Al-Oushoush et al. *Journal of Inequalities and Applications* (2024) 2024:122
https://doi.org/10.1186/s13660-024-03202-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Some new dynamic inequalities for B-monotone functions with respect to time scales nabla calculus

Nizar Kh. Al-Oushoush[1*], Laith E. Azar[2], Essam Awwad[3,4], Mario Krnić[5] and Ahmed I. Saied[4,6]

*Correspondence:
aloushoush@bau.edu.jo
[1]Department of Mathematics, Faculty of Science, Al-Balqa Applied University, P.O. Box 19117, Salt, Jordan
Full list of author information is available at the end of the article

Abstract

Motivated by certain results known from the literature, in this paper we give several new dynamic inequalities for B-monotone functions with respect to time scales nabla calculus. If the time scale represents the set of real numbers, our results reduce to integral inequalities known from the literature. On the other hand, in the setting of positive integers, we obtain new discrete inequalities for B-monotone sequences.

Mathematics Subject Classification: Primary 26D10; 26D15; secondary 34N05; 47B38; 39A12

Keywords: B-monotone function; Time scales; Inequality; Concave function

1 Introduction

In [16] Heinig and Maligranda showed that for $\omega \geq 0$, which is decreasing on (ι, ϱ), and $\Theta \geq 0$, which is increasing on (ι, ϱ), with $\Theta(\iota) = 0$, $-\infty \leq \iota < \varrho \leq \infty$, the inequality

$$\int_{\iota}^{\varrho} \omega(\varkappa) d\Theta(\varkappa) \leq \left(\int_{\iota}^{\varrho} \omega^{\gamma}(\varkappa) d\left[\Theta^{\gamma}(\varkappa)\right] \right)^{\frac{1}{\gamma}}, \tag{1}$$

holds for every $\gamma \in (0,1]$, while for $1 \leq \gamma < \infty$ the inequality is reversed. It has been also showed that if ω is increasing on (ι, ϱ) and Θ is decreasing on (ι, ϱ) with $\Theta(\varrho) = 0$, then the inequality

$$\int_{\iota}^{\varrho} \omega(\varkappa) d\left[-\Theta(\varkappa)\right] \leq \left(\int_{\iota}^{\varrho} \omega^{\gamma}(\varkappa) d\left[-\Theta^{\gamma}(\varkappa)\right] \right)^{\frac{1}{\gamma}}, \tag{2}$$

holds, where $\gamma \in (0,1]$. In the same paper, Heinig and Maligranda showed that if $0 < p \leq q < \infty$, and u, v are nonnegative functions, then $\exists D > 0$ such that

$$\left[\int_{0}^{\infty} u(\varkappa) \omega^{q}(\varkappa) d\varkappa \right]^{\frac{1}{q}} \leq D \left[\int_{0}^{\infty} v(\varkappa) \omega^{p}(\varkappa) d\varkappa \right]^{\frac{1}{p}}, \tag{3}$$

for a nonnegative decreasing function ω iff

$$\left[\int_0^{\aleph} u(\varkappa)d\varkappa\right]^{\frac{1}{q}} \leq D\left[\int_0^{\aleph} v(\varkappa)d\varkappa\right]^{\frac{1}{p}}, \quad \forall\aleph > 0.$$

In addition, they showed that inequality (3) holds for all nonnegative increasing functions ω and $0 < p \leq q < \infty$ iff

$$\left[\int_{\aleph}^{\infty} u(\varkappa)d\varkappa\right]^{\frac{1}{q}} \leq D\left[\int_{\aleph}^{\infty} v(\varkappa)d\varkappa\right]^{\frac{1}{p}}, \quad \forall\aleph > 0.$$

In 1997, Pečarić et al. [19] extended the results from [16] to the case of B-monotone functions. Recall that if $s \leq \aleph$ implies $\omega(\aleph) \leq B\omega(s)$ for all $s, \aleph \in [\iota, \varrho]$, then ω is a B-decreasing function on $[\iota, \varrho]$. On the other hand, if $s \leq \aleph$ implies $\omega(s) \leq B\omega(\aleph)$, $s, \aleph \in [\iota, \varrho]$, then ω is B-increasing. Clearly, if $B = 1$, then the notion of B-monotonicity reduces to the usual monotonicity.

Pečarić et al. [19] showed that if $\varpi : [0, \infty) \to \mathbb{R}$ is a concave, nonnegative, and differentiable function such that $\varpi(0) = 0$, ω is B-decreasing, $B \geq 1$, and Θ is increasing on $[\iota, \varrho]$, such that $\Theta(\iota) = 0$, then the inequality

$$\varpi\left(B\int_{\iota}^{\varrho} \omega(\varkappa)d\Theta(\varkappa)\right) \leq B\int_{\iota}^{\varrho} \varpi'\left[\omega(\varkappa)\Theta(\varkappa)\right]\left(\omega(\varkappa)d\Theta(\varkappa)\right), \tag{4}$$

holds. In addition, if ω is B-increasing with $B \geq 1$ and Θ is increasing on $[\iota, \varrho]$ such that $\Theta(\iota) = 0$, then the inequality

$$\varpi\left(\frac{1}{B}\int_{\iota}^{\varrho} \omega(\varkappa)d\Theta(\varkappa)\right) \geq \frac{1}{C}\int_{\iota}^{\varrho} \varpi'\left[\omega(\varkappa)\Theta(\varkappa)\right]\omega(\varkappa)d\Theta(\varkappa), \tag{5}$$

holds. Analogous results have been derived for the case of a decreasing function Θ. More precisely, if ω is B-increasing, $B \geq 1$, and Θ is decreasing on $[\iota, \varrho]$ with $\Theta(\varrho) = 0$, then the inequality

$$\varpi\left(B\int_{\iota}^{\varrho} \omega(\varkappa)d\left[-\Theta(\varkappa)\right]\right) \leq B\int_{\iota}^{\varrho} \varpi'\left[\omega(\varkappa)\Theta(\varkappa)\right]\omega(\varkappa)d\left[-\Theta(\varkappa)\right], \tag{6}$$

holds, while for a B-decreasing function ω, $B \geq 1$, and a decreasing function Θ such that $\Theta(\varrho) = 0$, one has the inequality

$$\varpi\left(\frac{1}{B}\int_{\iota}^{\varrho} \omega(\varkappa)d\left[-\Theta(\varkappa)\right]\right) \geq \frac{1}{B}\int_{\iota}^{\varrho} \varpi'\left[\omega(\varkappa)\Theta(\varkappa)\right]\omega(\varkappa)d\left[-\Theta(\varkappa)\right]. \tag{7}$$

Hilger in his seminal work [17] introduced the time scale theory and established a new dynamic inequality with a general domain called a time scale $\mathbb{T}$. For the dynamic inequalities, see monographs [1, 2, 9], as well as papers [3, 5–8, 11–14, 18, 20–27].

In [24], Saker et al. introduced the inequalities (4)–(7) on time scales. More precisely, the time scale variant of (4) asserts that if ω is B-decreasing on $[\iota, \varrho]_{\mathbb{T}}$, $B \geq 1$, and Θ is increasing on $[\iota, \varrho]_{\mathbb{T}}$ with $\Theta(\iota) = 0$, then

$$\varpi\left(B\int_{\iota}^{\varrho} \omega(\varkappa)\Theta^{\Delta}(\varkappa)\Delta\varkappa\right) \leq B\int_{\iota}^{\varrho} \omega(\varkappa)\Theta^{\Delta}(\varkappa)\varpi'\left[\omega(\varkappa)\Theta(\varkappa)\right]\Delta\varkappa, \tag{8}$$

where $[\iota,\varrho]_{\mathbb{T}}$ stands for $[\iota,\varrho]\cap\mathbb{T}$. Similarly, the time scale analogue of (5) asserts that if ω is B-increasing on $[\iota,\varrho]_{\mathbb{T}}$, $B\geq 1$, and Θ is increasing on $[\iota,\varrho]_{\mathbb{T}}$ with $\Theta(\iota)=0$, then

$$\varpi\left(\frac{1}{B}\int_{\iota}^{\varrho}\omega(\varkappa)\Theta^{\Delta}(\varkappa)\Delta x\right)\geq\frac{1}{B}\int_{\iota}^{\varrho}\omega(\varkappa)\Theta^{\Delta}(\varkappa)\varpi'\left[\omega^{\sigma}(\varkappa)\Theta^{\sigma}(\varkappa)\right]\Delta\varkappa. \tag{9}$$

In [24], one can also find time scale variants of the inequalities (6) and (7).

This paper seeks to extend inequalities (1), (2), and (4)–(7) via the time scales nabla calculus. We will introduce the difference between the results (8) and (9) on delta calculus in [24] and the nabla calculus. In particular, we will employ the corresponding chain rule formula, as well as the basic properties of B-monotone functions. Of course, the results that we will derive contain classical results and the usual monotonicity.

The paper is structured as follows: Sect. 2 involves the basic definitions and properties on time scales that will be important in proving our main results. Section 3 contains the extended inequalities (1), (2), and (4)–(7) via the time scales nabla calculus. It turns out that if $\mathbb{T}=\mathbb{R}$, our results reduce to inequalities presented in this Introduction, while for $\mathbb{T}=\mathbb{N}$, the obtained relations are new.

2 Basic lemmas

In this section, we present the basic notions and facts of nabla calculus. For a comprehensive insight of the time scales calculus, the reader is referred to the monographs [9, 10] by Bohner and Peterson.

The product and the quotient of nabla differentiable functions are also nabla differentiable functions. More precisely, if ω, $\Theta:\mathbb{T}\to\mathbb{R}$ are nabla differentiable at $\aleph\in\mathbb{T}$, then

$$(\omega\Theta)^{\nabla}(\aleph)=\omega^{\nabla}(\aleph)\Theta(\aleph)+\omega^{\rho}(\aleph)\Theta^{\nabla}(\aleph)=\omega(\aleph)\Theta^{\nabla}(\aleph)+\omega^{\nabla}(\aleph)\Theta^{\rho}(\aleph)$$

and

$$\left(\frac{\omega}{\Theta}\right)^{\nabla}(\aleph)=\frac{\omega^{\nabla}(\aleph)\Theta(\aleph)-\omega(\aleph)\Theta^{\nabla}(\aleph)}{\Theta(\aleph)\Theta^{\rho}(\aleph)}$$

provided that $\Theta(\aleph)\Theta^{\rho}(\aleph)\neq 0$. For the proofs of the above properties, the reader is referred to [4]. The function $\omega:\mathbb{T}\to\mathbb{R}$ is increasing (decreasing) if $\omega^{\nabla}(\varkappa)>0$, $\varkappa\in\mathbb{T}$ ($\omega^{\nabla}(\varkappa)<0$, $\varkappa\in\mathbb{T}$).

On the other hand, the chain rule for the nabla derivative asserts that if $\omega:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and $\Theta:\mathbb{T}\to\mathbb{R}$ is continuous and nabla differentiable, then $\omega\circ\Theta:\mathbb{T}\to\mathbb{R}$ is nabla differentiable and there exists $d\in[\rho(\aleph),\aleph]$ such that

$$(\omega\circ\Theta)^{\nabla}(t)=\omega'(\Theta(d))\,\Theta^{\nabla}(\aleph). \tag{10}$$

Relation (10) has been established in [15].

The properties of integration in nabla calculus state that if ι, ϱ, $c\in\mathbb{T}$, α, $\beta\in\mathbb{R}$ and ω, $\Theta:\mathbb{T}\to\mathbb{R}$ are ld-continuous, then

(i) $\int_{\iota}^{\varrho}[\alpha\omega(\aleph)+\beta\Theta(\aleph)]\nabla\aleph=\alpha\int_{\iota}^{\varrho}\omega(\aleph)\nabla\aleph+\beta\int_{\iota}^{\varrho}\Theta(\aleph)\nabla\aleph$,

(ii) $\int_{\iota}^{\varrho}\omega(\aleph)\nabla\aleph=-\int_{\varrho}^{\iota}\omega(\aleph)\nabla\aleph$,

(iii) $\int_{\iota}^{c}\omega(\aleph)\nabla\aleph=\int_{\iota}^{\varrho}\omega(\aleph)\nabla\aleph+\int_{\varrho}^{c}\omega(\aleph)\nabla\aleph$;

see [9, Theorem 8.47]. The properties of integration transformation on time scales to the analogue in continuous and discrete forms state that for $\iota, \varrho \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$, which is ld-continuous, the following properties hold:

(i) If $\mathbb{T} = \mathbb{R}$, then $\int_\iota^\varrho f(\aleph)\nabla\aleph = \int_\iota^\varrho f(\aleph)d\aleph$.

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$\int_\iota^\varrho f(\aleph)\nabla\aleph = \begin{cases} \sum_{t=\iota+1}^{\varrho} f(\aleph) & \text{if } \iota < \varrho, \\ 0 & \text{if } \iota = \varrho, \\ -\sum_{t=\varrho+1}^{\iota} f(\aleph) & \text{if } \iota < \varrho; \end{cases}$$

see [9, Theorem 8.48].

In addition, the integration by parts formula in nabla calculus asserts that

$$\int_\iota^\varrho \omega(\aleph)\Theta^\nabla(\aleph)\nabla\aleph = \omega(\aleph)\Theta(\aleph)|_\iota^\varrho - \int_\iota^\varrho \omega^\nabla(\aleph)\Theta^\rho(\aleph)\nabla\aleph,$$

provided $\omega, \Theta : \mathbb{T} \to \mathbb{R}$ are ld-continuous functions, see [9, Theorem 8.47].

3 Main results

If nothing else is explicitly stated, we assume that all the considered functions are nonnegative, ld-continuous, ∇-differentiable, and locally ∇-integrable on $[\iota, \infty)_\mathbb{T}$. Furthermore, all the considered integrals are well defined.

In addition, throughout this section, $\varpi : [0, \infty) \to \mathbb{R}$ is a nonnegative, concave, and differentiable function such that $\varpi(0) = 0$.

Now, we are ready to state and prove our main results. Our first results is an extension of inequalities (1) and (4) in the setting of time scales nabla calculus.

Theorem 1 *Let $\mathbb{T}$ be a time scale with $\iota, \varrho \in \mathbb{T}$, and let ω be B-decreasing on $[\iota, \varrho]_\mathbb{T}$, where $B \geq 1$. If Θ is increasing on $[\iota, \varrho]_\mathbb{T}$ and $\Theta(\iota) = 0$, then the following inequality holds*:

$$\varpi\left(B\int_\iota^\varrho \omega(\varkappa)\Theta^\nabla(\varkappa)\nabla\varkappa\right) \leq B\int_\iota^\varrho \omega(\varkappa)\Theta^\nabla(\varkappa)\varpi'\left[\omega^\rho(\varkappa)\Theta^\rho(\varkappa)\right]\nabla\varkappa. \tag{11}$$

Proof Denoting

$$\Upsilon(\aleph) := \varpi\left(B\int_\iota^\aleph \omega(\varkappa)\Theta^\nabla(\varkappa)\nabla\varkappa\right) - B\int_\iota^\aleph \omega(\varkappa)\Theta^\nabla(\varkappa)\varpi'\left[\omega^\rho(\varkappa)\Theta^\rho(\varkappa)\right]\nabla\varkappa$$

and

$$\beth(\aleph) := B\int_\iota^\aleph \omega(\varkappa)\Theta^\nabla(\varkappa)\nabla\varkappa,$$

we get

$$\Upsilon(\aleph) := \varpi(\beth(\aleph)) - B\int_\iota^\aleph \omega(\varkappa)\Theta^\nabla(\varkappa)\varpi'\left[\omega^\rho(\varkappa)\Theta^\rho(\varkappa)\right]\nabla\varkappa.$$

Now, since ω is B-decreasing, it follows that $\omega(\aleph) \leq B\omega(\varkappa)$, for $\aleph \geq x$. Moreover, taking into account that Θ is increasing and $\Theta(\iota) = 0$, we obtain

$$\begin{aligned}\beth(\aleph) = \int_{\iota}^{\aleph} B\omega(\varkappa)\Theta^{\nabla}(\varkappa)\nabla\varkappa \geq \int_{\iota}^{\aleph} \omega(\aleph)\Theta^{\nabla}(\varkappa)\nabla\varkappa = \omega(\aleph)\int_{\iota}^{\aleph} \Theta^{\nabla}(\varkappa)\nabla\varkappa \\ = \omega(\aleph)\left[\Theta(\aleph) - \Theta(\iota)\right] = \omega(\aleph)\Theta(\aleph).\end{aligned} \tag{12}$$

On the other hand, applying (10) to $\varpi(\beth(\aleph))$, we see that there exists $\zeta \in [\rho(\aleph), \aleph]$ such that

$$\varpi^{\nabla}(\beth(\aleph)) = \varpi'(\beth(\zeta))\beth^{\nabla}(\aleph). \tag{13}$$

Now, since Θ is an increasing function, the definition of $\beth$ yields the relation

$$\beth^{\nabla}(\aleph) = B\omega(\aleph)\Theta^{\nabla}(\aleph) \geq 0, \tag{14}$$

which implies that $\beth$ is increasing on $[\iota, \varrho]_{\mathbb{T}}$ and, consequently, $\beth(\zeta) \geq \beth^{\rho}(\aleph)$, for $\zeta \geq \rho(\aleph)$. In addition, since ϖ is concave on $[0, \infty)$, it follows that ϖ' is decreasing on $[0, \infty)$ and so

$$\varpi'(\beth(\zeta)) \leq \varpi'(\beth^{\rho}(\aleph)). \tag{15}$$

Taking into account (13)–(15), we arrive at the relation

$$\varpi^{\nabla}(\beth(\aleph)) \leq B\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\beth^{\rho}(\aleph)). \tag{16}$$

Clearly, relation (12) implies the inequality $\varpi'(\beth^{\rho}(\aleph)) \leq \varpi'(\omega^{\rho}(\aleph)\Theta^{\rho}(\aleph))$, that is,

$$B\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\beth^{\rho}(\aleph)) \leq B\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\omega^{\rho}(\aleph)\Theta^{\rho}(\aleph)),$$

since ω is nonnegative and Θ is increasing. Hence, taking into account (16), we have that

$$\varpi^{\nabla}(\beth(\aleph)) \leq B\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\omega^{\rho}(\aleph)\Theta^{\rho}(\aleph)). \tag{17}$$

Taking into account the definition of Υ, it follows that

$$\Upsilon^{\nabla}(\aleph) = \varpi^{\nabla}(\beth(\aleph)) - B\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'\left[\omega^{\rho}(\aleph)\Theta^{\rho}(\aleph)\right],$$

which, together with (17), yields $\Upsilon^{\nabla}(\aleph) \leq 0$. Consequently, Υ is decreasing on $[\iota, \varrho]_{\mathbb{T}}$ and $\Upsilon(\varrho) \leq \Upsilon(\iota)$, since $\varrho > \iota$.

Finally, since $\varpi(0) = 0$, we have that $\Upsilon(\varrho) \leq \Upsilon(\iota) = \varpi(0) = 0$, i.e.,

$$\varpi\left(B\int_{\iota}^{\varrho} \omega(\varkappa)\Theta^{\nabla}(\varkappa)\nabla\varkappa\right) \leq B\int_{\iota}^{\varrho} \omega(\varkappa)\Theta^{\nabla}(\varkappa)\varpi'\left[\omega^{\rho}(\varkappa)\Theta^{\rho}(\varkappa)\right]\nabla\varkappa,$$

as claimed. The proof is now complete. □

Remark 1 In particular, if $\mathbb{T} = \mathbb{R}$ and $\rho(\varkappa) = \varkappa$, our Theorem 1 reduces to the integral inequality (4) proved by Pečarić et al. [19]. Moreover, if $\mathbb{T} = \mathbb{R}$, $B = 1$, $\varpi(\aleph) = \aleph^p$, $0 < p \leq 1$, our inequality (11) becomes the integral inequality (1) proved by Heinig and Maligranda in [16].

Remark 2 Considering Theorem 1 applied to $\mathbb{T} = \mathbb{N}_0$ and $\iota = 0$, we obtain the following discrete inequality:

$$\varpi\left(B\sum_{k=1}^{N}\omega_k\nabla\Theta_k\right) \leq B\sum_{k=1}^{N}\left(\omega_k\nabla\Theta_k\right)\varpi'\left[\omega_{k-1}\Theta_{k-1}\right].$$

Remark 3 Considering Theorem 1 applied to $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, we obtain the following discrete inequality:

$$\begin{aligned}
&\varpi\left(B\sum_{\varkappa=q\iota}^{\varrho}\left(1-\frac{1}{q}\right)\varkappa\omega(\varkappa)\nabla_q\Theta(\varkappa)\right)\\
&\quad\leq B\sum_{\varkappa=q\iota}^{\varrho}\left(1-\frac{1}{q}\right)\varkappa\omega(\varkappa)\nabla_q\Theta(\varkappa)\varpi'\left[\omega(\varkappa/q)\Theta(\varkappa/q)\right].
\end{aligned}$$

Our next result is a dynamic extension of the integral inequality (5) due to Pečarić et al. [19].

Theorem 2 *Let $\iota, \varrho \in \mathbb{T}$, and let ω be B-increasing on $[\iota, \varrho]_{\mathbb{T}}$, where $B \geq 1$. If Θ is increasing on $[\iota, \varrho]_{\mathbb{T}}$ and $\Theta(\iota) = 0$, then the inequality*

$$\varpi\left(\frac{1}{B}\int_{\iota}^{\varrho}\omega(\varkappa)\Theta^{\nabla}(\varkappa)\nabla x\right) \geq \frac{1}{B}\int_{\iota}^{\varrho}\omega(\varkappa)\Theta^{\nabla}(\varkappa)\varpi'\left[\omega(\varkappa)\Theta(\varkappa)\right]\nabla\varkappa, \tag{18}$$

holds.

Proof Let

$$\Upsilon(\aleph) := \varpi\left(\frac{1}{B}\int_{\iota}^{\aleph}\omega(\varkappa)\Theta^{\nabla}(\varkappa)\nabla\varkappa\right) - \frac{1}{B}\int_{\iota}^{\aleph}\omega(\varkappa)\Theta^{\nabla}(\varkappa)\varpi'\left[\omega(\varkappa)\Theta(\varkappa)\right]\nabla\varkappa$$

and

$$\beth(\aleph) := \frac{1}{B}\int_{\iota}^{\aleph}\omega(\varkappa)\Theta^{\nabla}(\varkappa)\nabla\varkappa,$$

that is,

$$\Upsilon(\aleph) = \varpi\left(\beth(\aleph)\right) - \frac{1}{B}\int_{\iota}^{\aleph}\omega(\varkappa)\Theta^{\nabla}(\varkappa)\varpi'\left[\omega(\varkappa)\Theta(\varkappa)\right]\nabla\varkappa. \tag{19}$$

Since ω is B-increasing, we observe that $\omega(\varkappa) \leq B\omega(\aleph)$, for $x \leq \aleph$. Furthermore, having in mind that Θ is increasing and $\Theta(\iota) = 0$, it follows that

$$\int_{\iota}^{\aleph} \omega(\varkappa)\Theta^{\nabla}(\varkappa)\nabla\varkappa \leq \int_{\iota}^{\aleph} B\omega(\aleph)\Theta^{\nabla}(\varkappa)\nabla\varkappa = B\omega(\aleph)\int_{\iota}^{\aleph} \Theta^{\nabla}(\varkappa)\nabla\varkappa$$
$$= B\omega(\aleph)\left[\Theta(\aleph) - \Theta(\iota)\right] = B\omega(\aleph)\Theta(\aleph),$$

which yields the relation

$$\beth(\aleph) \leq \omega(\aleph)\Theta(\aleph), \tag{20}$$

due to the definition of the function $\beth$.

On the other hand, using (10) to $\varpi(\beth(\aleph))$, it follows that there exists $\zeta \in [\rho(\aleph), \aleph]$ such that

$$\varpi^{\nabla}(\beth(\aleph)) = \varpi'(\beth(\zeta))\beth^{\nabla}(\aleph). \tag{21}$$

Clearly, since Θ is increasing, we have that

$$\beth^{\nabla}(\aleph) = \frac{1}{B}\omega(\aleph)\Theta^{\nabla}(\aleph) \geq 0, \tag{22}$$

which means that $\beth(\aleph)$ is increasing on $[\iota, \varrho]_{\mathbb{T}}$. Consequently, if $\zeta \leq \aleph$, then $\beth(\zeta) \leq \beth(\aleph)$, and so

$$\varpi'(\beth(\zeta)) \geq \varpi'(\beth(\aleph)), \tag{23}$$

due to the concavity of function ϖ. Now, taking into account (21)–(23), we get

$$\varpi^{\nabla}(\beth(\aleph)) \geq \frac{1}{B}\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\beth(\aleph)). \tag{24}$$

In addition, by virtue of (20), we have that $\varpi'(\beth(\aleph)) \geq \varpi'(\omega(\aleph)\Theta(\aleph))$, and so (note that ω is nonnegative and Θ is increasing)

$$\frac{1}{B}\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\beth(\aleph)) \geq \frac{1}{B}\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\omega(\aleph)\Theta(\aleph)),$$

which, together with (24), yields the relation

$$\varpi^{\nabla}(\beth(\aleph)) \geq \frac{1}{B}\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\omega(\aleph)\Theta(\aleph)).$$

Now, taking into account (19), we obtain

$$\Upsilon^{\nabla}(\aleph) = \varpi^{\nabla}(\beth(\aleph)) - \frac{1}{B}\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'\left[\omega(\aleph)\Theta(\aleph)\right]$$

and, consequently, $\Upsilon^{\nabla}(\aleph) \geq 0$, due to the previous inequality.

This means that Υ is increasing and so $\Upsilon(\varrho) \geq \Upsilon(\iota) = \varpi(0) = 0$, i.e.,

$$\varpi\left(\frac{1}{B}\int_{\iota}^{\varrho}\omega(\varkappa)\Theta^{\nabla}(\varkappa)\nabla x\right) \geq \frac{1}{B}\int_{\iota}^{\varrho}\omega(\varkappa)\Theta^{\nabla}(\varkappa)\varpi'\left[\omega(\varkappa)\Theta(\varkappa)\right]\nabla\varkappa,$$

which represents (18). □

Remark 4 If $\mathbb{T} = \mathbb{R}$, our Theorem 2 reduces to the integral inequality (5) established by Pečarić et al. [19], while for $\mathbb{T} = \mathbb{N}_0$ and $\iota = 0$, inequality (18) can be rewritten in the following form:

$$\varpi\left(\frac{1}{B}\sum_{k=1}^{N}\omega_k\nabla\Theta_k\right) \geq \frac{1}{B}\sum_{k=1}^{N}\left(\omega_k\nabla\Theta_k\right)\varpi'\left[\omega_k\Theta_k\right].$$

Remark 5 Considering Theorem 2 applied to $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, we obtain the following discrete inequality:

$$\varpi\left(\frac{1}{B}\sum_{\varkappa=q\iota}^{\varrho}\left(1-\frac{1}{q}\right)\varkappa\omega(\varkappa)\nabla_q\Theta(\varkappa)\right)$$
$$\geq \frac{1}{B}\sum_{\varkappa=q\iota}^{\varrho}\left(1-\frac{1}{q}\right)\varkappa\omega(\varkappa)\nabla_q\Theta(\varkappa)\varpi'[\omega(\varkappa)\Theta(\varkappa)].$$

In the sequel, we establish time scales versions of the integral inequalities (2) and (6) from the Introduction.

Theorem 3 *Let $\iota, \varrho \in \mathbb{T}$, ω be B-increasing on $[\iota,\varrho]_{\mathbb{T}}$, $B \geq 1$, and let Θ be decreasing on $[\iota,\varrho]_{\mathbb{T}}$. If $\Theta(\varrho) = 0$, then the inequality*

$$\varpi\left(B\int_{\iota}^{\varrho}\omega(\varkappa)\left[-\Theta(\varkappa)\right]^{\nabla}\nabla\varkappa\right) \leq B\int_{\iota}^{\varrho}\omega(\varkappa)\left[-\Theta^{\nabla}(\varkappa)\right]\varpi'\left[\omega(\varkappa)\Theta(\varkappa)\right]\nabla\varkappa, \tag{25}$$

holds.

Proof Let

$$\Upsilon(\aleph) = -\varpi\left(\beth^*(\aleph)\right) - B\int_{\aleph}^{\varrho}\omega(\varkappa)\Theta^{\nabla}(\varkappa)\varpi'\left[\omega(\varkappa)\Theta(\varkappa)\right]\nabla\varkappa,$$

where

$$\beth^*(\aleph) := B\int_{\aleph}^{\varrho}\omega(\varkappa)\left[-\Theta(\varkappa)\right]^{\nabla}\nabla\varkappa.$$

It is easy to see that

$$\beth^*(\aleph) \geq \omega(\aleph)\Theta(\aleph). \tag{26}$$

Namely, since ω is B-increasing and Θ is decreasing with $\Theta(\varrho) = 0$, we have that

$$B\int_{\aleph}^{\varrho} \omega(\varkappa)\left[-\Theta(\varkappa)\right]^{\nabla} \nabla\varkappa \geq \int_{\aleph}^{\varrho} \omega(\aleph)\left[-\Theta(\varkappa)\right]^{\nabla} \nabla\varkappa = \omega(\aleph)\int_{\aleph}^{\varrho} \left[-\Theta(\varkappa)\right]^{\nabla} \nabla\varkappa$$
$$= \omega(\aleph)\left[\Theta(\aleph) - \Theta(\varrho)\right] = \omega(\aleph)\Theta(\aleph),$$

so assertion (26) holds.

Now, by virtue of the chain rule (10), there exists $\zeta \in [\rho(\aleph), \aleph]$ such that

$$\varpi^{\nabla}(\beth^{*}(\aleph)) = \varpi'(\beth^{*}(\zeta))\left[\beth^{*}(\aleph)\right]^{\nabla}. \tag{27}$$

Clearly, $\beth^{*}$ is decreasing on $[\iota, \varrho]_{\mathbb{T}}$ since $[\beth^{*}(\aleph)]^{\nabla} = B\omega(\aleph)\Theta^{\nabla}(\aleph) \leq 0$, which, together with the concavity of function ϖ, implies the relation

$$\varpi'(\beth^{*}(\zeta)) \leq \varpi'(\beth^{*}(\aleph)), \tag{28}$$

provided that $\zeta \leq \aleph$. Therefore, relations (27) and (28) imply the inequality

$$\varpi^{\nabla}(\beth^{*}(\aleph)) \geq B\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\beth^{*}(\aleph)). \tag{29}$$

On the other hand, taking into account (26), we have that $\varpi'(\beth^{*}(\aleph)) \leq \varpi'(\omega(\aleph)\Theta(\aleph))$, and so

$$B\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\beth^{*}(\aleph)) \geq B\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\omega(\aleph)\Theta(\aleph)),$$

that is,

$$\varpi^{\nabla}(\beth^{*}(\aleph)) \geq B\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\omega(\aleph)\Theta(\aleph)), \tag{30}$$

due to (29).

Finally, since $\Upsilon^{\nabla}(\aleph) = -\varpi^{\nabla}(\beth^{*}(\aleph)) + B\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'[\omega(\aleph)\Theta(\aleph)]$, taking into account (30), it follows that $\Upsilon^{\nabla}(\aleph) \leq 0$, i.e., Υ is decreasing on $[\iota, \varrho]_{\mathbb{T}}$. Consequently, we have that $\Upsilon(\iota) \geq \Upsilon(\varrho) = -\varpi(0) = 0$, that is,

$$\varpi\left(B\int_{\iota}^{\varrho} \omega(\varkappa)\left[-\Theta(\varkappa)\right]^{\nabla} \nabla\varkappa\right) \leq B\int_{\iota}^{\varrho} \omega(\varkappa)\left[-\Theta^{\nabla}(\varkappa)\right]\varpi'[\omega(\varkappa)\Theta(\varkappa)]\nabla\varkappa,$$

which represents (25), as claimed. □

Remark 6 If $\mathbb{T} = \mathbb{R}$, our Theorem 3 represents the inequality (6) proved by Pečarić et al. [19]. In addition, if $B = 1$, $\varpi(\aleph) = \aleph^{p}$, and $0 < p \leq 1$, we get (2) established by Heinig and Maligranda [16], while for $\mathbb{T} = \mathbb{N}_0$ and $\iota = 0$, relation (25) reduces to the following discrete inequality:

$$\varpi\left(B\sum_{k=1}^{N} \omega_k \nabla\left[-\Theta_k\right]\right) \leq B\sum_{k=1}^{N} \left(\omega_k \nabla\left[-\Theta_k\right]\right)\varpi'[\omega_k\Theta_k].$$

Remark 7 Considering Theorem 3 applied to $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, we obtain the following discrete inequality:

$$\varpi\left(B\sum_{\varkappa=q\iota}^{\varrho}\left(1-\frac{1}{q}\right)\varkappa\omega(\varkappa)(-\nabla_q\Theta(\varkappa))\right)$$
$$\leq B\sum_{\varkappa=q\iota}^{\varrho}\left(1-\frac{1}{q}\right)\varkappa\omega(\varkappa)(-\nabla_q\Theta(\varkappa))\varpi'[\omega(\varkappa)\Theta(\varkappa)]. \tag{31}$$

In order to conclude our discussion, it remains to give a dynamic extension of the inequality (7).

Theorem 4 *Let $\iota, \varrho \in \mathbb{T}$, ω be B-decreasing on $[\iota,\varrho]_{\mathbb{T}}$, $B \geq 1$, and let Θ be decreasing on $[\iota,\varrho]_{\mathbb{T}}$. Then the inequality*

$$\varpi\left(\frac{1}{B}\int_{\iota}^{\varrho}\omega(\varkappa)\left[-\Theta(\varkappa)\right]^{\nabla}\nabla\varkappa\right) \geq \frac{1}{B}\int_{\iota}^{\varrho}\omega(\varkappa)\left[-\Theta(\varkappa)\right]^{\nabla}\varpi'[\omega^{\rho}(\varkappa)\Theta^{\rho}(\varkappa)]\nabla\varkappa, \tag{32}$$

holds, provided that $\Theta(\varrho) = 0$.

Proof Let

$$\Upsilon(\aleph) = -\varpi\left(\daleth^{*}(\aleph)\right) + \frac{1}{B}\int_{\aleph}^{\varrho}\omega(\varkappa)\left[-\Theta(\varkappa)\right]^{\nabla}\varpi'\left[\omega^{\rho}(\varkappa)\Theta^{\rho}(\varkappa)\right]\nabla\varkappa,$$

where

$$\daleth^{*}(\aleph) := \frac{1}{B}\int_{\aleph}^{\varrho}\omega(\varkappa)\left[-\Theta(\varkappa)\right]^{\nabla}\nabla\varkappa.$$

Since ω is B-decreasing, we have that

$$\int_{\aleph}^{\varrho}\omega(\varkappa)\left[-\Theta(\varkappa)\right]^{\nabla}\nabla\varkappa \leq \int_{\aleph}^{\varrho}B\omega(\aleph)\left[-\Theta(\varkappa)\right]^{\nabla}\nabla\varkappa = B\omega(\aleph)\int_{\aleph}^{\varrho}\left[-\Theta(\varkappa)\right]^{\nabla}\nabla\varkappa$$
$$= B\omega(\aleph)\left[\Theta(\aleph)-\Theta(\varrho)\right] = B\omega(\aleph)\Theta(\aleph),$$

and so

$$\daleth^{*}(\aleph) \leq \omega(\aleph)\Theta(\aleph). \tag{33}$$

Utilizing (10) to $\varpi\left(\daleth^{*}(\aleph)\right)$, we conclude that there exists $d \in [\rho(\aleph), \aleph]$ such that

$$\varpi^{\nabla}(\daleth^{*}(\aleph)) = \varpi'(\daleth^{*}(d))\left[\daleth^{*}(\aleph)\right]^{\nabla}. \tag{34}$$

It should be noticed here that $\daleth^{*}$ is decreasing on $[\iota,\varrho]_{\mathbb{T}}$ since

$$\left[\daleth^{*}(\aleph)\right]^{\nabla} = \frac{1}{B}\omega(\aleph)\Theta^{\nabla}(\aleph) \leq 0,$$

which, together with the concavity of ϖ, yields the relation

$$\varpi'(\beth^*(d)) \geq \varpi'(\beth^*(\rho(\aleph)). \tag{35}$$

Therefore, comparing (34) and (35), we get

$$\varpi^{\nabla}(\beth^*(\aleph)) \leq \frac{1}{B}\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\beth^*(\rho(\aleph)). \tag{36}$$

On the other hand, since (33) holds, we have that $\varpi'(\beth^*(\rho(\aleph)) \geq \varpi'(\omega^{\rho}(\aleph)\Theta^{\rho}(\aleph))$, i.e.,

$$\frac{1}{B}\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\beth^*(\rho(\aleph))) \leq \frac{1}{B}\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\omega^{\rho}(\aleph)\Theta^{\rho}(\aleph)),$$

that is,

$$\varpi^{\nabla}(\beth^*(\aleph)) \leq \frac{1}{B}\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'(\omega^{\rho}(\aleph)\Theta^{\rho}(\aleph)), \tag{37}$$

due to (36).

At last, since $\Upsilon^{\nabla}(\aleph) = -\varpi^{\nabla}(\beth^*(\aleph)) + \frac{1}{B}\omega(\aleph)\Theta^{\nabla}(\aleph)\varpi'[\omega^{\rho}(\aleph)\Theta^{\rho}(\aleph)]$, relation (37) implies that Υ is increasing on $[\iota,\varrho]_{\mathbb{T}}$. Consequently, it follows that

$$\Upsilon(\iota) \leq \Upsilon(\varrho) = -\varpi(0) = 0,$$

which can be rewritten as

$$\varpi\left(\frac{1}{B}\int_{\iota}^{\varrho} \omega(\varkappa)\,[-\Theta(\varkappa)]^{\nabla}\,\nabla\varkappa\right) \geq \frac{1}{B}\int_{\iota}^{\varrho} \omega(\varkappa)\,[-\Theta(\varkappa)]^{\nabla}\,\varpi'[\omega^{\rho}(\varkappa)\Theta^{\rho}(\varkappa)]\nabla\varkappa,$$

i.e., we obtain (32). The proof is now complete. □

Remark 8 By putting $\mathbb{T} = \mathbb{R}$ and $\rho(\varkappa) = \varkappa$ in (32), we get (7) of Pečarić et al. [19]. Furthermore, if $\mathbb{T} = \mathbb{N}_0$, $\rho(k) = k-1$, and $\iota = 0$, our Theorem 4 provides the following inequality:

$$\varpi\left(\frac{1}{B}\sum_{k=1}^{N}\omega_k\nabla\,[-\Theta_k]\right) \geq \frac{1}{B}\sum_{k=1}^{N}\omega_k\nabla\,[-\Theta_k]\,\varpi'[\omega_{k-1}\Theta_{k-1}].$$

Remark 9 Considering Theorem 4 applied to $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, we obtain the following discrete inequality:

$$\varpi\left(\frac{1}{B}\sum_{\varkappa=q\iota}^{\varrho}\left(1-\frac{1}{q}\right)\varkappa\omega(\varkappa)(-\nabla_q\Theta(\varkappa))\right)$$
$$\geq \frac{1}{B}\sum_{\varkappa=q\iota}^{\varrho}\left(1-\frac{1}{q}\right)\varkappa\omega(\varkappa)(-\nabla_q\Theta(\varkappa))\varpi'[\omega(\varkappa/q)\Theta(\varkappa/q)].$$

4 Conclusion and future work

This paper presents several new dynamic inequalities for B-monotone functions in time scales nabla calculus and also several different inequalities in discrete calculus. In the future, we will establish their analogues in diamond alpha calculus.

Author contributions
All the authors contributed equally to this paper.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Ethical approval
Not applicable.

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics, Faculty of Science, Al-Balqa Applied University, P.O. Box 19117, Salt, Jordan. [2]Department of Mathematics, Faculty of Science, Al al-Bayt University, P. O. Box 130095, Mafraq, Jordan. [3]Department of Mathematics, College of Science, Gurayat, Jouf University, Jouf, Saudi Arabia. [4]Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt. [5]Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia. [6]Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 040 01, Košice, Slovakia.

Received: 9 October 2023 Accepted: 10 September 2024 Published online: 16 September 2024

References

1. Agarwal, R.P., O'Regan, D., Saker, S.H.: Dynamic Inequalities on Time Scales. Springer, Berlin (2014)
2. Agarwal, R.P., O'Regan, D., Saker, S.H.: Hardy Type Inequalities on Time Scales. Springer, Berlin (2016)
3. Ahammad, N.A., Rasheed, H.U., El-Deeb, A.A., Almarri, B., Shah, N.A.: A numerical intuition of activation energy in transient micropolar nanofluid flow configured by an exponentially extended plat surface with thermal radiation effects. Mathematics **10**(21), 1–20 (2022). 4046
4. Anderson, D., Bullock, J., Erbe, L., Peterson, A., Tran, H.: Nabla dynamic equations on time scales. Panam. Math. J. **13**(1), 1–47 (2003)
5. Awwad, E., Saied, A.: Some weighted dynamic inequalities of Hardy type with kernels on time scales nabla calculus. J. Math. Inequal. **18**, 457–475 (2024)
6. Awwad, E., Saied, A.I.: Some new multidimensional Hardy-type inequalities with general kernels on time scales. J. Math. Inequal. **16**(1), 393–412 (2022)
7. Bibi, R., Bohner, M., Pečarić, J., Varošanec, S.: Minkowski and Beckenbach–Dresher inequalities and functionals on time scales. J. Math. Inequal. **7**(3), 299–312 (2013)
8. Bohner, M., Georgiev, S.G.: Multiple Integration on Time Scales. Multivariable Dynamic Calculus on Time Scales. Springer, Cham (2016)
9. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)
10. Bohner, M., Peterson, A.: Advances in Dynamic Equations on Time Scales. Springer, Berlin (2002)
11. El-Deeb, A.A., El-Sennary, H.A., Khan, Z.A.: Some reverse inequalities of Hardy type on time scales. Adv. Differ. Equ. **2020**, 402 (2020)
12. El-Deeb, A.A., Elsennary, H.A., Baleanu, D.: Some new Hardy-type inequalities on time scales. Adv. Differ. Equ. **2020**, 441 (2020)
13. El-Deeb, A.A., Rashid, S.: On some new double dynamic inequalities associated with Leibniz integral rule on time scales. Adv. Differ. Equ. **2021**, 125 (2021)
14. El-Deeb, A.A., Xu, H., Abdeldaim, A., Wang, G.: Some dynamic inequalities on time scales and their applications. Adv. Differ. Equ. **2019**(1), 130 (2019)
15. Güvenilir, A.F., Kaymakçalan, B., Pelen, N.N.: Constantin's inequality for nabla and diamond-alpha derivative. J. Inequal. Appl. **2015**(167), 167 (2015)
16. Heinig, H., Maligranda, L.: Weighted inequalities for monotone and concave functions. Stud. Math. **116**(2), 133–165 (1995)
17. Hilger, S.: Analysis on measure chains—a unified approach to continuous and discrete calculus. Results Math. **18**(1), 18–56 (1990)
18. Oguntuase, J.A., Persson, L.E.: Time scales Hardy-type inequalities via superquadracity. Ann. Funct. Anal. **5**(2), 61–73 (2014)
19. Pečarić, J., Perić, I., Persson, L.E.: Integral inequalities for monotone functions. J. Math. Anal. Appl. **215**(1), 235–251 (1997)
20. Řehak, P.: Hardy inequality on time scales and its application to half-linear dynamic equations. J. Inequal. Appl. **2005**(5), 495–507 (2005)
21. Rezk, H.M., Albalawi, W., Abd El-Hamid, H.A., Saied, A.I., Bazighifan, O., Mohamed, M.S., Zakarya, M.: Hardy–Leindler-type inequalities via conformable delta fractional calculus. J. Funct. Spaces **2022**, 1–10 (2022)
22. Rezk, H.M., Saied, A.I., AlNemer, G., Zakarya, M.: On Hardy–Knopp type inequalities with kernels via time scale calculus. J. Math. **2022**, 1–13 (2022)
23. Saker, S.H., Alzabut, J., Saied, A.I., O'Regan, D.: New characterizations of weights on dynamic inequalities involving a Hardy operator. J. Inequal. Appl. **2021**(1), 73 (2021)

24. Saker, S.H., Awwad, E., Saied, A.I.: Some new dynamic inequalities involving monotonic functions on time scales. J. Funct. Spaces **2019**, 1–11 (2019)
25. Saker, S.H., Saied, A.I., Anderson, D.R.: Some new characterizations of weights in dynamic inequalities involving monotonic functions. Qual. Theory Dyn. Syst. **20**(2), 1–22 (2021)
26. Saker, S.H., Saied, A.I., Krnić, M.: Some new dynamic Hardy-type inequalities with kernels involving monotone functions. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **114**, 1–16 (2020)
27. Saker, S.H., Saied, A.I., Krnić, M.: Some new weighted dynamic inequalities for monotone functions involving kernels. Mediterr. J. Math. **17**(2), 1–18 (2020)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.