Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Polynomial decay of the energy of solutions of coupled wave equations with locally boundary fractional dissipation

Amina Chaili[1], Abderrahmane Beniani[1], Ahmed Bchatnia[2*] and Suleman Alfalqi[3]

*Correspondence:
ahmed.bchatnia@fst.utm.tn
[2]Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis El Manar, LR Analyse Non-Linéaire et Géométrie, LR21ES08, El Manar 2, 2092, Tunis, Tunisia
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate a system of coupled wave equations featuring boundary fractional damping applied to a portion of the domain. We first establish the well-posedness of the system, proving the existence and uniqueness of solutions through semi-group theory. While the system does not exhibit exponential stability, we demonstrate its strong stability. Furthermore, leveraging Arendt and Batty's general criteria and certain geometric conditions, we prove a polynomial rate of energy decay for the solutions.

Mathematics Subject Classification: 35L05; 34K37

Keywords: Wave equations; Fractional derivatives; $\mathcal{C}_0$ semi-group; General decay

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with boundary Γ of class C^2, and let $\{\Gamma_1, \Gamma_2\}$ be a partition of Γ. We are concerned with the energy decay property of the solutions of the following system:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + l(u(x,t) - v(x,t)) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - \Delta v(x,t) + l(v(x,t) - u(x,t)) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \Gamma_1 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu} = -\rho_1 \partial_t^{\alpha,\eta} u(x,t), & (x,t) \in \Gamma_2 \times (0,+\infty), \\ \frac{\partial v}{\partial \nu} = -\rho_2 \partial_t^{\alpha,\eta} v(x,t), & (x,t) \in \Gamma_2 \times (0,+\infty), \end{cases} \tag{1.1}$$

where l, ρ_1, ρ_2 are positive constants, and the initial conditions are

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,$$

$$v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega.$$

The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo fractional derivative of order α, $0 < \alpha < 1$, with respect to the time t. It is defined as follows:

$$\partial_t^{\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{df}{ds}(s)\, ds, \quad \eta \geq 0.$$

In recent years, the scientific community has experienced a growing interest in unraveling the intricate dynamics and practical applications of wave equations. The behavior of waves, whether occurring naturally, such as seismic waves in the earth's crust or in engineered systems like acoustic waves in materials, captivates both researchers and practitioners alike. Considerable research efforts have been devoted to investigating wave equations with diverse damping types and exploring their stability and controllability. These waves emerge when a vibrating source disrupts the surrounding medium. Researchers have shown a keen interest in addressing damping-related challenges, whether local or global and have illustrated different forms of stability.

In [16], B. Mbodje explored the asymptotic behavior of solutions within the system:

$$\begin{cases} \partial_t^2 u(x,t) - u_x^2(x,t) = 0,\ (x,t) \in (0,1) \times (0,+\infty), \\ u(0,t) = 0, \\ \partial_x u(1,t) = -k\partial_t^{\alpha,\eta} u(1,t),\ \alpha \in (0,1), \eta \geq 0, k > 0, \\ u(x,0) = u_0(x), \\ \partial_t u(x,0) = v_0(x). \end{cases}$$

He demonstrated that the corresponding semi-group lacks exponential stability, showing only strong asymptotic stability. Additionally, the system's energy diminishes over time, approaching a decay proportional to t^{-1} as time extends to infinity.

In [5], Akil and Wehbe considered a multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \Gamma_0 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu}(x,t) + \gamma\, \partial_t^{\alpha,\eta} u(x,t) = 0, & (x,t) \in \Gamma_1 \times (0,+\infty). \end{cases}$$

They demonstrated the system's strong stability while establishing that it lacks uniform stability. Additionally, they derived a polynomial energy decay for smooth solutions of the form $t^{-\frac{1}{1-\alpha}}$. This analysis assumes specific geometric conditions for the boundary control region and leverages the exponential decay of the wave equation with standard damping.

In [9], Atoui and Benaissa examined a transmission problem involving waves under a nonlocal boundary control:

$$\begin{cases} \rho_1 u_{tt}(x,t) - \tau_1 u_{xx}(x,t) = 0, & (x,t) \in (0,l_0) \times (0,+\infty), \\ \rho_2 v_{tt}(x,t) - \tau_2 v_{xx}(x,t) = 0, & (x,t) \in (l_0,L) \times (0,+\infty), \\ u(l_0,t) = v(l_0,t), \rho_2\tau_1 u_x(l_0,t) = \rho_1\tau_2 v_x(l_0,t), & \forall t \in (0,+\infty), \\ u(0,t) = 0, \quad \tau_2 v_x(L,t) + \gamma\rho_2\tau_2 \partial_t^{\alpha,\eta} v(L,t) = 0, & \forall t \in (0,+\infty). \end{cases}$$

They established that the energy decay in this context is characterized by polynomial decay rather than exponential. Employing the spectrum method, they demonstrated the absence of exponential stability. Furthermore, they applied the Borichev-Tomilov theorem to ascertain the specific polynomial decay rate.

Recently, in [13], Beniani et al. examined a system comprising coupled wave equations featuring a diffusive internal control of a general nature:

$$\begin{cases} \partial_{tt}u - \Delta_x u + \zeta \int_{-\infty}^{+\infty} \varrho(\omega)\phi(x,\omega,t)\,d\omega + \beta v = 0, \\ \partial_{tt}v - \Delta_x v + \zeta \int_{-\infty}^{+\infty} \varrho(\omega)\varphi(x,\omega,t)\,d\omega + \beta u = 0, \\ u = v = 0, \text{ on } \partial\Omega, \\ \phi_t(x,\omega,t) + (\omega^2+\eta)\phi(x,\omega,t) - \partial_t u \varrho(\omega) = 0, \\ \varphi_t(x,\omega,t) + (\omega^2+\eta)\varphi(x,\omega,t) - \partial_t v \varrho(\omega) = 0, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \\ v(x,0) = v_0(x), \quad \partial_t v(x,0) = v_1(x), \\ \phi(x,\omega,0) = \phi_0(x,\omega), \quad \text{and} \quad \varphi(x,\omega,0) = \varphi_0(x,\omega). \end{cases}$$

They showed the absence of exponential stability and explored the asymptotic stability of the model, establishing a general decay rate that depends on the density function ϱ. The references [2–4, 6, 17] compile a series of published works that underpin the mathematical formulation of problems related to fractional differential equations and provide insights into the decay rate of the energy of the solution to (1.1).

This paper is organized as follows. In Sect. 2, we reformulate the system (1.1) into an augmented system by coupling the wave equations with compatible diffusion equations. Subsequently, we establish the well-posedness of our system using a semi-group approach. In Sect. 3, we show that the system lacks exponential stability. In Sect. 4, we demonstrate the strong stability of our system, even in the absence of resolvent compactness, by combining a general criterion of Arendt and Batty with Holmgren's theorem, and we show a general decay rate result.

2 Preliminary results and well-posedness

We first recall the following result due to [5]:

Theorem 2.1 [5] *Let μ be the function*:

$$\mu(\xi) = |\xi|^{(2\alpha-n)/2}, \quad \xi \in \mathbb{R}^n, 0 < \alpha < 1.$$

Then, the relationship between the "input" U and the "output" O of the following system:

$$\partial_t \varphi(\xi,x,t) + \left(|\xi|^2+\eta\right)\varphi(\xi,x,t) - U(x,t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}^n, \eta \geq 0, t > 0,$$

$$\varphi(\xi,x,0) = 0,$$

$$O(x,t) = \frac{2\sin(\alpha\pi)\Gamma\left(\frac{n}{2}+1\right)}{n\pi^{\frac{n}{2}+1}} \int_{\mathbb{R}^n} \mu(\xi)\varphi(\xi,x,t)\,d\xi,$$

is given by

$$O = I^{1-\alpha,\eta}U = D^{\alpha,\eta}U,$$

where

$$\left[I^{\alpha,\eta}U\right](x,t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}U(x,\tau)\,d\tau.$$

Now, let us recall some results that will be needed later.

Lemma 2.2 [1, 5, 12] *If* $\lambda\in D=\{\lambda\in\mathbb{C}\mid\Re\lambda+\eta>0\}\cup\{\Im\lambda\neq 0\}$, *then*

$$\int_{\mathbb{R}^n}\frac{\mu^2(\xi)}{|\lambda|+\eta+|\xi|^2}\,d\xi=\frac{n\pi^{\frac{n}{2}+1}}{2\sin(\alpha\pi)\,\Gamma\left(\frac{n}{2}+1\right)}\left(|\lambda|+\eta\right)^{\alpha-1}.$$

and

$$\left(\int_{\mathbb{R}^n}\frac{\mu^2(\xi)}{\left(|\lambda|+\eta+|\xi|^2\right)^2}\,d\xi\right)^{\frac{1}{2}}=\left(\frac{n\pi^{\frac{n}{2}}}{2\Gamma\left(\frac{n}{2}+1\right)}\int_1^{+\infty}\frac{(y-1)^{\alpha}}{y^2}\,dy\right)^{\frac{1}{2}}\left(|\lambda|+\eta\right)^{\frac{\alpha}{2}-1}.$$

Lemma 2.3 [5] *Let* $\eta\geq 0$, *then we have*

$$\int_{\mathbb{R}^n}\frac{|\xi|^{2\alpha-n+2}}{\left(1+|\xi|^2+\eta\right)^2}\,d\xi<+\infty.$$

Using the previous theorem, system (1.1) can be rewritten as the following augmented model:

$$\begin{cases} u_{tt}(x,t)-\Delta u(x,t)+l(u(x,t)-v(x,t))=0, & (x,t)\in\Omega\times(0,+\infty),\\ v_{tt}(x,t)-\Delta v(x,t)+l(v(x,t)-u(x,t))=0, & (x,t)\in\Omega\times(0,+\infty),\\ \partial_t\varphi(\xi,x,t)+\left(|\xi|^2+\eta\right)\varphi(\xi,x,t) & \\ \quad-\mu(\xi)\,\partial_t u(x,t)=0, & (x,t)\in\Gamma_2\times(0,+\infty),\ \xi\in\mathbb{R}^n,\\ \partial_t\psi(\xi,x,t)+\left(|\xi|^2+\eta\right)\psi(\xi,x,t) & \\ \quad-\mu(\xi)\,\partial_t v(x,t)=0, & (x,t)\in\Gamma_2\times(0,+\infty),\ \xi\in\mathbb{R}^n,\\ u(x,t)=v(x,t)=0, & (x,t)\in\Gamma_1\times(0,+\infty),\\ \frac{\partial u}{\partial\nu}+\rho_1 C\int_{\mathbb{R}^n}\mu(\xi)\,\varphi(\xi,x,t)\,d\xi=0, & (x,t)\in\Gamma_2\times(0,+\infty),\\ \frac{\partial v}{\partial\nu}+\rho_2 C\int_{\mathbb{R}^n}\mu(\xi)\,\psi(\xi,x,t)\,d\xi=0, & (x,t)\in\Gamma_2\times(0,+\infty), \end{cases}\tag{2.1}$$

with the following initial conditions:

$$\begin{aligned} &u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in\Omega,\\ &v(x,0)=v_0(x),\quad v_t(x,0)=v_1(x),\quad x\in\Omega,\\ &\varphi(\xi,x,0)=0,\quad \psi(\xi,x,0)=0,\quad \xi\in\mathbb{R}^n, \end{aligned}$$

and $C=\frac{2\sin(\alpha\pi)\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}+1}}$.

We define the energy of the solution of (2.1) by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_{\Omega}\left(|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right.\\ &+ \left.|v_t(x,t)|^2 + |\nabla v(x,t)|^2 + l|u(x,t)-v(x,t)|^2\right)dx \\ &+ \frac{\rho_1 C}{2}\int_{\Gamma_2}\int_{\mathbb{R}^n}|\varphi(\xi,x,t)|^2 d\xi d\Gamma + \frac{\rho_2 C}{2}\int_{\Gamma_2}\int_{\mathbb{R}^n}|\psi(\xi,x,t)|^2 d\xi d\Gamma. \end{aligned} \tag{2.2}$$

For all $t \geq 0$, we have the following energy identity:

Lemma 2.4 *Let $U = (u, v, y, z, \varphi, \psi)$ be a regular solution of problem (2.1). Then, the functional energy defined in equation (2.2) satisfies*

$$\begin{aligned} \frac{d}{dt}E(t) = &-\rho_1 C\int_{\Gamma_2}\int_{\mathbb{R}^n}\left(|\xi|^2+\eta\right)|\varphi(\xi,x,t)|^2 d\xi d\Gamma \\ &- \rho_2 C\int_{\Gamma_2}\int_{\mathbb{R}^n}\left(|\xi|^2+\eta\right)|\psi(\xi,x,t)|^2 d\xi d\Gamma. \end{aligned}$$

Proof Multiplying equations $(2.1)_1$ and $(2.1)_2$ by u_t and v_t, respectively, using integration by parts over Ω, equations $(2.1)_6$ and $(2.1)_7$, applying Green's formula, and adding the two equations, we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega}(|u_t(x,t)|^2 + |v_t(x,t)|^2 + |\nabla u(x,t)|^2 + |\nabla v(x,t)|^2 + l|(u-v)(x,t)|^2)dx\right) \\ &+\rho_1 C\int_{\Gamma_2}\left(\int_{\mathbb{R}^n}\mu(\xi)\varphi(\xi,x,t)d\xi\right)u_t(x,t)d\Gamma \\ &+\rho_2 C\int_{\Gamma_2}\left(\int_{\mathbb{R}^n}\mu(\xi)\psi(\xi,x,t)d\xi\right)v_t(x,t)d\Gamma \\ &= 0. \end{aligned} \tag{2.3}$$

Multiplying equations $(2.1)_3$ and $(2.1)_4$ by $\rho_1 C\varphi$ and $\rho_2 C\psi$, respectively, using integration over $\Gamma_2 \times \mathbb{R}^n$, and adding the two equations, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\rho_1 C\int_{\Gamma_2}\int_{\mathbb{R}^n}|\varphi(\xi,x,t)|^2 d\xi d\Gamma + \rho_2 C\int_{\Gamma_2}\int_{\mathbb{R}^n}|\psi(\xi,x,t)|^2 d\xi d\Gamma\right) \\ &+ \rho_1 C\int_{\Gamma_2}\int_{\mathbb{R}^n}(|\xi|^2+\eta)|\varphi(\xi,x,t)|^2 d\xi d\Gamma \\ &+ \rho_2 C\int_{\Gamma_2}\int_{\mathbb{R}^n}(|\xi|^2+\eta)|\psi(\xi,x,t)|^2 d\xi d\Gamma \\ &- \rho_1 C\int_{\Gamma_2}\left(\int_{\mathbb{R}^n}\mu(\xi)\varphi(\xi,x,t)d\xi\right)u_t(x,t)d\Gamma \\ &- \rho_2 C\int_{\Gamma_2}\left(\int_{\mathbb{R}^n}\mu(\xi)\psi(\xi,x,t)d\xi\right)v_t(x,t)d\Gamma \\ &= 0. \end{aligned} \tag{2.4}$$

Combining equations (2.3) and (2.4), we obtain

$$\begin{aligned} \frac{d}{dt}E(t) = &-\rho_1 C\int_{\Gamma_2}\int_{\mathbb{R}^n}\left(|\xi|^2+\eta\right)|\varphi(\xi,x,t)|^2 d\xi d\Gamma \\ &-\rho_2 C\int_{\Gamma_2}\int_{\mathbb{R}^n}\left(|\xi|^2+\eta\right)|\psi(\xi,x,t)|^2 d\xi d\Gamma. \end{aligned}$$

This completes the proof of the lemma. □

We now discuss the well-posedness of (2.1). For this purpose, we define the following Hilbert space (the energy space):

$$\mathcal{H} = (H^1_{\Gamma_1}(\Omega))^2 \times (L^2(\Omega))^2 \times (L^2(\mathbb{R}^n \times \Gamma_2))^2,$$

where $H^1_{\Gamma_1}(\Omega)$ is given by

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma_1\}.$$

For $U = (u, v, y, z, \varphi, \psi)^T$ and $U_1 = (u_1, v_1, y_1, z_1, \varphi_1, \psi_1)^T$, we define the following inner product in $\mathcal{H}$

$$\begin{aligned}\langle U, U_1\rangle_{\mathcal{H}} &= \int_\Omega \left(y\overline{y}_1 + z\overline{z}_1 + \nabla u \nabla \overline{u}_1 + \nabla v \nabla \overline{v}_1 + l(u-v)(\overline{u}_1 - \overline{v}_1)\right) dx \\ &\quad + \rho_1 C \int_{\Gamma_2}\int_{\mathbb{R}^n} \varphi(\xi,x)\overline{\varphi}_1(\xi,x)\, d\xi\, d\Gamma + \rho_2 C \int_{\Gamma_2}\int_{\mathbb{R}^n} \psi(\xi,x)\overline{\psi}_1(\xi,x)\, d\xi\, d\Gamma. \end{aligned} \tag{2.5}$$

We then reformulate (2.1) into a semi-group setting. Introducing the vector function $U = (u, v, y, z, \varphi, \psi)^T$, system (2.1) is equivalent to

$$\begin{cases} U' = \mathcal{A}U, & t > 0, \\ U(0) = U_0, \end{cases} \tag{2.6}$$

where $U_0 := (u_0, v_0, u_1, v_1, \varphi_0, \psi_0)^T$. The operator $\mathcal{A}$ is linear and defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ y \\ z \\ \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} y \\ z \\ \Delta u - l(u-v) \\ \Delta v - l(v-u) \\ -\left(|\xi|^2 + \eta\right)\varphi + y|_{\Gamma_2}\mu(\xi) \\ -\left(|\xi|^2 + \eta\right)\psi + z|_{\Gamma_2}\mu(\xi) \end{pmatrix}. \tag{2.7}$$

The domain of $\mathcal{A}$ is then

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, y, z, \varphi, \psi)^T \in \mathcal{H} : u, v \in H^1_{\Gamma_1}(\Omega), \Delta u, \Delta v \in L^2(\Omega), \\ |\xi|\varphi, |\xi|\psi \in L^2(\mathbb{R}^n \times \Gamma_2), \\ -\left(|\xi|^2 + \eta\right)\varphi + y|_{\Gamma_2}\mu(\xi) \in L^2(\mathbb{R}^n \times \Gamma_2), \\ \frac{\partial u}{\partial \nu}|_{\Gamma_2} = -\rho_1 C \int_{\mathbb{R}^n} \mu(\xi)\varphi(\xi,x) d\xi, \\ -\left(|\xi|^2 + \eta\right)\psi + y|_{\Gamma_2}\mu(\xi) \in L^2(\mathbb{R}^n \times \Gamma_2), \\ \frac{\partial v}{\partial \nu}|_{\Gamma_2} = -\rho_2 C \int_{\mathbb{R}^n} \mu(\xi)\psi(\xi,x) d\xi. \end{array} \right\}. \tag{2.8}$$

We have the following theorem of existence and uniqueness.

Theorem 2.5

1. *If* $U_0 \in D(\mathcal{A})$, *then system* (2.1) *has a unique strong solution*

$$U \in \mathcal{C}^0(\mathbb{R}_+, D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+, \mathcal{H}).$$

2. *If $U_0 \in \mathcal{H}$, then system (2.1) has a unique weak solution*

$$U \in \mathcal{C}^0(\mathbb{R}_+, \mathcal{H}).$$

Proof First, we prove that the operator $\mathcal{A}$ is dissipative. For any $U \in D(\mathcal{A})$, we have

$$\begin{aligned}\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\rho_1 C \int_{\Gamma_2}\int_{\mathbb{R}^n} \left(|\xi|^2+\eta\right)|\varphi(\xi,x)|^2 d\xi d\Gamma \\ -\rho_2 C \int_{\Gamma_2}\int_{\mathbb{R}^n} \left(|\xi|^2+\eta\right)|\psi(\xi,x)|^2 d\xi d\Gamma = 0.\end{aligned} \tag{2.9}$$

Hence, $\mathcal{A}$ is dissipative. We will show that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. Given $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}$, we prove that there exists $U \in D(\mathcal{A})$ satisfying

$$(\lambda I - \mathcal{A})U = F. \tag{2.10}$$

Equation (2.10) is equivalent to

$$\begin{cases} \lambda u - y = f_1, \\ \lambda v - z = f_2, \\ \lambda y - \Delta u + l(u-v) = f_3, \\ \lambda z - \Delta v + l(v-u) = f_4, \\ \lambda\varphi + \left(|\xi|^2+\eta\right)\varphi - y|_{\Gamma_2}\mu(\xi) = f_5, \\ \lambda\psi + \left(|\xi|^2+\eta\right)\psi - z|_{\Gamma_2}\mu(\xi) = f_6. \end{cases} \tag{2.11}$$

Then, from $(2.11)_1$ and $(2.11)_2$, we find that

$$\begin{cases} y = \lambda u - f_1, \\ z = \lambda v - f_2. \end{cases} \tag{2.12}$$

It is clear that $y \in H^1_{\Gamma_1}(\Omega)$ and $z \in H^1_{\Gamma_1}(\Omega)$. Furthermore, from $(2.11)_5$ and $(2.11)_6$, we can find φ and ψ as

$$\begin{cases} \varphi = \frac{f_5(x,\xi)}{\lambda+|\xi|^2+\eta} + \frac{\lambda u_{/\Gamma_2}\mu(\xi)}{\lambda+|\xi|^2+\eta} - \frac{f_1(x)\mu(\xi)}{\lambda+|\xi|^2+\eta}, \\ \psi = \frac{f_6(x,\xi)}{\lambda+|\xi|^2+\eta} + \frac{\lambda v_{/\Gamma_2}\mu(\xi)}{\lambda+|\xi|^2+\eta} - \frac{f_2(x)\mu(\xi)}{\lambda+|\xi|^2+\eta}. \end{cases} \tag{2.13}$$

By inserting $(2.12)_1$ into $(2.11)_3$ and $(2.12)_2$ into $(2.11)_4$, we get

$$\begin{cases} \lambda^2 u - \Delta u + l(u-v) = f_3 + \lambda f_1, \\ \lambda^2 v - \Delta v + l(v-u) = f_4 + \lambda f_2. \end{cases} \tag{2.14}$$

Solving system (2.14) is equivalent to finding $u, v \in H^1_{\Gamma_1}(\Omega) \cap H^2(\Omega)$ such that

$$\begin{cases} \int_\Omega \left(\lambda^2 u\chi - \Delta u\chi + l(u-v)\chi\right) dx = \int_\Omega \left(f_3 + \lambda f_1\right)\chi\, dx, \\ \int_\Omega \left(\lambda^2 v\zeta - \Delta v\zeta + l(v-u)\zeta\right) dx = \int_\Omega \left(f_4 + \lambda f_2\right)\zeta\, dx \end{cases} \tag{2.15}$$

for all $\chi,\zeta \in H^1_{\Gamma_1}(\Omega)$. From (2.15), one can see that the functions u and v satisfy the following system:

$$\begin{cases} \int_\Omega \left(\lambda^2 u\chi + \nabla u \nabla \chi + l(u-v)\chi\right) dx + \rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi)\varphi(\xi,x,t)d\xi\right) \chi \, d\Gamma \\ = \int_\Omega \left(f_3 + \lambda f_1\right) \chi \, dx, \\ \int_\Omega \left(\lambda^2 v\zeta + \nabla v \nabla \zeta + l(v-u)\zeta\right) dx + \rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi)\psi(\xi,x,t)d\xi\right) \zeta \, d\Gamma \\ = \int_\Omega \left(f_4 + \lambda f_2\right) \zeta \, dx. \end{cases} \tag{2.16}$$

Using (2.13) in (2.16), we get

$$\begin{cases} \int_\Omega \left(\lambda^2 u\chi + \nabla u \nabla \chi + l(u-v)\chi\right) dx + \rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi)\frac{f_5(\xi) + y|_{\Gamma_2}\mu(\xi)}{\lambda + |\xi|^2 + \eta} d\xi\right) \chi \, d\Gamma \\ = \int_\Omega \left(f_3 + \lambda f_1\right) \chi \, dx, \\ \int_\Omega \left(\lambda^2 v\zeta + \nabla v \nabla \zeta + l(v-u)\zeta\right) dx + \rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi)\frac{f_6(\xi) + z|_{\Gamma_2}\mu(\xi)}{\lambda + |\xi|^2 + \eta} d\xi\right) \zeta \, d\Gamma \\ = \int_\Omega \left(f_4 + \lambda f_2\right) \zeta \, dx. \end{cases} \tag{2.17}$$

By inserting (2.12) into (2.17), we obtain

$$\begin{cases} \int_\Omega \left(\lambda^2 u\chi + \nabla u \nabla \chi + l(u-v)\chi\right) dx + \rho_1 \lambda C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu^2(\xi)\frac{u|_{\Gamma_2}}{\lambda + |\xi|^2 + \eta} d\xi\right) \chi \, d\Gamma \\ = \int_\Omega \left(f_3 + \lambda f_1\right) \chi \, dx - \rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi)\frac{f_5(\xi) - f_1|_{\Gamma_2}(\xi)\mu(\xi)}{\lambda + |\xi|^2 + \eta} d\xi\right) \chi(x) d\Gamma, \\ \int_\Omega \left(\lambda^2 v\zeta + \nabla v \nabla \zeta + l(v-u)\zeta\right) dx + \rho_2 \lambda C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu^2(\xi)\frac{v|_{\Gamma_2}}{\lambda + |\xi|^2 + \eta} d\xi\right) \zeta \, d\Gamma \\ = \int_\Omega \left(f_4 + \lambda f_2\right) \zeta \, dx - \rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi)\frac{f_6(\xi) - f_2|_{\Gamma_2}(\xi)\mu(\xi)}{\lambda + |\xi|^2 + \eta} d\xi\right) \zeta(x) d\Gamma. \end{cases}$$

Then,

$$\begin{cases} \quad \int_\Omega \left(\lambda^2 u\chi + \nabla u \nabla \chi + l(u-v)\chi\right) dx + \int_\Omega \left(\lambda^2 v\zeta + \nabla v \nabla \zeta + l(v-u)\zeta\right) dx \\ + \rho_1 \lambda C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu^2(\xi)\frac{u|_{\Gamma_2}}{\lambda + |\xi|^2 + \eta} d\xi\right) \chi \, d\Gamma \\ + \rho_2 \lambda C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu^2(\xi)\frac{v|_{\Gamma_2}}{\lambda + |\xi|^2 + \eta} d\xi\right) \zeta \, d\Gamma \\ = \int_\Omega \left(f_3 + \lambda f_1\right) \chi \, dx + \int_\Omega \left(f_4 + \lambda f_2\right) \zeta \, dx \\ - \rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi)\frac{f_5(\xi) - f_1|_{\Gamma_2}(\xi)\mu(\xi)}{\lambda + |\xi|^2 + \eta} d\xi\right) \chi(x) d\Gamma \\ - \rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi)\frac{f_6(\xi) - f_2|_{\Gamma_2}(\xi)\mu(\xi)}{\lambda + |\xi|^2 + \eta} d\xi\right) \zeta(x) d\Gamma. \end{cases} \tag{2.18}$$

Consequently, problem (2.18) is equivalent to the problem

$$a\left((u,v),(\chi,\zeta)\right) = L\left(\chi,\zeta\right), \tag{2.19}$$

where the bilinear form $a : \left(H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)\right)^2 \longrightarrow \mathbb{C}$ and the linear form $L : H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega) \longrightarrow \mathbb{C}$ are defined by

$$\begin{aligned}
a\left((u,v),(\chi,\zeta)\right) = & \int_\Omega \left(\lambda^2 u\chi + \nabla u \nabla \chi + l(u-v)\chi\right) dx \\
& + \int_\Omega \left(\lambda^2 v\zeta + \nabla v \nabla \zeta + l(v-u)\zeta\right) dx \\
& + \rho_1 \lambda C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu^2(\xi) \frac{u|_{\Gamma_2}}{\lambda + |\xi|^2 + \eta} d\xi \right) \chi \, d\Gamma \\
& + \rho_2 \lambda C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu^2(\xi) \frac{v|_{\Gamma_2}}{\lambda + |\xi|^2 + \eta} d\xi \right) \zeta \, d\Gamma,
\end{aligned}$$

and

$$\begin{aligned}
L(\chi,\zeta) = & \int_\Omega \left(f_3 + \lambda f_1\right) \chi \, dx + \int_\Omega \left(f_4 + \lambda f_2\right) \zeta \, dx \\
& - \rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi) \frac{f_5(\xi) - f_1|_{\Gamma_2}(\xi)\mu(\xi)}{\lambda + |\xi|^2 + \eta} d\xi \right) \chi(x) d\Gamma \\
& - \rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi) \frac{f_6(\xi) - f_2|_{\Gamma_2}(\xi)\mu(\xi)}{\lambda + |\xi|^2 + \eta} d\xi \right) \zeta(x) d\Gamma.
\end{aligned}$$

It is easy to verify that a is continuous and coercive, and L is continuous. Applying the Lax-Milgram's theorem, we infer that for all $(\chi,\zeta) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ problem (2.19) has a unique solution $(u,v) \in H^1_{\Gamma_1} \times H^1_{\Gamma_1}$. Moreover, by the regularity theory for the linear elliptic equations, it follows that $u, v \in H^2(\Omega)$.

To establish the existence of U in $D(A)$, it is necessary to demonstrate that $\varphi(x;\xi)$, $\psi(x;\xi)$, $|\xi|\varphi(x;\xi)$, and $|\xi|\psi(x;\xi)$ belong to $L^2(\Gamma_2 \times \mathbb{R}^n)$. From (2.13), we obtain:

$$\begin{aligned}
\int_{\Gamma_2} \int_{\mathbb{R}^n} |\varphi(x;\xi)|^2 \, d\xi \, d\Gamma_2 \leq & \; 3 \int_{\Gamma_2} \int_{\mathbb{R}^n} \frac{|f_5(x;\xi)|^2}{\left(1 + |\xi|^2 + \eta\right)^2} d\xi \, d\Gamma_2 \\
& + 3 \left(\int_{\Gamma_2} \left(|\lambda u|^2 + |f_1|^2\right) d\Gamma_2 \right) \int_{\mathbb{R}^n} \frac{|\xi|^{2\alpha - n}}{\left(1 + |\xi|^2 + \eta\right)^2} d\xi.
\end{aligned}$$

Using Lemma 2.2, it is evident that:

$$\int_{\mathbb{R}^n} \frac{|\xi|^{2\alpha-n}}{\left(1 + |\xi|^2 + \eta\right)^2} d\xi \leq \int_{\mathbb{R}^n} \frac{|\xi|^{2\alpha-n}}{1 + |\xi|^2 + \eta} d\xi < +\infty.$$

Moreover, considering that $f_5 \in L^2(\Gamma_2 \times \mathbb{R}^n)$, we have:

$$\int_{\Gamma_2} \int_{\mathbb{R}^n} \frac{|f_5(x;\xi)|^2}{\left(1 + |\xi|^2 + \eta\right)^2} d\xi \, d\Gamma_2 \leq \frac{1}{(1+\eta)^2} \int_{\Gamma_2} \int_{\mathbb{R}^n} |f_5(x;\xi)|^2 \, d\xi \, d\Gamma_2 < +\infty.$$

Therefore, applying the trace theorem, we conclude that $\varphi(x,\xi) \in L^2(\Gamma_2 \times \mathbb{R}^n)$. Further, from (2.13), we derive:

$$\begin{aligned}
\int_{\Gamma_2} \int_{\mathbb{R}^n} |\xi \varphi(x;\xi)|^2 \, d\xi \, d\Gamma_2 \leq & \; 3 \int_{\Gamma_2} \int_{\mathbb{R}^n} \frac{|\xi f_5(x;\xi)|^2}{\left(1 + |\xi|^2 + \eta\right)^2} d\xi \, d\Gamma_2 \\
& + 3 \left(\int_{\Gamma_2} \left(|\lambda u|^2 + |f_1|^2\right) d\Gamma_2 \right) \int_{\mathbb{R}^n} \frac{|\xi|^{2\alpha - n + 2}}{\left(1 + |\xi|^2 + \eta\right)^2} d\xi.
\end{aligned}$$

Using the trace theorem and Lemma 2.3, we obtain:

$$\left(\int_{\Gamma_2} \left(|\lambda u|^2 + |f_1|^2\right) d\Gamma_2\right) \int_{\mathbb{R}^n} \frac{|\xi|^{2\alpha-n+2}}{\left(1+|\xi|^2+\eta\right)^2} d\xi < +\infty.$$

Given that $|\xi|^2 < |\xi|^2 + \eta + 1$ and $f_5 \in L^2(\Gamma_2 \times \mathbb{R}^n)$, we find:

$$\int_{\Gamma_2}\int_{\mathbb{R}^n} \frac{|\xi f_5(x;\xi)|^2}{\left(1+|\xi|^2+\eta\right)^2} d\xi\, d\Gamma_2 \le \int_{\Gamma_2}\int_{\mathbb{R}^n} |f_5(x;\xi)|^2\, d\xi\, d\Gamma_2 < +\infty.$$

Thus, $|\xi\varphi| \in L^2(\Gamma_2 \times \mathbb{R}^n)$.

Similarly, it can be shown that $\psi(x;\xi), |\xi\psi(x;\xi)| \in L^2(\Gamma_2 \times \mathbb{R}^n)$.

Finally, there exists a unique $U := (u, v, y, z, \varphi, \psi) \in D(\mathcal{A})$ that solves $(\lambda I - \mathcal{A})U = F$.

Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. Finally, the result of Theorem 2.5 follows from the Lumer-Phillips theorem. □

3 Lack of exponential stability

With the objective of demonstrating the lack of exponential stability, we need the following theorem:

Theorem 3.1 [15, 18] *Let $S(t) = e^{\mathcal{A}t}$ be a $\mathcal{C}_0$ semi-group of contractions on the Hilbert space. Then, $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{3.1}$$

and

$$\overline{\lim_{|\beta|\longrightarrow\infty}} \left\|(i\beta I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < +\infty. \tag{3.2}$$

Our main result in this section is the following:

Theorem 3.2 *The semi-group generated by the operator $\mathcal{A}$ is not exponentially stable in the energy space $\mathcal{H}$.*

Proof Let $-\alpha_n^2 = (i\alpha_n)^2$ be a sequence of eigenvalues corresponding to the sequence of normalized eigenfunctions u_n of the operator Δ, such that $|\alpha_n| \to \infty$ as $n \to \infty$ and

$$\begin{cases} \Delta u_n = -\alpha_n^2 u_n, & \text{in } \Omega, \\ u_n = 0, & \text{on } \Gamma_1. \end{cases} \tag{3.3}$$

Our objective is to demonstrate, given certain conditions, that if $i\alpha_n$ satisfies (3.1), then (3.2) does not hold. In other words, we aim to establish the existence of an infinite number of eigenvalues of $\mathcal{A}$ approaching the imaginary axis, which prevents the exponential stability of the wave system (2.1). To begin, we compute the characteristic equation yielding the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector

$U = (u, v, y, z, \varphi, \psi)^T$. Then, $\mathcal{A}U = \lambda U$ is equivalent to:

$$\begin{cases} \lambda u - y = 0, \\ \lambda v - z = 0, \\ \lambda y - \Delta u + l(u - v) = 0, \\ \lambda z - \Delta v + l(v - u) = 0, \\ \lambda\varphi + \left(|\xi|^2 + \eta\right)\varphi - y|_{\Gamma_2}\mu(\xi) = 0, \\ \lambda\psi + \left(|\xi|^2 + \eta\right)\psi - z|_{\Gamma_2}\mu(\xi) = 0. \end{cases}$$

We observe that if we consider the decomposition given by $\Phi := u + v$, $\Theta := y + z$ and $\Lambda := \varphi + \psi$, we have:

$$\begin{cases} \lambda\Phi - \Theta = 0, \\ \lambda\Theta - \Delta\Phi = 0, \\ \lambda\Lambda + \left(|\xi|^2 + \eta\right)\Lambda - \Theta|_{\Gamma_2}\mu(\xi) = 0. \end{cases} \tag{3.4}$$

Taking $\theta := u - v$, $\kappa := y - z$ and $\sigma := \varphi - \psi$, we have:

$$\begin{cases} \lambda\theta - \kappa = 0, \\ \lambda\kappa - \Delta\theta + 2l\theta = 0, \\ \lambda\sigma + \left(|\xi|^2 + \eta\right)\sigma - \kappa|_{\Gamma_2}\mu(\xi) = 0. \end{cases} \tag{3.5}$$

The problem presented in (3.5) can be reformulated as

$$V_t = \mathcal{A}_1 V, \quad V(0) = V_0,$$

where $V_0 = (\theta_0, \kappa_0, \sigma_0)^T$, and $\mathcal{A}_1 : D(\mathcal{A}) \subset H \to H$ is defined as follows

$$\mathcal{A}_1(\theta, \kappa, \sigma) = \left(\kappa, \Delta\theta - 2l\theta, -\left(|\xi|^2 + \eta\right)\sigma + \kappa|_{\Gamma_2}\mu(\xi)\right).$$

Moreover, we note that

$u := \frac{1}{2}(\Phi + \theta)$, $v := \frac{1}{2}(\Phi - \theta)$, $y := \frac{1}{2}(\Theta + \kappa)$, $z := \frac{1}{2}(\Theta - \kappa)$, $\varphi := \frac{1}{2}(\Lambda + \sigma)$ and $\psi := \frac{1}{2}(\Lambda - \sigma)$.

We define the Hilbert space

$$H = H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\mathbb{R}^n),$$

equipped with the following inner product

$$\begin{cases} \langle V_1, V_2\rangle_H = \displaystyle\int_\Omega \left(\Theta_1\overline{\Theta_2} + \nabla\Phi_1\nabla\overline{\Phi_2}\right)dx + (\rho_1 + \rho_2)\, C\int_{\Gamma_2}\int_{\mathbb{R}^n} \Lambda_1\overline{\Lambda_2}\, d\xi\, d\Gamma, \\ \langle W_1, W_2\rangle_H = \displaystyle\int_\Omega \left(\theta_1\overline{\theta_2} + \nabla\kappa_1\nabla\overline{\kappa_2} + l\theta_1\overline{\theta_2}\right)dx + (\rho_1 + \rho_2)\, C\int_{\Gamma_2}\int_{\mathbb{R}^n} \sigma_1\overline{\sigma_2}\, d\xi\, d\Gamma, \end{cases}$$

where $V_1 = (\Phi_1, \Theta_1, \Lambda_1)$, $V_2 = (\Phi_2, \Theta_2, \Lambda_2)$, $W_1 = (\theta_1, \kappa_1, \sigma_1)$, and $W_2 = (\theta_2, \kappa_2, \sigma_2)$. Note that the inner product $\langle U_1, U_2\rangle_{\mathcal{H}}$ defined in (2.5) satisfies the equality

$$\langle U_1, U_2\rangle_{\mathcal{H}} = \frac{1}{2}\left(\langle V_1, V_2\rangle_H + \langle W_1, W_2\rangle_H\right).$$

Now, we must proceed to solve the problems (3.4) and (3.5). From $(3.5)_1$, we have

$$\kappa = \lambda\theta. \tag{3.6}$$

Inserting (3.6) in $(3.5)_2$, we get

$$\Delta\theta = \left(\lambda^2 + 2l\right)\theta. \tag{3.7}$$

From (3.3) and (3.7), we obtain the existence of a sequence of eigenvalues λ_n of $\mathcal{A}_1$ corresponding to the sequence α_n, such that

$$-\alpha_n^2\theta_n = \Delta\theta_n = \left(\lambda_n^2 + 2l\right)\theta_n,$$

then, we obtain

$$\alpha_n^2 = -\lambda_n^2 - 2l.$$

By taking $\sigma_n = \dfrac{\mu(\xi)}{|\xi|^2 + \eta + i\alpha_n}\theta_n|_{\Gamma_2}$ and $V_n = \left(\dfrac{\theta_n}{i\alpha_n}, \theta_n, \sigma_n\right)^T$. We have $V_n \in D(\mathcal{A}_1)$. Then, a direct computation gives

$$\mathcal{A}_1 \begin{pmatrix} \dfrac{\theta_n}{i\alpha_n} \\ \theta_n \\ \sigma_n \end{pmatrix} = \begin{pmatrix} \theta_n \\ i\alpha_n\theta_n - \dfrac{2l\theta_n}{i\alpha_n} \\ i\alpha_n\sigma_n \end{pmatrix}.$$

It follows that

$$(i\alpha_n I - \mathcal{A}_1)\, V_n = \begin{pmatrix} 0 \\ \dfrac{2l\theta_n}{i\alpha_n} \\ 0 \end{pmatrix}.$$

Using the fact that

$$2l \left\| \frac{\theta_n}{\alpha_n} \right\| \longrightarrow 0 \quad \text{as} \quad |\alpha_n| \longrightarrow \infty,$$

and the fact that θ_n is a normalized eigenfunction of the operator Δ, for each $n \in \mathbb{N}$, one gets

$$\| (i\alpha_n I - \mathcal{A}_1)^{-1} \|_{\mathcal{L}(\mathcal{H})} \longrightarrow \infty \quad \text{as} \quad |\alpha_n| \longrightarrow \infty.$$

This completes the proof. □

4 Stability

4.1 Strong stability of the system

In this part, we use the general criteria of Arendt and Batty (see [8]), following which a $\mathcal{C}_0$-semi-group of contractions $e^{\mathcal{A}t}$ in a Banach space is strongly stable if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contains only a countable number of elements.

Our main result in this part is the following theorem.

Theorem 4.1 [7] *Suppose that* $\eta \geq 0$. *The* $\mathcal{C}_0$*-semi-group* $e^{\mathcal{A}t}$ *is strongly stable in* $\mathcal{H}$; *i.e, for all* $U_0 \in \mathcal{H}$, *the solution of* (2.6) *satisfies*

$$\lim_{t \longrightarrow \infty} \left\| e^{\mathcal{A}t} U_0 \right\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 4.1, we need the following lemmas:

Lemma 4.2 $\mathcal{A}$ *has no eigenvalues on* $i\mathbb{R}$.

Proof We distinguish two cases, $i\lambda = 0$ and $i\lambda \neq 0$.

Case 1. Solving $\mathcal{A}U = 0$ leads to $U = 0$, thanks to the boundary conditions in (2.8). Hence, $i\lambda = 0$ is not an eigenvalue of $\mathcal{A}$.

Case 2. We will argue by contradiction. Let us suppose that there exists $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $U \neq 0$, such that $\mathcal{A}U = i\lambda U$. Then, we get

$$\begin{cases} i\lambda u - y = 0, \\ i\lambda v - z = 0, \\ i\lambda y - \Delta u + l(u - v) = 0, \\ i\lambda z - \Delta v + l(v - u) = 0, \\ i\lambda \varphi + \left(|\xi|^2 + \eta\right) \varphi - y|_{\Gamma_2} \mu(\xi) = 0, \\ i\lambda \psi + \left(|\xi|^2 + \eta\right) \psi - z|_{\Gamma_2} \mu(\xi) = 0. \end{cases} \tag{4.1}$$

Then, from (2.9), we have

$$\varphi \equiv 0, \quad \psi \equiv 0.$$

Hence, from (2.8) and $(4.1)_5$ and $(4.1)_6$, we obtain

$$\begin{cases} \frac{\partial u}{\partial \nu} = 0 \text{ and } y = 0, \text{ on } \Gamma_2, \\ \frac{\partial v}{\partial \nu} = 0 \text{ and } z = 0, \text{ on } \Gamma_2, \end{cases}$$

and so $u = v = 0$ on Γ_2.

By eliminating y and z, the system (4.1) implies that

$$\begin{cases} -\lambda^2 u - \Delta u + l(u - v) = 0, & \text{on} \quad \Omega, \\ -\lambda^2 v - \Delta v + l(v - u) = 0, & \text{on} \quad \Omega, \\ u = v = 0, & \text{on} \quad \Gamma_1, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & \text{on} \quad \Gamma_2. \end{cases}$$

Using the unique continuation theorem, one gets $u = v \equiv 0$ on Ω and therefore $U = 0$.

This completes the proof. □

Lemma 4.3 *For* $\lambda \neq 0$ *or* $\lambda = 0$ *and* $\eta \neq 0$ *the operator* $i\lambda I - \mathcal{A}$ *is surjective.*

Proof *Case* 1: $\lambda \neq 0$.

We will demonstrate that the operator $i\lambda I - \mathcal{A}$ is surjective for $\lambda \neq 0$. Letting $F = (f_1,f_2,f_3,f_4,f_5,f_6) \in \mathcal{H}$, we seek for $X = (u,v,y,z,\varphi,\psi) \in D(\mathcal{A})$ solution of the following equation

$$(i\lambda I - \mathcal{A})X = F.$$

Equivalently, we have

$$\begin{cases} i\lambda u - y = f_1, \\ i\lambda v - z = f_2, \\ i\lambda y - \Delta u + l(u-v) = f_3, \\ i\lambda z - \Delta v + l(v-u) = f_4, \\ i\lambda \varphi + \left(|\xi|^2 + \eta\right)\varphi - y|_{\Gamma_2}\mu(\xi) = f_5, \\ i\lambda \psi + \left(|\xi|^2 + \eta\right)\psi - z|_{\Gamma_2}\mu(\xi) = f_6. \end{cases} \tag{4.2}$$

From $(4.2)_1$ and $(4.2)_2$, we find that

$$\begin{cases} y = i\lambda u - f_1, \\ z = i\lambda v - f_2. \end{cases} \tag{4.3}$$

By inserting $(4.3)_1$ into $(4.2)_3$ and $(4.3)_2$ into $(4.2)_4$, we get

$$\begin{cases} -\lambda^2 u - \Delta u + l(u-v) = f_3 + i\lambda f_1, \\ -\lambda^2 v - \Delta v + l(v-u) = f_4 + i\lambda f_2. \end{cases} \tag{4.4}$$

Solving system (4.4) is equivalent to finding $(u,v) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ such that

$$\begin{cases} \int_\Omega \left(-\lambda^2 u\chi - \Delta u\chi + l(u-v)\chi\right) dx = \int_\Omega \left(f_3 + i\lambda f_1\right)\chi\, dx, \\ \int_\Omega \left(-\lambda^2 v\zeta - \Delta v\zeta + l(v-u)\zeta\right) dx = \int_\Omega \left(f_4 + i\lambda f_2\right)\zeta\, dx, \end{cases} \tag{4.5}$$

for all $(\chi,\zeta) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.

From (4.5), one can see that the functions u and v satisfy the following system

$$\begin{cases} \int_\Omega \left(-\lambda^2 u\chi + \nabla u\nabla\chi + l(u-v)\chi\right) dx + \rho_1 C\int_{\Gamma_2}\left(\int_{\mathbb{R}^n}\mu(\xi)\varphi(\xi,x,t)d\xi\right)\chi\, d\Gamma \\ = \int_\Omega \left(f_3 + i\lambda f_1\right)\chi\, dx, \\ \int_\Omega \left(-\lambda^2 v\zeta + \nabla v\nabla\zeta + l(v-u)\zeta\right) dx + \rho_2 C\int_{\Gamma_2}\left(\int_{\mathbb{R}^n}\mu(\xi)\psi(\xi,x,t)d\xi\right)\zeta\, d\Gamma \\ = \int_\Omega \left(f_4 + i\lambda f_2\right)\zeta\, dx. \end{cases}$$

Then, using (4.2)$_5$ and (4.2)$_6$, we get

$$\begin{cases} \int_\Omega \left(-\lambda^2 u\chi + \nabla u \nabla \chi + l(u-v)\chi\right) dx + \rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi) \frac{f_5(\xi) + y|_{\Gamma_2}\mu(\xi)}{i\lambda + |\xi|^2 + \eta} d\xi \right) \chi \, d\Gamma \\ = \int_\Omega \left(f_3 + i\lambda f_1\right) \chi \, dx, \\ \int_\Omega \left(-\lambda^2 v\zeta + \nabla v \nabla \zeta + l(v-u)\zeta\right) dx + \rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi) \frac{f_6(\xi) + z|_{\Gamma_2}\mu(\xi)}{i\lambda + |\xi|^2 + \eta} d\xi \right) \zeta \, d\Gamma \\ = \int_\Omega \left(f_4 + i\lambda f_2\right) \zeta \, dx. \end{cases} \tag{4.6}$$

By inserting (4.3) into (4.6), we obtain

$$\begin{cases} \int_\Omega \left(-\lambda^2 u\chi + \nabla u \nabla \chi + l(u-v)\chi\right) dx + i\lambda\rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu^2(\xi) \frac{u|_{\Gamma_2}}{i\lambda + |\xi|^2 + \eta} d\xi \right) \chi \, d\Gamma \\ = \int_\Omega \left(f_3 + i\lambda f_1\right) \chi \, dx - \rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi) \frac{f_5(\xi) - f_1|_{\Gamma_2}(\xi)\mu(\xi)}{i\lambda + |\xi|^2 + \eta} d\xi \right) \chi(x) d\Gamma, \\ \int_\Omega \left(-\lambda^2 v\zeta + \nabla v \nabla \zeta + l(v-u)\zeta\right) dx + i\lambda\rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu^2(\xi) \frac{v|_{\Gamma_2}}{i\lambda + |\xi|^2 + \eta} d\xi \right) \zeta \, d\Gamma \\ = \int_\Omega \left(f_4 + i\lambda f_2\right) \zeta \, dx - \rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi) \frac{f_6(\xi) - f_2|_{\Gamma_2}(\xi)\mu(\xi)}{i\lambda + |\xi|^2 + \eta} d\xi \right) \zeta(x) d\Gamma. \end{cases}$$

Then,

$$\begin{cases} \int_\Omega \left(-\lambda^2 u\chi + \nabla u \nabla \chi + l(u-v)\chi\right) dx + \int_\Omega \left(-\lambda^2 v\zeta + \nabla v \nabla \zeta + l(v-u)\zeta\right) dx, \\ +i\lambda\rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu^2(\xi) \frac{u|_{\Gamma_2}}{i\lambda + |\xi|^2 + \eta} d\xi \right) \chi \, d\Gamma, \\ +i\lambda\rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu^2(\xi) \frac{v|_{\Gamma_2}}{i\lambda + |\xi|^2 + \eta} d\xi \right) \zeta \, d\Gamma, \\ = \int_\Omega \left(f_3 + i\lambda f_1\right) \chi \, dx + \int_\Omega \left(f_4 + i\lambda f_2\right) \zeta \, dx, \\ -\rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi) \frac{f_5(\xi) - f_1|_{\Gamma_2}(\xi)\mu(\xi)}{i\lambda + |\xi|^2 + \eta} d\xi \right) \chi(x) d\Gamma, \\ -\rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi) \frac{f_6(\xi) - f_2|_{\Gamma_2}(\xi)\mu(\xi)}{i\lambda + |\xi|^2 + \eta} d\xi \right) \zeta(x) d\Gamma. \end{cases} \tag{4.7}$$

We can rewrite (4.7) as

$$-L_\lambda\Big((u,v),(\chi,\zeta)\Big) + \langle (u,v),(\chi,\zeta)\rangle_{H^1_{\Gamma_1}(\Omega)} = \ell(\chi,\zeta), \tag{4.8}$$

where

$$\langle (u,v),(\chi,\zeta)\rangle_{H^1_{\Gamma_1}(\Omega)} = \int_\Omega \left(\nabla u \nabla \chi + \nabla v \nabla \zeta + l(u-v)(\chi-\zeta)\right) dx,$$

$$\begin{aligned} L_\lambda\Big((u,v),(\chi,\zeta)\Big) = \lambda^2 \int_\Omega (u\chi + v\zeta)\, dx - i\lambda\rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu^2(\xi) \frac{u|_{\Gamma_2}}{i\lambda + |\xi|^2 + \eta} d\xi \right) \chi \, d\Gamma \\ - i\lambda\rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu^2(\xi) \frac{v|_{\Gamma_2}}{i\lambda + |\xi|^2 + \eta} d\xi \right) \zeta \, d\Gamma, \end{aligned}$$

and

$$
\begin{aligned}
\ell(\chi,\zeta) = & \int_\Omega \left(f_3 + i\lambda f_1\right)\chi\,dx + \int_\Omega \left(f_4 + i\lambda f_2\right)\zeta\,dx \\
& - \rho_1 C \int_{\Gamma_2}\left(\int_{\mathbb{R}^n}\mu(\xi)\frac{f_5(\xi) - f_1|_{\Gamma_2}(\xi)\mu(\xi)}{i\lambda + |\xi|^2 + \eta}d\xi\right)\chi(x)d\Gamma \\
& - \rho_2 C \int_{\Gamma_2}\left(\int_{\mathbb{R}^n}\mu(\xi)\frac{f_6(\xi) - f_2|_{\Gamma_2}(\xi)\mu(\xi)}{i\lambda + |\xi|^2 + \eta}d\xi\right)\zeta(x)d\Gamma.
\end{aligned}
$$

Using the continuous embedding from $L^2(\Omega)$ into $H^{-1}(\Omega)$ and the compactness embedding from $H^1_{\Gamma_1}(\Omega)$ into $L^2(\Omega)$, we deduce that the operator L_λ is compact from $L^2(\Omega)$ into $L^2(\Omega)$. Consequently, by the Fredholm alternative, proving the existence of (u,v) solution of (4.8) reduces to proving that 1 is not an eigenvalue of L_λ for $\ell \equiv 0$. Let us suppose that 1 is an eigenvalue, then there exists $(u,v) \neq 0$ such that

$$
L_\lambda\Big((u,v),(\chi,\zeta)\Big) = \langle (u,v),(\chi,\zeta)\rangle_{H^1_{\Gamma_1}(\Omega)}, \ \forall(\chi,\zeta) \in \Big(H^1_{\Gamma_1}(\Omega)\Big)^2. \tag{4.9}
$$

In particular, for $(\chi,\zeta) = (u,v)$, it follows that

$$
\begin{aligned}
& \lambda^2\left(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\right) - i\lambda\rho_1 C\,\|u\|\|^2_{L^2(\Gamma_2)}\int_{\mathbb{R}^n}\frac{\mu^2(\xi)}{i\lambda + |\xi|^2 + \eta}d\xi \\
& - i\lambda\rho_2 C\,\|v\|\|^2_{L^2(\Gamma_2)}\int_{\mathbb{R}^n}\frac{\mu^2(\xi)}{i\lambda + |\xi|^2 + \eta}d\xi \\
& = \|\nabla u\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} + l\,\|u - v\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.10}
$$

Taking the imaginary part in (4.10), we deduce that

$$
u = v = 0, \ \text{on} \quad \Gamma_2.
$$

Considering u and v are within $D(\mathcal{A})$, we infer that

$$
u = v = 0, \ \text{on}\ \Gamma.
$$

From (4.9), we obtain

$$
\begin{cases}
-\lambda^2 u - \Delta u + l(u - v) = 0, \text{ on } \Omega, \\
-\lambda^2 v - \Delta v + l(v - u) = 0, \text{ on } \Omega, \\
u = v = 0, \qquad\qquad \text{on } \Gamma.
\end{cases}
$$

Utilizing the analogous reasoning as presented in Lemma 4.2 based on the boundary geometric condition, one derives $u = v = 0$. Then, $U = 0$. Hence, $i\lambda I - \mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}^*$.

Case 2: $\lambda = 0$ and $\eta \neq 0$.

System (4.2) is reduced to the following system:

$$\begin{cases} -y = f_1, \\ -z = f_2, \\ -\Delta u + l(u-v) = f_3, \\ -\Delta v + l(v-u) = f_4, \\ \left(|\xi|^2 + \eta\right)\varphi - y|_{\Gamma_2}\mu(\xi) = f_5, \\ \left(|\xi|^2 + \eta\right)\psi - z|_{\Gamma_2}\mu(\xi) = f_6. \end{cases}$$

Consequently, from (4.7), we obtain

$$\begin{cases} \int_\Omega (\nabla u \nabla \chi + l(u-v)\chi)\, dx + \int_\Omega (\nabla v \nabla \zeta + l(v-u)\zeta)\, dx, \\ = \int_\Omega f_3 \chi\, dx + \int_\Omega f_4 \zeta\, dx - \rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi) \frac{f_5(\xi) - f_1|_{\Gamma_2}(\xi)\mu(\xi)}{|\xi|^2 + \eta} d\xi \right) \chi(x) d\Gamma, \\ -\rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi) \frac{f_6(\xi) - f_2|_{\Gamma_2}(\xi)\mu(\xi)}{|\xi|^2 + \eta} d\xi \right) \zeta(x) d\Gamma. \end{cases} \tag{4.11}$$

Thus, system (4.11) can be written as the problem

$$a_\eta\left((u,v),(\chi,\zeta)\right) = L_\eta(\chi,\zeta), \tag{4.12}$$

where the bilinear form $a_\eta : \left(H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)\right)^2 \to \mathbb{R}$ and the linear form $L_\eta : H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega) \to \mathbb{R}$ are defined as follows

$$a_\eta\left((u,v),(\chi,\zeta)\right) = \int_\Omega (\nabla u \nabla \chi + l(u-v)\chi)\, dx + \int_\Omega (\nabla v \nabla \zeta + l(v-u)\zeta)\, dx,$$

and

$$\begin{cases} L_\eta(\chi,\zeta) = \int_\Omega f_3 \chi\, dx + \int_\Omega f_4 \zeta\, dx \\ \qquad - \rho_1 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi) \frac{f_5(\xi) - f_1|_{\Gamma_2}(\xi)\mu(\xi)}{|\xi|^2 + \eta} d\xi \right) \chi(x) d\Gamma \\ \qquad - \rho_2 C \int_{\Gamma_2} \left(\int_{\mathbb{R}^n} \mu(\xi) \frac{f_6(\xi) - f_2|_{\Gamma_2}(\xi)\mu(\xi)}{|\xi|^2 + \eta} d\xi \right) \zeta(x) d\Gamma. \end{cases}$$

It is clear that the bilinear form a_η is continuous and coercive, and the linear form L_η is continuous. Then, by Lax-Milgram's theorem, the variational problem (4.12) admits a unique solution $(u,v) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$, for all $(\chi,\zeta) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$. Then, we deduce from (4.11) that $(u,v) \in H^2(\Omega) \times H^2(\Omega)$. Consequently, the operator $\mathcal{A}$ is surjective. The proof of the lemma is thus complete. □

The following lemma is similar to Lemma 2.10 in [5], whose proof we have followed with slight modifications to fit our context.

Lemma 4.4 *Assume that* $\eta = 0$. *Then,* $0 \in \sigma(\mathcal{A})$.

Proof We argue by contradiction. We suppose that $0 \in \rho(\mathcal{A})$. Let consider $\omega_k \in H^1_{\Gamma_1}(\Omega)$ be an eigenfunction of the system:

$$\begin{cases} -\Delta \omega_k = \mu_k^2 \omega_k, & \text{in } \Omega, \\ \omega_k = 0, & \text{on } \Gamma_1, \\ \frac{\partial \omega_k}{\partial \nu} = 0, & \text{on } \Gamma_2. \end{cases}$$

Now, we define the vector $F = (0, \omega_k, 0, 0, 0, 0) \in \mathcal{H}$. Assume that there exists $U = (u, v, y, z, \varphi, \psi) \in D(\mathcal{A})$ such that

$$\mathcal{A}U = F.$$

It follows that

$$\begin{cases} z = \omega_k, & \text{in } \Omega, \\ y = 0, & \text{in } \Omega, \\ -|\xi|^2 \psi + \mu(\xi) z = 0, & \text{on } \Gamma_2, \end{cases}$$

and we deduce that $\psi(x,\xi) = |\xi|^{\frac{2\alpha-n-4}{2}} \omega_{k/\Gamma_2}$. We easy can check that, for $\alpha \in]0,1[$, the function $\psi(x,\xi) \notin L^2(\Gamma_2 \times \mathbb{R}^n)$. So, the assumption of the existence of U is false, and consequently the operator $\mathcal{A}$ is not invertible. □

Proof of Theorem 4.1. By Lemma 4.2, the operator $\mathcal{A}$ has no pure imaginary eigenvalues and by Lemma 4.3 $R(i\lambda - \mathcal{A}) = \mathcal{H}$ for all $\lambda \in \mathbb{R}^*$ and $R(i\lambda - \mathcal{A}) = \mathcal{H}$ for $\lambda = 0$ and for all $\eta > 0$. Therefore, the closed graph theorem of Banach implies that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ if $\eta > 0$ and $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ if $\eta = 0$. This completes the proof. □

4.2 The rate of decay of the $\mathcal{C}_0$ semi-group

Our main result in this part concerns the polynomial decay of the energy of the solution of (1.1) under a geometric condition. For this purpose, we consider the following hypothesis:

Denote ν the outward unit normal vector to Γ. Fix x_0 in Ω and define $m(x) = x - x_0$. We assume that

$$m \cdot \nu \leq 0 \text{ on } \Gamma_1 \quad \text{and} \quad m \cdot \nu \geq m_0 \text{ on } \Gamma_2, \tag{4.13}$$

where m_0 is a positive constant. We say that the boundary Γ satisfies the boundary multiplier geometric control condition (MGC).

Theorem 4.5 *Assume that $\eta > 0$ and the boundary multiplier geometric condition (MGC) (4.13) holds then for all initial data $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of U_0, such that the energy of the solution U of (1.1) satisfies the following estimation*:

$$E(t) \leq \frac{C}{t^{\frac{2}{1-\alpha}}} \|U_0\|_{D(\mathcal{A})}.$$

To prove Theorem 4.5, we establish a particular resolvent estimate based on a result of Batty in [10, 11] and Borichev and Tomilov in [14].

Theorem 4.6 *Assume that $\mathcal{A}$ is the generator of a strongly continuous semi-group of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ on the energy space $\mathcal{H}$. If $i\mathbb{R}\subset\rho(\mathcal{A})$, then for a fixed $\ell>0$, the following conditions are equivalent.*

1. $\sup_{\lambda\in\mathbb{R}}\left\|(i\lambda I-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})}=O\left(|\lambda|^{\ell}\right)$,
2. $\left\|e^{t\mathcal{A}}U_0\right\|_{\mathcal{H}}\leq\frac{C}{t^{\frac{1}{\ell}}}$, $\quad\forall t>0$, $U_0\in D(\mathcal{A})$, *for some* $C>0$.

We are now able to prove the energy of smooth solution of system (1.1) decays polynomial to 0 as t goes to infinity.

Proof of Theorem 4.5 As a consequence of Theorem 4.6, one has to prove that the operator $\mathcal{A}$ defined by (2.7) and (2.8) satisfies:

$$\limsup_{\lambda\in\mathbb{R},|\lambda|\to+\infty}\|\left(i\lambda I-\mathcal{A}\right)^{-1}\|_{\mathcal{L}(\mathcal{H})}=O(|\lambda|^{1-\alpha}).$$

For clarity, we divide the proof into several steps.

Step 1. By contradiction, suppose that

$$\limsup_{\lambda\in\mathbb{R},|\lambda|\to\infty}\|\left(i\lambda I-\mathcal{A}\right)^{-1}\|_{\mathcal{L}(\mathcal{H})}=\infty.$$

Then, there exists a sequence of real number $\lambda_n>0$ with $\lambda_n\to\infty$, and a sequence of vectors $(U_n)_n\in D(\mathcal{A})$ with

$$\|U_n\|_{\mathcal{H}}=1, \tag{4.14}$$

such that

$$\lambda_n^{1-\alpha}(i\lambda_n I-\mathcal{A})U_n:=F_n=o(1),\qquad\text{in }\mathcal{H}. \tag{4.15}$$

For simplicity, we drop in the next the index n. Our goal is to derive from (4.15) that U converges to zero, which consists of a contradiction.

Note that (4.15) is equivalent to

$$\begin{cases} i\lambda u-y=\dfrac{f_1}{\lambda^{1-\alpha}},\\ i\lambda v-z=\dfrac{f_2}{\lambda^{1-\alpha}},\\ i\lambda y-\Delta u+l(u-v)=\dfrac{f_3}{\lambda^{1-\alpha}},\\ i\lambda z-\Delta v+l(v-u)=\dfrac{f_4}{\lambda^{1-\alpha}},\\ i\lambda\varphi+\left(|\xi|^2+\eta\right)\varphi-y|_{\Gamma_2}\mu(\xi)=\dfrac{f_5}{\lambda^{1-\alpha}},\\ i\lambda\psi+\left(|\xi|^2+\eta\right)\psi-z|_{\Gamma_2}\mu(\xi)=\dfrac{f_6}{\lambda^{1-\alpha}}, \end{cases} \tag{4.16}$$

where

$$F=(f_1,f_2,f_3,f_4,f_5,f_6)\in\mathcal{H}.$$

Taking the inner product in $\mathcal{H}$ of (4.15) with

$$U = (u, v, y, z, \varphi, \psi)^T,$$

and using the fact that

$$\mathcal{R}e\,\langle (i\lambda - \mathcal{A})U, U\rangle_{\mathcal{H}} = \rho_1 C \int_{\Gamma_2}\int_{\mathbb{R}^n} \left(|\xi|^2 + \eta\right) |\varphi(\xi, x, t)|^2 d\xi\, d\Gamma + \rho_2 C \int_{\Gamma_2}\int_{\mathbb{R}^n} \left(|\xi|^2 + \eta\right) |\psi(\xi, x, t)|^2 d\xi\, d\Gamma,$$

we obtain

$$\| \left(|\xi|^2 + \eta\right)^{1/2} \varphi \|_{L^2(\mathbb{R}^n \times \Gamma_2)} = \frac{o(1)}{\lambda^{\frac{1-\alpha}{2}}} \text{ and } \| \left(|\xi|^2 + \eta\right)^{1/2} \psi \|_{L^2(\mathbb{R}^n \times \Gamma_2)} = \frac{o(1)}{\lambda^{\frac{1-\alpha}{2}}},$$

and we deduce that

$$\begin{aligned} &\|\varphi\|_{L^2(\mathbb{R}^n \times \Gamma_2)} = \frac{o(1)}{\lambda^{\frac{1-\alpha}{2}}}, \quad \|\psi\|_{L^2(\mathbb{R}^n \times \Gamma_2)} = \frac{o(1)}{\lambda^{\frac{1-\alpha}{2}}} \\ &\left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(\Gamma_2)} = \frac{o(1)}{\lambda^{\frac{1-\alpha}{2}}} \text{ and } \left\|\frac{\partial v}{\partial \nu}\right\|_{L^2(\Gamma_2)} = \frac{o(1)}{\lambda^{\frac{1-\alpha}{2}}}. \end{aligned} \tag{4.17}$$

Now, multiplying equation $(4.16)_5$ by $(i\lambda + |\xi|^2 + \eta)^{-1-n}$, integrating over $\mathbb{R}^n$, and applying the Cauchy-Schwartz inequality, one gets:

$$\begin{aligned} &|y_{/\Gamma_2}| \int_{\mathbb{R}^n} \frac{|\xi|^{\alpha+\frac{n}{2}}}{\left(|\lambda| + |\xi|^2 + \eta\right)^{n+1}} d\xi \\ &\leq \left(\int_{\mathbb{R}^n} \frac{|\xi|^{2n-2}}{\left(|\lambda| + |\xi|^2 + \eta\right)^{2n}} d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\xi \varphi(x,\xi)|^2 d\xi\right)^{\frac{1}{2}} \\ &+ \frac{1}{|\lambda|^{1-\alpha}} \left(\int_{\mathbb{R}^n} \frac{|\xi|^{2n}}{\left(|\lambda| + |\xi|^2 + \eta\right)^{2n+2}} d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \left|f_5(x,\xi)\right|^2 d\xi\right)^{\frac{1}{2}}. \end{aligned}$$

Using Young's inequality, Lemma A.3 in [5], and integrating over Γ_2, we deduce

$$\|y_{/\Gamma_2}\|_{L^2(\Gamma_2)} = o(1). \tag{4.18}$$

Similarly, we obtain

$$\|z_{/\Gamma_2}\|_{L^2(\Gamma_2)} = o(1). \tag{4.19}$$

Note also that from $(4.16)_1$ and $(4.16)_2$, we deduce

$$\|u\|_{L^2} = \frac{O(1)}{\lambda} \text{ and } \|v\|_{L^2} = \frac{O(1)}{\lambda}. \tag{4.20}$$

Step 2. By eliminating y and z, system (4.16) implies that

$$\begin{cases} -\lambda^2 u - \Delta u + l(u-v) = \dfrac{f_3 + i\lambda f_1}{\lambda^{1-\alpha}}, & \text{on } \Omega, \\ -\lambda^2 v - \Delta v + l(v-u) = \dfrac{f_4 + i\lambda f_2}{\lambda^{1-\alpha}}, & \text{on } \Omega, \\ u = v = 0, \quad on \quad \Gamma_1, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{o(1)}{\lambda^{\frac{1-\alpha}{2}}} \text{ and } \dfrac{\partial v}{\partial \nu} = \dfrac{o(1)}{\lambda^{\frac{1-\alpha}{2}}} & \text{in} \quad L^2(\Gamma_2). \end{cases} \tag{4.21}$$

By taking $\phi = u + v$, we get

$$\begin{cases} \lambda^2 \phi + \Delta \phi = \dfrac{-f_3 - f_4 - i\lambda f_1 - i\lambda f_2}{\lambda^{1-\alpha}}, \text{ on } \Omega, \\ \phi = 0, \text{ on } \Gamma_1, \\ \dfrac{\partial \phi}{\partial \nu} = \dfrac{o(1)}{\lambda^{\frac{1-\alpha}{2}}} \text{ in } L^2(\Gamma_2). \end{cases} \tag{4.22}$$

First, multiplying equation $(4.22)_1$ by $(n-1)\bar{\phi}$, integrating over Ω, and then using green formula and the boundary conditions, we get

$$\begin{aligned} \lambda^2(n-1)\int_\Omega |\phi|^2 dx - (n-1)\int_\Omega |\nabla\phi|^2 dx + (n-1)\int_{\Gamma_2} \frac{\partial \Phi}{\partial \nu}\bar{\Phi} d\Gamma \\ = (n-1)\int_\Omega \frac{-f_3 - f_4 - i\lambda f_1 - i\lambda f_2}{\lambda^{1-\alpha}}\overline{\Phi} dx. \end{aligned}$$

Using (4.17), (4.20), $(4.22)_4$ and the fact that $\|f_1\|_{H^1_{\Gamma_1}(\Omega)} = o(1)$, $\|f_2\|_{H^1_{\Gamma_1}(\Omega)} = o(1)$, $\|f_3\|_{L^2(\Omega)} = o(1)$ and $\|f_4\|_{L^2(\Omega)} = o(1)$, we deduce

$$(n-1)\int_\Omega |\lambda\phi|^2 dx - (n-1)\int_\Omega |\nabla\phi|^2 dx = \frac{o(1)}{\lambda^{\frac{1-\alpha}{2}}}. \tag{4.23}$$

Next, multiplying equation $(4.22)_1$ by $2(m\cdot\nabla\bar{\phi})$ and integrating over Ω, one gets:

$$\begin{aligned} 2\lambda^2 \int_\Omega \phi\,(m\cdot\nabla\bar{\phi})dx + 2\int_\Omega \Delta\phi\,(m\cdot\nabla\bar{\phi})dx \\ = 2\int_\Omega \frac{-f_3 - f_4 - i\lambda f_1 - i\lambda f_2}{\lambda^{1-\alpha}}(m\cdot\nabla\bar{\phi})dx = \frac{o(1)}{\lambda^{\frac{1-\alpha}{2}}}. \end{aligned} \tag{4.24}$$

Applying the integration by parts to the first integral in the left-hand side in equation (4.24), we get

$$2\lambda^2\int_\Omega \phi\,(m\cdot\nabla\bar{\phi})dx = -n\int_\Omega |\lambda\phi|^2 dx + \lambda^2\int_{\Gamma_2}(m\cdot\nu)|\phi|^2 d\Gamma. \tag{4.25}$$

Using the green formula on the second integral in the left-hand side in equation (4.24), we gets

$$2\int_\Omega \Delta\phi\,(m\cdot\nabla\bar{\phi})dx = (n-2)\int_\Omega |\nabla\phi|^2 dx + 2\int_\Gamma \frac{\partial\phi}{\partial\nu}(m\cdot\nabla\bar{\phi})d\Gamma - \int_\Gamma (m.\nu)|\nabla\Phi|^2 d\Gamma. \tag{4.26}$$

Inserting (4.25) and (4.26) in (4.24), we deduce

$$-n\int_{\Omega}|\lambda\phi|^2dx+(n-2)\int_{\Omega}|\nabla\phi|^2dx+\lambda^2\int_{\Gamma_2}(m\cdot\nu)|\phi|^2d\Gamma+2\int_{\Gamma}\frac{\partial\phi}{\partial\nu}(m\cdot\nabla\bar{\phi})d\Gamma - \int_{\Gamma}(m.\nu)|\nabla\Phi|^2d\Gamma=\frac{o(1)}{\lambda^{\frac{1-\alpha}{2}}}. \tag{4.27}$$

Combining (4.23) and (4.27), we obtain

$$\int_{\Omega}|\lambda\phi|^2dx+\int_{\Omega}|\nabla\phi|^2dx=\lambda^2\int_{\Gamma_2}(m\cdot\nu)|\phi|^2d\Gamma+2\int_{\Gamma}\frac{\partial\phi}{\partial\nu}(m\cdot\nabla\bar{\phi})d\Gamma - \int_{\Gamma}(m.\nu)|\nabla\Phi|^2d\Gamma+\frac{o(1)}{\lambda^{\frac{1-\alpha}{2}}}. \tag{4.28}$$

Using (4.18) and (4.19), we infer that

$$\lambda^2\int_{\Gamma_2}(m\cdot\nu)|\phi|^2d\Gamma=o(1). \tag{4.29}$$

On the other hand, it is easy to see that

$$2\int_{\Gamma}\frac{\partial\phi}{\partial\nu}(m\cdot\nabla\bar{\phi})d\Gamma-\int_{\Gamma}(m.\nu)|\nabla\Phi|^2d\Gamma=\int_{\Gamma_1}(m\cdot\nu)|\frac{\partial\Phi}{\partial\nu}|^2d\Gamma + 2\int_{\Gamma_2}\frac{\partial\phi}{\partial\nu}(m\cdot\nabla\bar{\phi})d\Gamma-\int_{\Gamma_2}(m.\nu)|\nabla\Phi|^2d\Gamma. \tag{4.30}$$

Inserting equations (4.29) and (4.30) in equation (4.28), we get

$$\int_{\Omega}|\lambda\phi|^2dx+\int_{\Omega}|\nabla\phi|^2dx=o(1)+\int_{\Gamma_1}(m\cdot\nu)|\frac{\partial\Phi}{\partial\nu}|^2d\Gamma+2\int_{\Gamma_2}\frac{\partial\phi}{\partial\nu}(m\cdot\nabla\bar{\phi})d\Gamma - \int_{\Gamma_2}(m.\nu)|\nabla\Phi|^2d\Gamma. \tag{4.31}$$

Letting $\epsilon>0$ and using the Young inequality, we get

$$\begin{aligned}2\Re\int_{\Gamma_2}\frac{\partial\phi}{\partial\nu}(m\cdot\nabla\bar{\phi})d\Gamma &\le \frac{\|m\|_\infty}{\epsilon}\int_{\Gamma_2}\left|\frac{\partial\phi}{\partial\nu}\right|^2d\Gamma+\epsilon\int_{\Gamma_2}|\nabla\Phi|^2d\Gamma\\ &=\epsilon\int_{\Gamma_2}|\nabla\Phi|^2d\Gamma+\frac{o(1)}{\lambda^{\frac{1-\alpha}{2}}}.\end{aligned} \tag{4.32}$$

Inserting equation (4.32) in equation (4.31) and using the (*MGC*) condition, we get

$$\int_{\Omega}|\lambda\phi|^2dx+\int_{\Omega}|\nabla\phi|^2dx\le(\epsilon-m_0)\int_{\Gamma_2}|\nabla\Phi|^2d\Gamma+o(1).$$

Taking $\epsilon<m_0$, we get

$$\int_{\Omega}|\lambda\phi|^2dx+\int_{\Omega}|\nabla\phi|^2dx\le o(1). \tag{4.33}$$

Thus, using equation (4.33), we obtain

$$\|\Phi\|_{H^1_{\Gamma_1}(\Omega)} = o(1). \tag{4.34}$$

Step 3. By replacing v by $\Phi - u$ in (4.21), we find

$$\begin{cases} (2l-\lambda^2)u - \Delta u = l\Phi + \dfrac{f_3 + i\lambda f_1}{\lambda^{1-\alpha}} \quad \text{on } \Omega, \\ u = 0 \quad \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{o(1)}{\lambda^{\frac{1-\alpha}{2}}} \quad \text{in} \quad L^2(\Gamma_2). \end{cases}$$

We proceed exactly as the beginning of the proof, and the system is verified by ϕ. However, in this case, we used $\lambda^2 - 2l$ instead of λ^2 and (4.34) to find

$$\|u\|_{H^1_{\Gamma_1}(\Omega)} = o(1) \text{ and consequently } \|v\|_{H^1_{\Gamma_1}(\Omega)} = o(1). \tag{4.35}$$

Step 4. On the other hand, multiplying $(4.16)_1$ by $\bar{u}$ and $(4.16)_2$ by $\bar{v}$ leads to:

$$\begin{cases} \displaystyle\int_\Omega i\lambda y\bar{u}\,dx - \int_\Omega \Delta u\bar{u}\,dx + l\int_\Omega (u-v)\bar{u}\,dx = \int_\Omega \frac{f_3\bar{u}}{\lambda^{1-\alpha}}\,dx, \\ \displaystyle\int_\Omega i\lambda z\bar{v}\,dx - \int_\Omega \Delta v\bar{v}\,dx + l\int_\Omega (v-u)\bar{v}\,dx = \int_\Omega \frac{f_4\bar{v}}{\lambda^{1-\alpha}}\,dx. \end{cases}$$

Then,

$$\begin{cases} \displaystyle -\int_\Omega y\overline{i\lambda u}\,dx + \int_\Omega |\nabla u|^2\,dx + \rho_1 C\int_{\Gamma_2}\left(\int_{\mathbb{R}^n} \mu(\xi)\varphi(\xi,x,t)d\xi\right)\bar{u}d\Gamma + l\int_\Omega (u-v)\bar{u}\,dx \\ \displaystyle\qquad = \int_\Omega \frac{f_3\bar{u}}{\lambda^{1-\alpha}}\,dx, \\ \displaystyle -\int_\Omega z\overline{i\lambda v}\,dx + \int_\Omega |\nabla v|^2\,dx + \rho_2 C\int_{\Gamma_2}\left(\int_{\mathbb{R}^n} \mu(\xi)\psi(\xi,x,t)d\xi\right)\bar{v}d\Gamma + l\int_\Omega (v-u)\bar{v}\,dx \\ \displaystyle\qquad = \int_\Omega \frac{f_4\bar{v}}{\lambda^{1-\alpha}}\,dx. \end{cases} \tag{4.36}$$

Replacing $(4.16)_1$ into $(4.36)_1$ and $(4.16)_2$ into $(4.36)_2$, we have

$$\begin{cases} \displaystyle -\int_\Omega y(\bar{y}+\overline{f_1})\,dx + \int_\Omega |\nabla u|^2\,dx + \rho_1 C\int_{\Gamma_2}\left(\int_{\mathbb{R}^n} \mu(\xi)\varphi(\xi,x,t)d\xi\right)\bar{u}d\Gamma \\ \displaystyle\qquad + l\int_\Omega (u-v)\bar{u}\,dx = \int_\Omega \frac{f_3\bar{u}}{\lambda^{1-\alpha}}\,dx, \\ \displaystyle -\int_\Omega z(\bar{z}+\overline{f_2})\,dx + \int_\Omega |\nabla v|^2\,dx + \rho_2 C\int_{\Gamma_2}\left(\int_{\mathbb{R}^n} \mu(\xi)\psi(\xi,x,t)d\xi\right)\bar{v}d\Gamma \\ \displaystyle\qquad + l\int_\Omega (v-u)\bar{v}\,dx = \int_\Omega \frac{f_4\bar{v}}{\lambda^{1-\alpha}}\,dx. \end{cases}$$

Consequently,

$$\begin{aligned}
-\int_\Omega \left(|y(x)|^2+|z(x)|^2\right)dx+\int_\Omega \left(|\nabla u|^2+|\nabla v|^2\right)dx+l\int_\Omega |u-v|^2dx \\
+\rho_1 C\int_{\Gamma_2}\left(\int_{\mathbb{R}^n}\mu(\xi)\varphi(\xi,t)d\xi\right)\bar{u}d\Gamma \\
+\rho_2 C\int_{\Gamma_2}\left(\int_{\mathbb{R}^n}\mu(\xi)\psi(\xi,t)d\xi\right)\bar{v}d\Gamma \\
=\int_\Omega \frac{f_3\bar{u}+y\overline{f_1}+f_4\bar{v}+z\overline{f_2}}{\lambda^{1-\alpha}}dx.
\end{aligned} \tag{4.37}$$

Using (4.14), (4.17), (4.35), (4.37) and the fact that $\|f_1\|_{H^1_{\Gamma_1}(\Omega)}=o(1)$, $\|f_2\|_{H^1_{\Gamma_1}(\Omega)}=o(1)$, $\|f_3\|_{L^2(\Omega)}=o(1)$ and $\|f_4\|_{L^2(\Omega)}=o(1)$, we deduce

$$\int_\Omega \left(|y(x)|^2+|z(x)|^2\right)dx=o(1). \tag{4.38}$$

Finally, using (4.14), (4.17), (4.35), and (4.38), one gets a contradiction.

Thus, the proof of Theorem 4.5 is complete. □

5 Conclusion

In this study, we have examined a coupled system of wave equations with boundary fractional dissipation applied locally. Our investigation began with establishing the well-posedness of the system through a semi-group approach, demonstrating the existence and uniqueness of solutions. While the system does not exhibit exponential stability, we confirmed its strong stability. Leveraging Arendt and Batty's criterion, we further showed that the energy of the system decays over time, following a polynomial rate. Moreover, we conjecture that the energy decay rate of type $t^{\frac{-2}{1-\alpha}}$ is optimal.

Acknowledgements

The authors extend their appreciation to the Deanship of Research and Graduate Studies at King Khaled University for funding this work through Large Research Project under grant number RGP2/37/45.

Author contributions

All authors contributed equally to this work.

Funding

Not applicable.

Data Availability

No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate

This was not required for the present study.

Competing interests

The authors declare no competing interests.

Author details

[1]Department of Mathematics, University Ain Témouchent, Belhadj Bouchaib, B.P. 284 RP, Ain Témouchent, 46000, Algeria. [2]Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis El Manar, LR Analyse Non-Linéaire et Géométrie, LR21ES08, El Manar 2, 2092, Tunis, Tunisia. [3]Department of Mathematics, Applied College in Mahayil, University of King Khalid, Abha, Saudi Arabia.

Received: 6 June 2024 Accepted: 2 September 2024 Published online: 12 September 2024

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