(H2)$ f satisfies

$$\left|f(t,x(\theta(t))) - f(t,y(\theta(t)))\right| \leq \alpha_1(t) e^{-\|x+y\|}$$

therefore we have

$\leq L_1 \left|x(\mu(t)) - y(\mu(t))\right| + L_2 \left|x(\theta(t)) - y(\theta(t))\right| \sigma(t) |\alpha_1(t)| e^{-\|x+y\|}$

$\leq \|x - y\| \left[L_1 + L_2 \|\sigma\| \|\alpha_1\| e^{-\|x+y\|} \right]$

Provided that

$$\nabla := \left[L_1 + L_2 \|\sigma\| \|\alpha_1\| e^{-\|x+y\|} \right] < 1 \tag{H3}$$

This shows that T has a unique fixed point, which is solution of integral equation (11).

We sum up all above discussion in the form of the following theorem.

Theorem *If k,f and g satisfy $(H1)$, $(H2)$ and $(H3)$, then a unique solution of integral equation (11) exists.*

7 Conclusion

In this article, we introduced the notion of a new extended cone b-metric-like space over Banach algebra. Some examples and properties were given in this space. We proved some fixed point results using a generalized Reich-type mapping. Furthermore, we proved a multivalued result using $\alpha^* - \psi$ contraction. Some related consequences and useful applications have been furnished.

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References

1. Huang, L.G., Zhang, X.: Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. **332**(2), 1468–1476 (2007)
2. Shatanawi, W., Mitrović, Z.D., Hussain, N., Radenović, S.: On generalized Hardy–Rogers type α-admissible mappings in cone b-metric spaces over Banach algebras. Symmetry **12**(1), 81 (2020)
3. Jeyaraman, M., Suganthi, M., Shatanawi, W.: Common fixed point theorems in intuitionistic generalized fuzzy cone metric spaces. Mathematics **8**(8), 1212 (2020)
4. Huang, H., Radenović, S.: Some fixed point results of generalized Lipschitz mappings on cone b-metric spaces over Banach algebras. J. Comput. Anal. Appl. **20**(3) (2016)
5. Du, W.S., Karapinar, E.: A note on cone b-metric and its related results: generalizations or equivalence? Fixed Point Theory Appl. **1**, 210 (2013)
6. Liu, H., Xu, S.: Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings. Fixed Point Theory Appl. **1**, 320 (2013)
7. Czerwik, S.: Contraction mappings in b-metric spaces. Acta Math. Inform. Univ. Ostrav. **1**(1), 5–11 (1993)
8. Kamran, T., Samreen, M., Ain, Q.U.L.: A generalization of b-metric space and some fixed point theorems. Mathematics **5**(2), 19 (2017)
9. Fernandez, J., Malviya, N., Shukla, S.: Cone b-metric-like spaces over Banach algebra and fixed point theorems with the application. Asian J. Math. Comput. Res. **18**, 49–66 (2017)
10. Reich, S.: Some remarks concerning contraction mappings. Can. Math. Bull. **14**(1), 121–124 (1971)
11. Samet, B., Vetro, C., Vetro, P.: Fixed point theorem for $\alpha - \psi$ contractive type mappings. Nonlinear Anal. **75**, 2154–2165 (2012)
12. Malhotra, S.K., Sharma, J.B., Shukla, S.: Fixed points of generalized α-admissible mappings in cone metric spaces with Banach algebra. Int. J. Anal. Appl. **9**(1), 9–18 (2015)
13. Hussain, N., Al-Solami, A.M., Kutbi, M.A.: Fixed points of generalized α-admissible mappings in cone b-metric spaces over Banach algebra. J. Math. Anal. **8**, 89–97 (2017)

14. Ullah, W., Samreen, M., Kamran, T.: Fixed points of mappings on extended cone b-metric space over real Banach algebra. Filomat **36**(3), 853–868 (2022)
15. Nadler, S.B.: Multi-valued contraction mappings. Pac. J. Math. **30**(2), 475–488 (1969)
16. Bakhtin, I.: The contraction mapping principle in quasi metric spaces. Func. An., Gos. Ped. Inst. Unianowsk **30**, 26–37 (1989)
17. Zada, B., Riaz, U.: Some fixed point results on multiplicative (b)-metric-like spaces. Turk. J. Anal. Number Theory **4**(5), 118–131 (2016)
18. Aydi, H.: Some coupled fixed point results on partial metric spaces. Int. J. Math. Math. Sci. **2011** (2011)
19. Klin-eam, C., Suanoom, C.: Dislocated quasi-b-metric spaces and fixed point theorems for cyclic contractions. Fixed Point Theory Appl. **2015**(1), 74 (2015)
20. Rahman, M.U., Sarwar, M.: Dislocated quasi b-metric space and fixed point theorems. Electron. J. Math. Anal. Appl. **4**(2), 16–24 (2016)
21. Jleli, M., Samet, B.: The Kannan's fixed point theorem in a cone rectangular metric space. J. Nonlinear Sci. Appl. **2**(3), 161–167 (2009)
22. Samreen, M., Kamran, T., Postolache, M.: Extended b-metric space, extended b-comparison function and nonlinear contractions. Univ. Buchar. Sci. Bull. Ser. Appl. Math. Phys. **4**, 21–28 (2018)
23. Rezapour, S., Hamlbarani, R.: Some notes on the paper "cone metric spaces and fixed point theorems of contractive mappings". J. Math. Anal. Appl. **345**(2), 719–724 (2008)
24. Kadelburg, Z., Radenović, S.: A note on various types of cones and fixed point results in cone metric spaces. Asian J. Math. Appl. (2013)
25. Kutbi, M.A., Ahmad, J., Al-Mazrooei, A.E., Hussain, N.: Multivalued fixed point theorems in cone b-metric spaces over Banach algebra with applications. J. Math. Anal. **9**(1), 52–64 (2018)
26. Shatanawi, W., Abodayeh, K., Mukheimer, A.: Some fixed point theorems in extended b-metric spaces. Politeh. Univ. Buchar. Sci. Bull. Ser. Appl. Math. Phys. **80**, 71–78 (2018)
27. Ali, K.K., Mukheimer, A., Younis, J.A., Abd El Salam, M.A., Aydi, H.: Spectral collocation approach with shifted Chebyshev sixth-kind series approximation for generalized space fractional partial differential equations. AIMS Math. **7**(5), 8622–8644 (2022)
28. Dhaniya, S., Kumar, A., Khan, A., Abdeljawad, T., Aydi, H.: Existence results for Langevin equations involving generalized Liouville-Caputo fractional derivatives with non-local boundary conditions. Alex. Eng. J. **90**, 153–160 (2024)
29. Ain, T.Q., Khan, A., Abdeljawad, T., Gomez-Aguilar, J.F., Riaz, S.: Dynamical study of varicella-zoster virus model in sense of Mittag-Leffler kernel. Int. J. Biomath. **17**(3), 2350027 (2024)
30. Samet, B., Vetro, C., Vetro, P.: Fixed point theorem for $\alpha - \psi$ contractive type mappings. Nonlinear Anal. **75**, 2154–2165 (2012)
31. Asl, J.H., Rezapour, S., Shahzad, N.: On fixed points of α-ψ-contractive multifunctions. Fixed Point Theory Appl. **2012**, 212 (2012)
32. Mehmood, N., Ahmad, N.: Banach Contraction Principle: A Centrurial Journey.
33. Rudin, W.: Functional Analysis. International Series in Pure and Applied Mathematics (1991)

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