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Global boundedness in an attraction–repulsion Chemotaxis system with nonlinear productions and logistic source

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Abstract

This paper deals with the attraction–repulsion chemotaxis system with nonlinear productions and logistic source,

$$u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\Phi(u)\nabla v) + \nabla\cdot(\Psi(u)\nabla w) + f(u),$$
$$v_t = \Delta v + \alpha u^k - \beta v, \qquad \tau w_t = \Delta w + \gamma u^l - \delta w, \quad \tau\in\{0,1\},$$

in a bounded domain $\Omega\subset\mathbb{R}^n$ ($n\geq 1$), subject to the homogeneous Neumann boundary conditions and initial conditions, where $D,\Phi,\Psi\in C^2[0,\infty)$ are nonnegative with $D(s)\geq(s+1)^p$ for $s\geq 0$, $\Phi(s)\leq\chi s^q$, $\xi s^g\leq\Psi(s)\leq\zeta s^j$, $s\geq s_0$, for $s_0>1$, the logistic source satisfies $f(s)\leq s(a-bs^d)$, $s>0$, $f(0)\geq 0$, and the nonlinear productions for the attraction and repulsion chemicals are described via αu^k and γu^l, respectively. When $k=l=1$, it is known that this system possesses a globally bounded solution in some cases. However, there has been no work in the case $k,l>0$. This paper develops the global boundedness of the solution to the system in some cases and extends the global boundedness criteria established by Tian, He, and Zheng (2016) for the attraction–repulsion chemotaxis system.

Keywords: Chemotaxis; Attraction–repulsion; Nonlinear productions; Logistic source; Boundedness; Fully parabolic

1 Introduction

In this paper, we consider the boundedness in the attraction–repulsion chemotaxis system

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\Phi(u)\nabla v) + \nabla\cdot(\Psi(u)\nabla w) + f(u), & (x,t)\in\Omega\times(0,T),\\ v_t = \Delta v + \alpha u^k - \beta v, & (x,t)\in\Omega\times(0,T),\\ \tau w_t = \Delta w + \gamma u^l - \delta w, & (x,t)\in\Omega\times(0,T),\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad \tau w(x,0)=w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\tau \in \{0,1\}$, ν denotes the outward normal vector to $\partial\Omega$, and the parameters $\alpha, \beta, \gamma, \delta, k, l > 0$. The nonlinear nonnegative functions D, Φ, Ψ satisfy

$$D, \Phi, \Psi \in C^2[0, \infty), \tag{1.2}$$

$$D(s) \geq (s+1)^p, \quad s \geq 0, \tag{1.3}$$

$$0 \leq \Phi(s) \leq \chi s^q, \quad s > 1, \tag{1.4}$$

$$\xi s^g \leq \Psi(s) \leq \zeta s^j, \quad s > 1, \tag{1.5}$$

with $\chi, \xi, \zeta > 0$ and $p, q, g, j \in \mathbb{R}$. The logistic source $f \in C^2[0, \infty)$ fulfills

$$f(0) \geq 0 \quad \text{and} \quad f(s) \leq s\big(a - bs^d\big), \quad s > 0, \tag{1.6}$$

where $a, b, d > 0$. In model (1.1) the functions u, v, and w represent the cell density and the concentrations of attractive and repulsive chemical substances, respectively. The productions of v and w in the model are both nonlinear of the forms αu^k and γu^l. This would substantially affect the boundedness of solutions.

Model (1.1) is one of many types of the chemotaxis systems proposed by Keller and Segel [1] (for guidance on various variants, we refer to Hillen and Painter [2]). For the attraction–repulsion system with linear productions and logistic source, i.e.,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) + f(u), & (x,t) \in \Omega \times (0,T), \\ \tau_1 v_t = \Delta v + \alpha u - \beta v, & (x,t) \in \Omega \times (0,T), \\ \tau_2 w_t = \Delta w + \gamma u - \delta w, & (x,t) \in \Omega \times (0,T), \end{cases} \tag{1.7}$$

where $\tau_i \in \{0,1\}$ $(i = 1,2)$, in the case $\tau_1 = \tau_2 = 1$ with $f(u) = u(a - bu)$, the global boundedness was established in [3–6]. Among them, Jin and Wang [5] dealt with the one-dimensional case. Also, the two- and three-dimensional settings were investigated by Jin and Liu [4] under the condition $\chi = \xi$. Furthermore, in the case $\tau_1 = \tau_2 = 0$ with $f(u) = u(a - bu)$ and $a = b$, Salako and Shen [7] derived the global boundedness under some special conditions. Moreover, with $f(u) \leq u(a - bu)$, Zhang and Li [8] proved that the problem possesses a globally bounded classical solution if one of the following holds: (a) $\alpha\chi - \gamma\xi \leq b$; (b) $n \leq 2$; (c) $\frac{n-2}{n}(\alpha\chi - \gamma\xi) \leq b$ with $n \geq 3$.

Lately, we turn our eyes into the chemotaxis system with nonlinear productions. Wang and Xiang [9] proved that for the chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & (x,t) \in \Omega \times (0,T), \\ 0 = \Delta v + \alpha u^k - \beta v, & (x,t) \in \Omega \times (0,T), \end{cases} \tag{1.8}$$

with $f(u) \leq u(a - bu^d)$ and $k, d > 0$, the solutions are globally bounded if either $d > k$ or $d = k$ with $\frac{kn-2}{kn}\chi < b$. Moreover, Hong, Tian, and-Zheng [10] extended the above criteria for the attraction–repulsion system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) + f(u), & (x,t) \in \Omega \times (0,T), \\ 0 = \Delta v + \alpha u^k - \beta v, & (x,t) \in \Omega \times (0,T), \\ 0 = \Delta w + \gamma u^l - \delta w, & (x,t) \in \Omega \times (0,T), \end{cases} \tag{1.9}$$

where $f(u) \leq u(a - bu^d)$, $k, l, d > 0$, $k < \max\{l, d, \frac{2}{n}\}$ or $k = \max\{l, d\} \geq \frac{2}{n}$ with the following assumptions: (a) $k = l = d$, $\frac{kn-2}{kn}(\alpha\chi - \gamma\xi) < b$; (b) $k = l > d$, $\alpha\chi - \gamma\xi < 0$; (c) $k = d > l$, $\frac{kn-2}{kn}\alpha\chi < b$.

Recently, Chiyo, Yokota, and Mizukami [11, 12] obtained some interesting results for a fully parabolic attraction–repulsion chemotaxis system with signal-dependent sensitivity.

Concerning the attraction–repulsion chemotaxis system (1.1), our main results are the following theorems.

Theorem 1 *Let* $\tau = 0, D, \Phi, \Psi$ *and* f *satisfy* (1.2)–(1.6) *with nonnegative initial data* $u_0 \in C(\bar{\Omega})$ *and* $v_0 \in W^{1,\sigma}(\bar{\Omega})$ $(\sigma > n)$.

(i) *If* $q + k < \max\{g + l, d + 1, \frac{2}{n} + p + 1\}$, *then Eq.* (1.1) *admits a globally bounded solution.*

(ii) *Assume* $q + k = \max\{g + l, d + 1\} \geq \frac{2}{n} + p + 1$ *and there exist* b_0, θ_0 *such that one of the following assumptions holds*:

(a) $q + k = g + l = d + 1$ *with* b *and* $\gamma\xi$ *sufficiently large such that* $b + \gamma\xi\theta_0 > 4b_0$;

(b) $q + k = g + l > d + 1$ *with* $\gamma\xi$ *sufficiently large such that* $\gamma\xi\theta_0 > 4b_0$;

(c) $q + k = d + 1 > g + l$ *with* b *sufficiently large such that* $b > 4b_0$.

Then the solution of Eq. (1.1) *is globally bounded.*

Theorem 2 *Let* $\tau = 1, D, \Phi, \Psi$, *and* f *satisfy* (1.2)–(1.6) *with nonnegative initial data* $u_0 \in C(\bar{\Omega})$ *and* $v_0, w_0 \in W^{1,\sigma}(\bar{\Omega})$ $(\sigma > n)$.

(i) *If* $q + k, g + l < \max\{d + 1, \frac{2}{n} + p + 1\}$, *then Eq.* (1.1) *admits a globally bounded solution.*

(ii) *Assume that* $q + k = d + 1$ *and* $q + k, g + l \geq \frac{2}{n} + p + 1$ *and that there exist* $b_1, b_2, b_3, \theta_1, \theta_2 > 0$ *such that one of the following assumptions holds*:

(a) $q + k = d + 1 = g + l$ *with* b *sufficiently large such that* $\frac{b - \xi\theta_2}{\chi\theta_1} > 1 + b_1$ *and* $\frac{b - \chi\theta_1}{\xi\theta_2} > 1 + b_2$;

(b) $q + k = d + 1 > g + l$ *with* b *sufficiently large such that* $\frac{b}{\chi\theta_1} > 2 + b_3$.

Then the solution of Eq. (1.1) *is globally bounded.*

Remark 1 Model (1.1) includes four mechanisms (a nonlinear diffusion, attraction, repulsion, and logistic source) and two nonlinear productions (components v and w). The behavior of the solution is determined by the interaction among them. It is known that besides the attraction and corresponding production, all the other benefit the global boundedness of solutions.

Remark 2 Theorem 1 illustrates how the nonlinear exponents $p, q, g, d, k, l > 0$ influence the evolution of solutions. More precisely, if the attraction is dominated by one of the other mechanisms ($q + k < \max\{g + l, d + 1, \frac{2}{n} + p + 1\}$), then the solution will be globally bounded. Under the balance situations with $q + k = \max\{g + l, d + 1\}$ and $q + k \geq \frac{2}{n} + p + 1$, the solution boundedness will be determined by some related coefficients. Furthermore, Theorem 1 extends the criteria for global boundedness established by Tian, He, and Zheng [13] for the attraction–repulsion chemotaxis system.

Remark 3 Theorem 2 also illustrates how the nonlinear exponents $p, q, g, d, k, l > 0$ influence the evolution of solutions. More precisely, if the attraction and repulsion are dominated by one of the other mechanisms ($q + k, g + l < \max\{d + 1, \frac{2}{n} + p + 1\}$), then the solution will be globally bounded. Under the balance situations with $q + k = d + 1$ and $q + k, g + l \geq \frac{2}{n} + p + 1$, the solution boundedness will be determined by some related coefficients. However, in two cases, $g + l > q + k \geq d + 1$ and$g + l = q + k > d + 1$, we have not found a satisfactory way to explain the behavior of the solution.

2 Preliminaries

In this section, we introduce some results on the local solutions, some integral estimates, and the maximal Sobolev regularity.

Lemma 1 (*See* [13, *Lemma* 2.1]) *Let* $\Omega \subset \mathbb{R}^n (n \geq 1)$ *be a bounded domain with smooth boundary,* $\tau = 0$*, and let* D*,* Φ*,* Ψ*, and* f *satisfy* (1.2)–(1.6)*. Then for nonnegative* $u_0 \in C^0(\bar{\Omega})$ *and* $v_0 \in W^{1,\sigma}(\Omega)(\sigma \geq n)$*, there exist nonnegative functions* $u, v, w \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))$ *with* $T_{\max} \in (0, \infty]$ *that classically solve* (1.1) *in* $\Omega \times (0, T_{\max})$*. Moreover, if* $T_{\max} < \infty$*, then*

$$\lim_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

The proof is similar to that of Lemma 1.1 in [14].

Lemma 2 *Let* $\Omega \subset \mathbb{R}^n (n \geq 1)$ *be a bounded domain with smooth boundary,* $\tau = 1$*, let* D*,* Φ*,* Ψ*, and* f *satisfy* (1.2)–(1.6)*, and let* $u_0 \in C^0(\bar{\Omega})$ *and* $v_0, w_0 \in W^{1,\infty}(\Omega)$ *be nonnegative with* $u_0 \not\equiv 0$*. Then there exist a maximal* $T_{\max} \in (0, \infty]$ *and a uniquely determined triplet* (u, v, w) *of nonnegative functions*

$$\begin{aligned} &u \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ &v, w \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap L^\infty_{loc}([0, T_{\max}); W^{1,\infty}) \end{aligned}$$

that classically solve (1.1) *in* $\Omega \times (0, T_{\max})$*. Moreover, if* $T_{\max} < \infty$*, then*

$$\limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

Some basic properties are derived as follows.

Lemma 3 [10] *Let* (u, v, w) *be a solution to* (1.1) *ensured by Lemma* 1*. Then for any* $l, \eta > 0$ *and* $\theta > 1$*, there is* $c_0 = c_0(\eta, \theta, l) > 0$ *such that*

$$\int_\Omega w^\theta \leq \eta \int_\Omega u^{l\theta} + c_0, \quad t \in (0, T_{\max}). \tag{2.1}$$

Moreover,

$$\int_\Omega u \leq \max\left\{\int_\Omega u_0, \left(\frac{a}{b}\right)^{\frac{1}{d}} |\Omega|\right\} := M.$$

Next, we prove a variation of the maximal Sobolev regularity. The idea is inspired by [10, Lemma 4.1] and the work presented in [15].

Lemma 4 *Let $\sigma > 1$. Consider the following equation*:

$$\begin{cases} \varsigma_t = \Delta\varsigma - \beta\varsigma + h, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial\varsigma}{\partial\nu} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \varsigma(x,0) = \varsigma_0(x), & x \in \Omega, \end{cases}$$

for any $\varsigma_0 \in W^{2,\sigma}(\Omega)$ $(\sigma > n)$, $\frac{\partial\varsigma_0}{\partial\nu} = 0$ on $\partial\Omega$, and all $h \in L^\sigma((0,T);L^\sigma(\Omega))$. Then it has a unique solution

$$\varsigma \in W^{1,\sigma}\big((0,T);L^\sigma(\Omega)\big) \cap L^\sigma\big((0,T);W^{2,\sigma}(\Omega)\big).$$

Moreover, if $t_0 \in [0,T)$, $\varsigma(\cdot,t_0) \in W^{2,\sigma}(\sigma > n)$ with $\frac{\partial\varsigma_0}{\partial\nu} = 0$, then there exists $C_\sigma > 0$ such that

$$\int_{t_0}^T \int_\Omega e^{\beta\sigma t}|\Delta\varsigma|^\sigma \le C_\sigma \int_{t_0}^T \int_\Omega e^{\beta\sigma t}h^\sigma + C_\sigma e^{\beta\sigma t_0}\big(\|\varsigma(\cdot,t_0)\|^\sigma_{L^\sigma(\Omega)} + \|\Delta\varsigma(\cdot,t_0)\|^\sigma_{L^\sigma(\Omega)}\big). \tag{2.2}$$

Proof Let $\bar{H}(x,t) = e^{\beta t}\varsigma(x,t)$. We have

$$\begin{cases} \bar{H}_t = \Delta\bar{H} + e^{\beta t}h, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial\bar{H}}{\partial\nu} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \bar{H}(x,0) = \bar{H}_0(x), & x \in \Omega. \end{cases}$$

By the standard Sobolev regularity there exists $C_\sigma > 0$ such that

$$\int_0^T \int_\Omega |\Delta\bar{H}|^\sigma \le C_\sigma \int_0^T \int_\Omega \big|e^{\beta t}h\big|^\sigma + C_\sigma\big(\|\bar{H}_0\|^\sigma_{L^\sigma(\Omega)} + \|\Delta\bar{H}_0\|^\sigma_{L^\sigma(\Omega)}\big),$$

and thus

$$\int_0^T \int_\Omega e^{\beta\sigma t}|\Delta\varsigma|^\sigma \le C_\sigma \int_0^T \int_\Omega e^{\beta\sigma t}h^\sigma + C_\sigma\big(\|\varsigma_0\|^\sigma_{L^\sigma(\Omega)} + \|\Delta\varsigma_0\|^\sigma_{L^\sigma(\Omega)}\big).$$

For any $t_0 > 0$, replacing $\varsigma(t)$ by $\varsigma(t+t_0)$, we get

$$\int_{t_0}^T \int_\Omega e^{\beta\sigma t}|\Delta\varsigma|^\sigma \le C_\sigma \int_{t_0}^T \int_\Omega e^{\beta\sigma t}h^\sigma + C_\sigma e^{\beta\sigma t_0}\big(\|\varsigma(\cdot,t_0)\|^\sigma_{L^\sigma(\Omega)} + \|\Delta\varsigma(\cdot,t_0)\|^\sigma_{L^\sigma(\Omega)}\big). \qquad \square$$

Given $t_0 \in (0,T_{\max})$ with $t_0 \le 1$, from the regularity principle stated by Lemma 1, we know that $u(\cdot,t_0), v(\cdot,t_0) \in C^2(\bar{\Omega})$ with $\frac{\partial v(\cdot,t_0)}{\partial\nu} = 0$ on $\partial\Omega$. So we can pick $\bar{M}_1 > 0$ such that

$$\begin{aligned} &\sup_{0\le t\le t_0}\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \bar{M}_1, \\ &\sup_{0\le t\le t_0}\|v(\cdot,t)\|_{L^\infty(\Omega)} \le \bar{M}_1 \quad \text{and} \quad \|\Delta v(\cdot,t_0)\|_{L^\infty(\Omega)} \le \bar{M}_1. \end{aligned} \tag{2.3}$$

Similarly, by Lemma 2 we know that $u(\cdot,t_0), v(\cdot,t_0), w(\cdot,t_0) \in C^2(\bar{\Omega})$ with $\frac{\partial v(\cdot,t_0)}{\partial \nu}, \frac{\partial w(\cdot,t_0)}{\partial \nu} = 0$ on $\partial\Omega$. So we can pick $\bar{M}_2 > 0$ such that

$$\begin{aligned}
&\sup_{0\le t\le t_0} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \bar{M}_2, \\
&\sup_{0\le t\le t_0} \|v(\cdot,t)\|_{L^\infty(\Omega)} \le \bar{M}_2 \quad \text{and} \quad \|\Delta v(\cdot,t_0)\|_{L^\infty(\Omega)} \le \bar{M}_2, \\
&\sup_{0\le t\le t_0} \|w(\cdot,t)\|_{L^\infty(\Omega)} \le \bar{M}_2 \quad \text{and} \quad \|\Delta w(\cdot,t_0)\|_{L^\infty(\Omega)} \le \bar{M}_2.
\end{aligned} \tag{2.4}$$

3 Proof of Theorem 1

In this section, we deal with the parabolic–parabolic–elliptic case (with $\tau = 0$) to prove Theorem 1. For simplicity, the variable of integration in an integral will be omitted without ambiguity; e.g., we write the integral $\int_\Omega f(x)\,dx$ as $\int_\Omega f(x)$. Hereafter, c_i, $i = 1,2,3,\ldots$, denote generic constants, which may change from one line to another.

Proof of Theorem 1 We first prove that for any $r > 1$, there is $c = c(r) > 0$ such that

$$\int_\Omega u^r \le c, \quad t \in (0, T_{\max}). \tag{3.1}$$

Without loss of generality, suppose $r > \max\{2, 1-q, 1-g, 1-p, 1-j\}$ and assume that $\nabla u\cdot\nabla v > 0$ and $\nabla u \cdot \nabla w > 0$. Taking u^{r-1} as a test function for the first equation of (1.1), we have

$$\begin{aligned}
\frac{1}{r}\frac{d}{dt}\int_\Omega u^r ={}& -(r-1)\int_\Omega u^{r-2}D(u)|\nabla u|^2 + (r-1)\int_\Omega u^{r-2}\Phi(u)\nabla u\cdot\nabla v \\
& -(r-1)\int_\Omega u^{r-2}\Psi(u)\nabla u\cdot\nabla w + a\int_\Omega u^r - b\int_\Omega u^{r+d} \\
\le{}& \chi(r-1)\int_\Omega u^{r+q-2}\nabla u\cdot\nabla v - \xi(r-1)\int_\Omega u^{r+g-2}\nabla u\cdot\nabla w \\
& + a\int_\Omega u^r - b\int_\Omega u^{r+d} \\
={}& -\frac{\chi(r-1)}{r+q-1}\int_\Omega u^{r+q-1}\Delta v + \frac{\xi(r-1)}{r+g-1}\int_\Omega u^{r+g-1}\Delta w + a\int_\Omega u^r \\
& - b\int_\Omega u^{r+d}, \quad t\in(t_0,T_{\max}),
\end{aligned} \tag{3.2}$$

and, combining it with the third equation of (1.1),

$$\begin{aligned}
\frac{1}{r}\frac{d}{dt}\int_\Omega u^r \le{}& -\frac{\chi(r-1)}{r+q-1}\int_\Omega u^{r+q-1}\Delta v - \frac{\gamma\xi(r-1)}{r+g-1}\int_\Omega u^{r+g+l-1} \\
& + \frac{\delta\xi(r-1)}{r+g-1}\int_\Omega u^{r+g-1}w + a\int_\Omega u^r - b\int_\Omega u^{r+d}, \quad t\in(t_0,T_{\max}).
\end{aligned}$$

By Young's inequality, for any $\varepsilon > 0$, there exists $c_1 = c_1(r,\varepsilon)$ such that

$$\frac{\delta\xi(r-1)}{r+g-1}\int_\Omega u^{r+g-1}w \le \frac{\varepsilon}{2}\int_\Omega u^{r+g+l-1} + c_1\int_\Omega w^{\frac{r+g+l-1}{l}}, \quad t\in(t_0,T_{\max}),$$

and combining this with (2.1) and letting $\eta = \frac{\varepsilon}{2c_1}$, we get

$$\frac{\delta\xi(r-1)}{r+g-1}\int_\Omega u^{r+g-1}w \le \varepsilon\int_\Omega u^{r+g+l-1} + c_2, \quad t\in(t_0,T_{\max}),$$

where $c_2 = c_2(r,\varepsilon) > 0$. By Young's inequality,

$$-\frac{\chi(r-1)}{r+q-1}\int_\Omega u^{r+q-1}\Delta v \le \frac{\chi(r-1)}{r+q-1}\left(\int_\Omega u^{r+q+k-1} + \int_\Omega |\Delta v|^{\frac{r+q+k-1}{k}}\right), \quad t\in(t_0,T_{\max}),$$

and thus

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u^r \le{}& \frac{\chi(r-1)}{r+q-1}\left(\int_\Omega u^{r+q+k-1} + \int_\Omega |\Delta v|^{\frac{r+q+k-1}{k}}\right)\\ &-\left(\frac{\gamma\xi(r-1)}{r+g-1}-\varepsilon\right)\int_\Omega u^{r+g+l-1}\\ &+a\int_\Omega u^r - b\int_\Omega u^{r+d} + c_2, \quad t\in(t_0,T_{\max}).\end{aligned} \tag{3.3}$$

Case 1: $q+k<\max\{g+l,d+1,\frac{2}{n}+p+1\}$.

Let $q+k<d+1$. By Young's inequality, for any $\eta_1>0$, there is $c_3=c_3(r,\eta_1)>0$ such that

$$\begin{aligned}&\frac{\chi(r-1)}{r+q-1}\left(\int_\Omega u^{r+q+k-1} + \int_\Omega \Delta v^{\frac{r+q+k-1}{k}}\right)\\ &\quad\le \frac{b}{2}\int_\Omega u^{r+d} + \eta_1\int_\Omega |\Delta v|^{\sigma_1} + c_3, \quad t\in(t_0,T_{\max}),\end{aligned}$$

where $\sigma_1 = \frac{r+d}{k}$. Taking $\varepsilon = \frac{\gamma\xi(r-1)}{r+g-1}$ in (3.3), we get

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u^r &\le \eta_1\int_\Omega |\Delta v|^{\sigma_1} - \frac{b}{2}\int_\Omega u^{r+d} + a\int_\Omega u^r + c_4\\ &= -\frac{\beta\sigma_1}{r}\int_\Omega u^r + \eta_1\int_\Omega |\Delta v|^{\sigma_1} - \frac{b}{2}\int_\Omega u^{r+d}\\ &\quad+\left(a+\frac{\beta\sigma_1}{r}\right)\int_\Omega u^r + c_4, \quad t\in(t_0,T_{\max}),\end{aligned}$$

where $c_4 = c_2 + c_3 > 0$. By Young's inequality,

$$\left(a+\frac{\beta\sigma_1}{r}\right)\int_\Omega u^r \le \frac{b}{4}\int_\Omega u^{r+d} + c_5, \quad t\in(t_0,T_{\max}),$$

where $c_5 = c_5(r) > 0$. Thus there exists $c_6 = c_4 + c_5 > 0$ such that

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r \le -\frac{\beta\sigma_1}{r}\int_\Omega u^r + \eta_1\int_\Omega |\Delta v|^{\sigma_1} - \frac{b}{4}\int_\Omega u^{r+d} + c_6, \quad t\in(t_0,T_{\max}),$$

and applying the variation-of-constants formula, we have

$$\begin{aligned}\frac{1}{r}\int_\Omega u^r &\le e^{-\beta\sigma_1(t-t_0)}\frac{1}{r}\int_\Omega u^r(\cdot,t_0)+\eta_1\int_{t_0}^t\int_\Omega e^{-\beta\sigma_1(t-s)}|\Delta v|^{\sigma_1}\\ &\quad-\frac{b}{4}\int_{t_0}^t\int_\Omega e^{-\beta\sigma_1(t-s)}u^{r+d}+c_6\int_{t_0}^t e^{-\beta\sigma_1(t-s)}\\ &\le \eta_1 e^{-\beta\sigma_1 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_1 s}|\Delta v|^{\sigma_1}-\frac{b}{4}e^{-\beta\sigma_1 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_1 s}u^{r+d}+c_7,\quad t\in(t_0,T_{\max}),\end{aligned}$$

where $c_7=\frac{1}{r}e^{\beta\sigma_1 t_0}\int_\Omega u^r(\cdot,t_0)+\frac{c_6}{\beta\sigma_1}(1+e^{\beta\sigma_1 t_0})$, which is independent of t. By (2.2) this yields that

$$\begin{aligned}\eta_1 e^{-\beta\sigma_1 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_1 s}|\Delta v|^{\sigma_1}&\le \eta_1 e^{-\beta\sigma_1 t}C_{\sigma_1}\alpha^{\sigma_1}\int_{t_0}^t\int_\Omega e^{\beta\sigma_1 s}u^{r+d}\\ &\quad+\eta_1 e^{-\beta\sigma_1 t}C_{\sigma_1}e^{\beta\sigma_1 t_0}\left\|v(\cdot,t_0)\right\|_{W^{2,\sigma_1}}^{\sigma_1},\quad t\in(t_0,T_{\max}),\end{aligned}$$

and thus

$$\begin{aligned}\frac{1}{r}\int_\Omega u^r &\le -\left(\frac{b}{4}-\eta_1 C_{\sigma_1}\alpha^{\sigma_1}\right)e^{-\beta\sigma_1 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_1 s}u^{r+d}\\ &\quad+\eta_1 e^{-\beta\sigma_1(t-t_0)}C_{\sigma_1}\left\|v(\cdot,t_0)\right\|_{W^{2,\sigma_1}}^{\sigma_1}+c_7,\quad t\in(t_0,T_{\max}),\end{aligned}\tag{3.4}$$

which gives (3.1) by taking $\eta_1=\frac{b}{4(c_{\sigma_1}\alpha^{\sigma_1})}$.

Let $q+k<g+l$. By Young's inequality, for any $\eta_2>0$, there is $c_8=c_8(r,\eta_2)>0$ such that

$$\begin{aligned}\frac{\chi(r-1)}{r+q-1}\left(\int_\Omega u^{r+q+k-1}+\int_\Omega \Delta v^{\frac{r+q+k-1}{k}}\right)&\le\frac{\gamma\xi(r-1)}{2(r+g-1)}\int_\Omega u^{r+g+l-1}\\ &\quad+\eta_2\int_\Omega|\Delta v|^{\frac{r+g+l-1}{k}}+c_8,\quad t\in(t_0,T_{\max}).\end{aligned}$$

By (3.3) with $\varepsilon=\frac{\gamma\xi(r-1)}{4(r+g-1)}$ we have

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u^r&\le\eta_2\int_\Omega|\Delta v|^{\sigma_2}-\frac{\gamma\xi(r-1)}{4(r+g-1)}\int_\Omega u^{r+g+l-1}+a\int_\Omega u^r+c_9\\ &=-\frac{\beta\sigma_2}{r}\int_\Omega u^r+\eta_2\int_\Omega|\Delta v|^{\sigma_2}-\frac{\gamma\xi(r-1)}{4(r+g-1)}\int_\Omega u^{r+g+l-1}\\ &\quad+\left(a+\frac{\beta\sigma_2}{r}\right)\int_\Omega u^r+c_9,\quad t\in(t_0,T_{\max}),\end{aligned}$$

where $c_9=c_2+c_8>0$ and $\sigma_2=\frac{r+g+l-1}{k}$. Since $g+l>d+1>0$, there is $c_{10}=c_{10}(r)>0$ such that

$$\left(a+\frac{\beta\sigma_2}{r}\right)\int_\Omega u^r\le\frac{\gamma\xi(r-1)}{8(r+g-1)}\int_\Omega u^{r+g+l-1}+c_{10},\quad t\in(t_0,T_{\max}).$$

Similarly to (3.4), we have

$$\frac{1}{r}\int_\Omega u^r \le -\left(\frac{\gamma\xi(r-1)}{8(r+g-1)} - \eta_2 C_{\sigma_2}\alpha^{\sigma_2}\right)e^{-\beta\sigma_2 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_2 s}u^{r+g+l-1} + \eta_2 e^{-\beta\sigma_2(t-t_0)}C_{\sigma_2}\left\|v(\cdot,t_0)\right\|_{W^{2,\sigma_2}}^{\sigma_2} + c_{12}, \quad t\in(t_0,T_{\max}),$$

where $c_{12} = \frac{1}{r}e^{\beta\sigma_2 t_0}\int_\Omega u^r(\cdot,t_0) + \frac{c_{11}}{\beta\sigma_2}(1+e^{\beta\sigma_2 t_0})$ independent of t, and $c_{11} = c_9 + c_{10} > 0$. Then (3.1) follows by taking $\eta_2 = \frac{\gamma\xi(r-1)}{8(r+g-1)c_{\sigma_2}\alpha^{\sigma_2}}$.

Let $q+k < \frac{2}{n}+p+1$. Without loss of generality, we suppose $q+k \ge \max\{g+l, d+1\}$. Take $(u+1)^{r+1}$ as a test function for the first equation in (1.1). Similarly, to obtain (3.2), we have

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int_\Omega (u+1)^r \\
&= -(r-1)\int_\Omega (u+1)^{r-2}D(u)|\nabla u|^2 + (r-1)\int_\Omega (u+1)^{r-2}\Phi(u)\nabla u\cdot\nabla v \\
&\quad - (r-1)\int_\Omega (u+1)^{r-2}\Psi(u)\nabla u\cdot\nabla w + a\int_\Omega u(u+1)^{r-1} - b\int_\Omega u^{d+1}(u+1)^{r-1} \\
&\le -(r-1)\int_\Omega (u+1)^{r-2}D(u)|\nabla u|^2 + \chi(r-1)\int_\Omega (u+1)^{r+q-2}\nabla u\cdot\nabla v \\
&\quad - \xi(r-1)\int_\Omega (u+1)^{r+g-2}\nabla u\cdot\nabla w + a\int_\Omega u(u+1)^{r-1} - b\int_\Omega u^{d+1}(u+1)^{r-1},
\end{aligned}$$

and then

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int_\Omega (u+1)^r \\
&\le -\frac{4(r-1)}{(r+p)^2}\int_\Omega \left|\nabla(u+1)^{\frac{r+p}{2}}\right|^2 - \frac{\chi(r-1)}{r+q-1}\int_\Omega (u+1)^{r+q-1}\Delta v \\
&\quad + \frac{\xi(r-1)}{r+g-1}\int_\Omega (u+1)^{r+g-1}\Delta w + a\int_\Omega u(u+1)^{r-1}, \quad t\in(t_0,T_{\max}).
\end{aligned} \tag{3.5}$$

By Young's inequality we have

$$-\frac{\chi(r-1)}{r+q-1}\int_\Omega u^{r+q-1}\Delta v \le \chi\left(\int_\Omega u^{r+q+k-1} + \int_\Omega |\Delta v|^{\frac{r+q+k-1}{k}}\right), \quad t\in(t_0,T_{\max}).$$

Similarly, replacing u in the second equation of (1.1) by $u+1$ to obtain (3.3), we have

$$\begin{aligned}
\frac{1}{r}\frac{d}{dt}\int_\Omega (u+1)^r \le & -\frac{4(r-1)}{(r+p)^2}\int_\Omega \left|\nabla(u+1)^{\frac{r+p}{2}}\right|^2 + \chi\int_\Omega |\Delta v|^{\frac{r+q+k-1}{k}} \\
& + \chi\int_\Omega (u+1)^{r+q+k-1} \\
& - \left(\frac{\gamma\xi(r-1)}{r+g-1} - \varepsilon\right)\int_\Omega u^{r+g+l-1} + a\int_\Omega u(u+1)^{r-1} \\
& + c_2, \quad t\in(t_0,T_{\max}).
\end{aligned}$$

Taking $\varepsilon = \frac{\gamma\xi(r-1)}{r+g-1}$, we further have

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega (u+1)^r \le & -\frac{4(r-1)}{(r+p)^2}\int_\Omega \left|\nabla(u+1)^{\frac{r+p}{2}}\right|^2 + \chi\int_\Omega |\Delta v|^{\frac{r+q+k-1}{k}} \\ & + \chi\int_\Omega (u+1)^{r+q+k-1} \\ & + a\int_\Omega u(u+1)^{r-1} + c_2, \quad t\in(t_0,T_{\max}).\end{aligned}$$

By the Gagliardo–Nirenberg inequality there exist $c_{13} = c_{13}(r) > 0$ and $c_{14} = c_{14}(r) > 0$ such that

$$\begin{aligned}\int_\Omega (u+1)^{r+q+k-1} &= \left\|(u+1)^{\frac{r+p}{2}}\right\|_{L^{\frac{2(r+q+k-1)}{r+p}}(\Omega)}^{\frac{2(r+q+k-1)}{r+p}} \\ &\le c_{13}\left\|\nabla(u+1)^{\frac{r+p}{2}}\right\|_{L^2(\Omega)}^{\frac{2(r+q+k-1)}{r+p}z}\left\|(u+1)^{\frac{r+p}{2}}\right\|_{L^{\frac{2}{r+p}}(\Omega)}^{\frac{2(r+q+k-1)}{r+p}(1-z)} \\ &\quad + c_{13}\left\|(u+1)^{\frac{r+p}{2}}\right\|_{L^{\frac{2}{r+p}}(\Omega)}^{\frac{2(r+q+k-1)}{r+p}} \\ &\le c_{14}\left\|\nabla(u+1)^{\frac{r+p}{2}}\right\|_{L^2(\Omega)}^{\frac{2(r+q+k-1)}{r+p}z} + c_{14}, \quad t\in(t_0,T_{\max}),\end{aligned}$$

where $z = (\frac{n(r+p)}{2} - \frac{n(r+p)}{2(r+q+k-1)})/(1-\frac{n}{2}+\frac{n(r+p)}{2}) \in (0,1)$.

Now let $q+k < \frac{2}{n}+p+1$. Then $\frac{2(r+q+k-1)}{r+p}z \le 2$. By Young's inequality, for any $\bar\eta > 0$,

$$\int_\Omega (u+1)^{r+q+k-1} \le \bar\eta\left\|\nabla(u+1)^{\frac{r+p}{2}}\right\|_{L^2(\Omega)}^2 + c_{15}, \quad t\in(t_0,T_{\max}), \tag{3.6}$$

where $c_{15} = c_{15}(r,\bar\eta) > 0$. Thus

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega (u+1)^r \le & -\left(\frac{4(r-1)}{\bar\eta(r+p)^2} - \chi\right)\int_\Omega (u+1)^{r+q+k-1} + \chi\int_\Omega |\Delta v|^{\sigma_3} \\ & + a\int_\Omega (u+1)^r + c_{16}, \quad t\in(t_0,T_{\max}),\end{aligned}$$

where $c_{16} = c_2 + c_{15} > 0$ and $\sigma_3 = \frac{r+q+k-1}{k}$. Applying the variation-of-constants formula and (2.2), we have

$$\begin{aligned}\frac{1}{r}\int_\Omega (u+1)^r \le & -\left(\frac{4(r-1)}{\bar\eta(r+p)^2} - \chi - a - \frac{\beta\sigma_3}{r} - \chi C_{\sigma_3}\alpha^{\sigma_3}\right) \\ & \times e^{-\beta\sigma_3 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_3 s}(u+1)^{r+q+k-1} \\ & + \chi e^{-\beta\sigma_3(t-t_0)}C_{\sigma_3}\left\|v(\cdot,t_0)\right\|_{W^{2,\sigma_3}(\Omega)}^{\sigma_3} + c_{17}, \quad t\in(t_0,T_{\max}),\end{aligned}$$

where $c_{17} = c_{17}(r,\bar\eta)$. This gives (3.1) with $\bar\eta$ small enough.

Case 2: $q+k = \max\{g+l, d+1\}$ and $q+k \ge \frac{2}{n}+p+1$.

(a) Let $q+k=g+l=d+1$. By (3.3) we have

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r \le \frac{\chi(r-1)}{r+q-1}\left(\int_\Omega u^{r+q+k-1}+\int_\Omega \Delta v^{\sigma_3}\right)-\left(b+\frac{\gamma\xi(r-1)}{r+g-1}-\varepsilon\right)\int_\Omega u^{r+q+k-1} + a\int_\Omega u^r + c_2, \quad t\in(t_0,T_{\max}).$$

By Young's inequality, for any $\eta_3>0$, there exists $c_{18}=c_{18}(r,\eta_3)>0$ such that

$$\frac{\chi(r-1)}{r+q-1}\left(\int_\Omega u^{r+q+k-1}+\int_\Omega \Delta v^{\sigma_3}\right)\le\left(\frac{\gamma\xi(r-1)}{2(r+g-1)}+\frac{b}{2}\right)\int_\Omega u^{r+q+k-1} + \eta_3\int_\Omega|\Delta v|^{\sigma_3}+c_{18}, \quad t\in(t_0,T_{\max}),$$

and thus we get

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u^r &\le -\left(\frac{b}{2}+\frac{\gamma\xi(r-1)}{2(r+g-1)}-\varepsilon\right)\int_\Omega u^{r+q+k-1}+\eta_3\int_\Omega|\Delta v|^{\sigma_3}+a\int_\Omega u^r+c_{19}\\ &= -\beta\frac{\sigma_3}{r}\int_\Omega u^r-\left(\frac{b}{2}+\frac{\gamma\xi(r-1)}{2(r+g-1)}-\varepsilon\right)\int_\Omega u^{r+q+k-1}+\eta_3\int_\Omega|\Delta v|^{\sigma_3}\\ &\quad+\left(a+\beta\frac{\sigma_3}{r}\right)\int_\Omega u^r+c_{19}, \quad t\in(t_0,T_{\max}),\end{aligned}$$

where $c_{19}=c_2+c_{18}$. By Young's inequality,

$$\left(a+\beta\frac{\sigma_3}{r}\right)\int_\Omega u^r\le\left(\frac{b}{4}+\frac{\gamma\xi(r-1)}{4(r+g-1)}\right)\int_\Omega u^{r+q+k-1}+c_{20}, \quad t\in(t_0,T_{\max}),$$

where $c_{20}=c_{20}(r)>0$. Then

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r\le-\beta\frac{\sigma_3}{r}\int_\Omega u^r-\left(\frac{b}{4}+\frac{\gamma\xi(r-1)}{4(r+g-1)}-\varepsilon\right)\int_\Omega u^{r+q+k-1} + \eta_3\int_\Omega|\Delta v|^{\sigma_3}+c_{21}, \quad t\in(t_0,T_{\max}),$$

where $c_{21}=c_{19}+c_{20}$. Applying the variation-of-constants formula and (2.2), we have

$$\begin{aligned}\frac{1}{r}\int_\Omega u^r &\le -\left(\frac{b}{4}+\frac{\gamma\xi(r-1)}{4(r+g-1)}-\varepsilon-\eta_3C_{\sigma_3}\alpha^{\sigma_3}\right)e^{-\beta\sigma_3 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_3 s}u^{r+q+k-1}\\ &\quad+\eta_3e^{-\beta\sigma_3(t-t_0)}C_{\sigma_3}\left\|v(\cdot,t_0)\right\|^{\sigma_3}_{W^{2,\sigma_3}(\Omega)}+c_{22}, \quad t\in(t_0,T_{\max}),\end{aligned} \tag{3.7}$$

where $c_{22}=c_{22}(r,\varepsilon)$. Let

$$b_0=b_0(r)=\inf_{\eta_3>0}\left(\eta_3C_{\sigma_3}\alpha^{\sigma_3}\right),$$

$$\theta_0=\theta_0(r)=\frac{r-1}{r+g-1}.$$

Then we can choose ε and η_3 small enough such that $\frac{1}{4}(b+\gamma\xi\theta_0)-\varepsilon-b_0>0$, provided that $b+\gamma\xi\theta_0>4b_0$, and, consequently, (3.1) is true.

(b) Let $q+k=g+l>d+1$. Then (3.3) becomes

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r \le \frac{\chi(r-1)}{r+q-1}\left(\int_\Omega u^{r+q+k-1}+\int_\Omega \Delta v^{\sigma_3}\right)-\left(\frac{\gamma\xi(r-1)}{r+g-1}-\varepsilon\right)\int_\Omega u^{r+q+k-1}$$
$$+a\int_\Omega u^r + c_2, \quad t\in(t_0,T_{\max}).$$

Following the same arguments as those for getting (3.7), we can find $c_{23}=c_{23}(r,\varepsilon)>0$ such that

$$\frac{1}{r}\int_\Omega u^r \le -\left(\frac{\gamma\xi(r-1)}{4(r+g-1)}-\varepsilon-\eta_3 C_{\sigma_3}\alpha^{\sigma_3}\right)e^{-\beta\sigma_3 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_3 s}u^{r+q+k-1}$$
$$+\eta_3 e^{-\beta\sigma_3(t-t_0)}C_{\sigma_3}\left\|v(\cdot,t_0)\right\|^{\sigma_3}_{W^{2,\sigma_3}(\Omega)}+c_{23}, \quad \in(t_0,T_{\max}).$$

Then we can choose ε and η_3 small enough such that $\frac{1}{4}\gamma\xi\theta_0-\varepsilon-b_0>0$, provided that $\gamma\xi\theta_0>4b_0$, and thus (3.1) is true.

(c) Let $q+k=d+1>g+l$. Taking $\varepsilon=\frac{\gamma\xi(r-1)}{r+g-1}$, by (3.3) we have

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r \le \frac{\chi(r-1)}{r+q-1}\left(\int_\Omega u^{r+q+k-1}+\int_\Omega \Delta v^{\sigma_3}\right)+a\int_\Omega u^r-b\int_\Omega u^{r+d}$$
$$+c_2, \quad t\in(t_0,T_{\max}).$$

Following the same arguments as those for getting (3.7), we can find $c_{24}=c_{24}(r)>0$ such that

$$\frac{1}{r}\int_\Omega u^r \le -\left(\frac{b}{4}-\eta_3 C_{\sigma_3}\alpha^{\sigma_3}\right)e^{-\beta\sigma_3 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_3 s}u^{r+q+k-1}$$
$$+\eta_3 e^{-\beta\sigma_3(t-t_0)}C_{\sigma_3}\left\|v(\cdot,t_0)\right\|^{\sigma_3}_{W^{2,\sigma_3}(\Omega)}+c_{24}, \quad t\in(t_0,T_{\max}).$$

Then we can choose η_3 small enough such that $\frac{1}{4}b-b_0>0$, provided that $b>4b_0$, and hence (3.1) is proved.

If $\nabla u\cdot\nabla v<0$ and $\nabla u\cdot\nabla w>0$, then similarly to (3.2), we derive

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u^r &= -(r-1)\int_\Omega u^{r-2}D(u)|\nabla u|^2+(r-1)\int_\Omega u^{r-2}\Phi(u)\nabla u\cdot\nabla v\\ &\quad -(r-1)\int_\Omega u^{r-2}\Psi(u)\nabla u\cdot\nabla w+a\int_\Omega u^r-b\int_\Omega u^{r+d}\\ &\le -\xi(r-1)\int_\Omega u^{r+g-2}\nabla u\cdot\nabla w+a\int_\Omega u^r-b\int_\Omega u^{r+d}\\ &= \frac{\xi(r-1)}{r+g-1}\int_\Omega u^{r+g-1}\Delta w+a\int_\Omega u^r-b\int_\Omega u^{r+d}, \quad t\in(t_0,T_{\max}).\end{aligned}$$

Combining THIS with the third equation of (1.1), we have

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r \le -\frac{\gamma\xi(r-1)}{r+g-1}\int_\Omega u^{r+g+l-1}+\frac{\delta\xi(r-1)}{r+g-1}\int_\Omega u^{r+g-1}w$$
$$+a\int_\Omega u^r-b\int_\Omega u^{r+d}, \quad t\in(t_0,T_{\max}).$$

By Young's inequality, for any $\varepsilon > 0$, there exists $c_{25} = c_{25}(r,\varepsilon)$ such that

$$\frac{\delta\xi(r-1)}{r+g-1}\int_\Omega u^{r+g-1}w \le \frac{\varepsilon}{2}\int_\Omega u^{r+g+l-1} + c_{25}\int_\Omega w^{\frac{r+g+l-1}{l}}, \quad t\in(t_0,T_{\max}).$$

Combining this with (2.1) and letting $\eta = \frac{\varepsilon}{2c_{25}}$, we get

$$\frac{\delta\xi(r-1)}{r+g-1}\int_\Omega u^{r+g-1}w \le \varepsilon\int_\Omega u^{r+g+l-1} + c_{26}, \quad t\in(t_0,T_{\max}),$$

where $c_{26} = c_{26}(r,\varepsilon) > 0$. Thus

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u^r \le{}& -\left(\frac{\gamma\xi(r-1)}{r+g-1} - \varepsilon\right)\int_\Omega u^{r+g+l-1} + a\int_\Omega u^r - b\int_\Omega u^{r+d}\\ &+ c_{26}, \quad t\in(t_0,T_{\max}).\end{aligned}$$

Taking $\varepsilon = \frac{\gamma\xi(r-1)}{r+g-1}$, by (3.3) we derive (3.1).

If $\nabla u\cdot\nabla v > 0$ and $\nabla u\cdot\nabla w < 0$, then we have

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u^r ={}& -(r-1)\int_\Omega u^{r-2}D(u)|\nabla u|^2 + (r-1)\int_\Omega u^{r-2}\Phi(u)\nabla u\cdot\nabla v\\ &-(r-1)\int_\Omega u^{r-2}\Psi(u)\nabla u\cdot\nabla w + a\int_\Omega u^r - b\int_\Omega u^{r+d}\\ \le{}& \chi(r-1)\int_\Omega u^{r+q-2}\nabla u\cdot\nabla v - \zeta(r-1)\int_\Omega u^{r+j-2}\nabla u\cdot\nabla w\\ &+ a\int_\Omega u^r - b\int_\Omega u^{r+d}\\ ={}& -\frac{\chi(r-1)}{r+q-1}\int_\Omega u^{r+q-1}\Delta v + \frac{\zeta(r-1)}{r+j-1}\int_\Omega u^{r+j-1}\Delta w + a\int_\Omega u^r\\ &- b\int_\Omega u^{r+d}, \quad t\in(t_0,T_{\max}).\end{aligned}$$

Similarly to the case where $\nabla u\cdot\nabla v > 0$ and $\nabla u\cdot\nabla w > 0$, we derive (3.1).

If $\nabla u\cdot\nabla v < 0$ and $\nabla u\cdot\nabla w < 0$, then we have

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u^r ={}& -(r-1)\int_\Omega u^{r-2}D(u)|\nabla u|^2 + (r-1)\int_\Omega u^{r-2}\Phi(u)\nabla u\cdot\nabla v\\ &-(r-1)\int_\Omega u^{r-2}\Psi(u)\nabla u\cdot\nabla w + a\int_\Omega u^r - b\int_\Omega u^{r+d}\\ \le{}& -\zeta(r-1)\int_\Omega u^{r+j-2}\nabla u\cdot\nabla w + a\int_\Omega u^r - b\int_\Omega u^{r+d}\\ ={}& \frac{\zeta(r-1)}{r+j-1}\int_\Omega u^{r+j-1}\Delta w + a\int_\Omega u^r - b\int_\Omega u^{r+d}, \quad t\in(t_0,T_{\max}).\end{aligned}$$

Similarly to the case where $\nabla u\cdot\nabla v < 0$ and $\nabla u\cdot\nabla w > 0$, we derive (3.1).

We have proved claim (3.1) for all cases of Theorem 1.

Furthermore, by a standard Alikakos–Moser iteration [16] and (2.3) we get that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t\in(0,T_{\max})$$

with some $C > 0$.

The boundedness of v can be obtained by the standard parabolic regularity theory.

By Lemma 1 we conclude $T_{\max} = \infty$. □

4 Proof of Theorem 2

In this section, we deal with the fully parabolic case (with $\tau = 1$) to prove Theorem 2.

Proof of Theorem 2 Just as in the proof of Theorem 1, we first claim that for any $r > 1$, there exists $c = c(r) > 0$ such that (3.1) holds for some cases. Without loss of generality, suppose $r > \max\{2, 1-q, 1-g, 1-p, 1-j\}$ and assume that $\nabla u \cdot \nabla v > 0$ and $\nabla u \cdot \nabla w > 0$.

Case 1: $q+k, g+l < \max\{d+1, \frac{2}{n}+p+1\}$.

Let $q+k < d+1$, $g+l < d+1$. By (3.2) and Young's inequality we have

$$\begin{aligned}
&-\frac{\chi(r-1)}{r+q-1}\int_\Omega u^{r+q-1}\Delta v \\
&\quad\le \frac{\chi(r-1)}{r+q-1}\left(\int_\Omega u^{r+q+k-1}+\int_\Omega|\Delta v|^{\frac{r+q+k-1}{k}}\right),\quad t\in(t_0,T_{\max}),\\
&\frac{\xi(r-1)}{r+g-1}\int_\Omega u^{r+g-1}\Delta w \\
&\quad\le \frac{\xi(r-1)}{r+g-1}\left(\int_\Omega u^{r+g+l-1}+\int_\Omega|\Delta w|^{\frac{r+g+l-1}{l}}\right),\quad t\in(t_0,T_{\max}).
\end{aligned}\tag{4.1}$$

Then, by Young's inequality again, for any $\eta_4 > 0$ and $\eta_5 > 0$, we have

$$\begin{aligned}
&\frac{\chi(r-1)}{r+q-1}\left(\int_\Omega u^{r+q+k-1}+\int_\Omega \Delta v^{\frac{r+q+k-1}{k}}\right)\\
&\quad\le \frac{b}{4}\int_\Omega u^{r+d}+\eta_4\int_\Omega|\Delta v|^{\frac{r+d}{k}}+c_{27},\quad t\in(t_0,T_{\max}),\\
&\frac{\xi(r-1)}{r+g-1}\left(\int_\Omega u^{r+g+l-1}+\int_\Omega|\Delta w|^{\frac{r+g+l-1}{l}}\right)\\
&\quad\le \frac{b}{4}\int_\Omega u^{r+d}+\eta_5\int_\Omega|\Delta w|^{\frac{r+d}{l}}+c_{28},\quad t\in(t_0,T_{\max}),
\end{aligned}$$

with $c_{27} = c_{27}(r) > 0$, $c_{28} = c_{28}(r) > 0$. Together with (3.2), this gives

$$\begin{aligned}
\frac{1}{r}\frac{d}{dt}\int_\Omega u^r &\le \eta_4\int_\Omega|\Delta v|^{\sigma_4}+\eta_5\int_\Omega|\Delta w|^{\sigma_5}-\frac{b}{2}\int_\Omega u^{r+d}+a\int_\Omega u^r+c_{29}\\
&= -(\beta\sigma_4+\delta\sigma_5)\frac{1}{r}\int_\Omega u^r+\eta_4\int_\Omega|\Delta v|^{\sigma_4}+\eta_5\int_\Omega|\Delta w|^{\sigma_5}-\frac{b}{2}\int_\Omega u^{r+d}\\
&\quad+\left(a+(\beta\sigma_4+\delta\sigma_5)\frac{1}{r}\right)\int_\Omega u^r+c_{29},\quad t\in(t_0,T_{\max}),
\end{aligned}$$

where $\sigma_4 = \frac{r+d}{k}$, $\sigma_5 = \frac{r+d}{l}$, and $c_{29} = c_{27} + c_{28} > 0$. By Young's inequality we have

$$\left(a+(\beta\sigma_4+\delta\sigma_5)\frac{1}{r}\right)\int_\Omega u^r\le\frac{b}{4}\int_\Omega u^{r+d}+c_{30},\quad t\in(t_0,T_{\max}),$$

with $c_{30} = c_{30}(r) > 0$. Thus

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r \le -(\beta\sigma_4+\delta\sigma_5)\frac{1}{r}\int_\Omega u^r + \eta_4\int_\Omega |\Delta v|^{\sigma_4} + \eta_5\int_\Omega |\Delta w|^{\sigma_5} - \frac{b}{4}\int_\Omega u^{r+d} + c_{31}, \quad t\in(t_0,T_{\max}),$$

for $c_{31} = c_{29} + c_{30} > 0$. Applying the variation-of-constants formula, we have

$$\begin{aligned}
\frac{1}{r}\int_\Omega u^r \le{}& e^{-(\beta\sigma_4+\delta\sigma_5)(t-t_0)}\frac{1}{r}\int_\Omega u^r(\cdot,t_0) + \eta_4 e^{-(\beta\sigma_4+\delta\sigma_5)t}\int_{t_0}^t\int_\Omega e^{(\beta\sigma_4+\delta\sigma_5)s}|\Delta v|^{\sigma_4}\\
&+ c_{31}e^{-(\beta\sigma_4+\delta\sigma_5)t}\int_{t_0}^t e^{(\beta\sigma_4+\delta\sigma_5)s} + \eta_5 e^{-(\beta\sigma_4+\delta\sigma_5)t}\int_{t_0}^t\int_\Omega e^{(\beta\sigma_4+\delta\sigma_5)s}|\Delta w|^{\sigma_5}\\
&- \frac{b}{4}e^{-(\beta\sigma_4+\delta\sigma_5)t}\int_{t_0}^t\int_\Omega e^{(\beta\sigma_4+\delta\sigma_5)s}u^{r+d}\\
\le{}& \eta_4 e^{-\beta\sigma_4 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_4 s}|\Delta v|^{\sigma_4} + \eta_5 e^{-\delta\sigma_5 t}\int_{t_0}^t\int_\Omega e^{\delta\sigma_5 s}|\Delta w|^{\sigma_5}\\
&- \frac{b}{8}e^{-(\beta\sigma_4+\delta\sigma_5)t+\delta\sigma_5 t_0}\int_{t_0}^t\int_\Omega e^{\beta\sigma_4 s}u^{r+d}\\
&- \frac{b}{8}e^{-(\beta\sigma_4+\delta\sigma_5)t+\beta\sigma_4 t_0}\int_{t_0}^t\int_\Omega e^{\delta\sigma_5 s}u^{r+d} + c_{32}\\
\le{}& \eta_4 e^{-\beta\sigma_4 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_4 s}|\Delta v|^{\sigma_4} + \eta_5 e^{-\delta\sigma_5 t}\int_{t_0}^t\int_\Omega e^{\delta\sigma_5 s}|\Delta w|^{\sigma_5}\\
&- \frac{b}{8}e^{-\beta\sigma_4 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_4 s}u^{r+d}\\
&- \frac{b}{8}e^{-\delta\sigma_5 t}\int_{t_0}^t\int_\Omega e^{\delta\sigma_5 s}u^{r+d} + c_{32}, \quad t\in(t_0,T_{\max}),
\end{aligned}$$

where $c_{32} = e^{(\beta\sigma_4+\delta\sigma_5)t_0}\frac{1}{r}\int_\Omega u^r(\cdot,t_0) + \frac{c_{31}}{\beta\sigma_4+\delta\sigma_5}(1+e^{(\beta\sigma_4+\delta\sigma_5)t_0})$. Then by the maximal Sobolev regularity (Lemma 2) we get

$$\begin{aligned}
\frac{1}{r}\int_\Omega u^r \le{}& -\left(\frac{b}{8} - \eta_4 C_{\sigma_4}\alpha^{\sigma_4}\right)e^{-\beta\sigma_4 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_4 s}u^{r+d}\\
&- \left(\frac{b}{8} - \eta_5 C_{\sigma_5}\gamma^{\sigma_5}\right)e^{-\delta\sigma_5 t}\int_{t_0}^t\int_\Omega e^{\delta\sigma_5 s}u^{r+d}\\
&+ \eta_4 e^{-\beta\sigma_4(t-t_0)}C_{\sigma_4}\left\|v(\cdot,t_0)\right\|_{W^{2,\sigma_4}}^{\sigma_4} + \eta_5 e^{-\delta\sigma_5(t-t_0)}C_{\sigma_5}\left\|w(\cdot,t_0)\right\|_{W^{2,\sigma_5}}^{\sigma_5}\\
&+ c_{32}, \quad t\in(t_0,T_{\max}).
\end{aligned}$$

This gives (3.1) by taking $\eta_4 = \frac{b}{8(c_{\sigma_4}\alpha^{\sigma_4})}$ and $\eta_5 = \frac{b}{8(c_{\sigma_5}\alpha^{\sigma_5})}$.

Now let $q+k < \frac{2}{n}+p+1$ and $g+l < \frac{2}{n}+p+1$. Without loss of generality, suppose $q+k \ge \max\{g+l, d+1\}$. By (3.5) and Young's inequality we have

$$-\frac{\chi(r-1)}{r+q-1}\int_\Omega u^{r+q-1}\Delta v \le \chi\left(\int_\Omega u^{r+q+k-1} + \int_\Omega |\Delta v|^{\frac{r+q+k-1}{k}}\right), \quad t\in(t_0,T_{\max}),$$

$$\frac{\xi(r-1)}{r+g-1}\int_\Omega u^{r+g-1}\Delta w \le \xi\left(\int_\Omega u^{r+g+l-1}+\int_\Omega |\Delta w|^{\frac{r+g+l-1}{l}}\right), \quad t\in(t_0,T_{\max}),$$

and thus, replacing u in the second and third equations of (1.1) by $u+1$, we have

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega (u+1)^r \le & -\frac{4(r-1)}{(r+p)^2}\int_\Omega \left|\nabla(u+1)^{\frac{r+p}{2}}\right|^2+\chi\int_\Omega |\Delta v|^{\frac{r+q+k-1}{k}}+\chi\int_\Omega (u+1)^{r+q+k-1}\\ &+\xi\int_\Omega |\Delta w|^{\frac{r+g+l-1}{l}}+\xi\int_\Omega (u+1)^{r+q+k-1}\\ &+a\int_\Omega u(u+1)^{r-1}, \quad t\in(t_0,T_{\max}).\end{aligned}$$

By (3.6) we have

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega (u+1)^r \le & -\left(\frac{4(r-1)}{\bar\eta(r+p)^2}-\chi-\xi\right)\int_\Omega (u+1)^{r+q+k-1}+\chi\int_\Omega |\Delta v|^{\sigma_6}\\ &+\xi\int_\Omega |\Delta w|^{\sigma_7}\\ &+a\int_\Omega (u+1)^r+c_{16}, \quad t\in(t_0,T_{\max}),\end{aligned}$$

where $\sigma_6=\frac{r+q+k-1}{k}$ and $\sigma_7=\frac{r+g+l-1}{l}$. Applying the variation-of-constants formula and (2.2), we have

$$\begin{aligned}\frac{1}{r}\int_\Omega (u+1)^r \le & -\left(\frac{4(r-1)}{\bar\eta(r+p)^2}-\chi C_{\sigma_6}\alpha^{\sigma_6}-\xi C_{\sigma_7}\gamma^{\sigma_7}-a\right)\\ &\times e^{-(\beta\sigma_6+\delta\sigma_7)t}\int_{t_0}^t\int_\Omega e^{(\beta\sigma_6+\delta\sigma_7)s}(u+1)^{r+q+k-1}\\ &+e^{-(\beta\sigma_6+\delta\sigma_7)(t-t_0)}\left(\chi C_{\sigma_6}\|v(\cdot,t_0)\|^{\sigma_6}_{W^{2,\sigma_6}(\Omega)}+\xi C_{\sigma_7}\|w(\cdot,t_0)\|^{\sigma_7}_{W^{2,\sigma_7}(\Omega)}\right)\\ &+c_{33}, \quad t\in(t_0,T_{\max}),\end{aligned}$$

with $c_{33}=c_{33}(r,\bar\eta)>0$. Letting $\bar\eta$ small enough, we obtain (3.1).

Case 2: $q+k=d+1$ and $q+k,g+l\ge\frac{2}{n}+p+1$.

(a) Let $q+k=d+1=g+l$. Similarly to (4.1), by Young's inequality and (3.2) we have

$$\begin{aligned}&-\frac{\chi(r-1)}{r+q-1}\int_\Omega u^{r+q-1}\Delta v \le \frac{\chi(r-1)}{r+q-1}\left(\int_\Omega u^{r+d}+\int_\Omega |\Delta v|^{\sigma_7}\right), \quad t\in(t_0,T_{\max}),\\ &\frac{\xi(r-1)}{r+g-1}\int_\Omega u^{r+g-1}\Delta w \le \frac{\xi(r-1)}{r+g-1}\left(\int_\Omega u^{r+d}+\int_\Omega |\Delta w|^{\sigma_5}\right), \quad t\in(t_0,T_{\max}),\end{aligned}$$

and thus

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u^r \le & -(b-\chi\theta_1-\xi\theta_2)\int_\Omega u^{r+d}+\chi\theta_1\int_\Omega |\Delta v|^{\sigma_7}+\xi\theta_2\int_\Omega |\Delta w|^{\sigma_5}+a\int_\Omega u^r\\ =& -(\beta\sigma_7+\delta\sigma_5)\frac{1}{r}\int_\Omega u^r-(b-\chi\theta_1-\xi\theta_2)\int_\Omega u^{r+d}+\chi\theta_1\int_\Omega |\Delta v|^{\sigma_7}\\ &+\xi\theta_2\int_\Omega |\Delta w|^{\sigma_5}+\left(a+(\beta\sigma_7+\delta\sigma_5)\frac{1}{r}\right)\int_\Omega u^r, \quad t\in(t_0,T_{\max}),\end{aligned}$$

where $\theta_1 = \theta_1(r,q) = \frac{r-1}{r+q-1}$ and $\theta_2 = \theta_2(r,g) = \frac{r-1}{r+g-1}$. By Young's inequality we have

$$\left(a + (\beta\sigma_7 + \delta\sigma_5)\frac{1}{r}\right)\int_\Omega u^r \le \frac{1}{2}(b - \chi\theta_1 - \xi\theta_2)\int_\Omega u^{r+d} + c_{34}, \quad t \in (t_0, T_{\max}),$$

with $c_{34} = c_{34}(r) > 0$. We directly have

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u^r \le& -(\beta\sigma_7 + \delta\sigma_5)\frac{1}{r}\int_\Omega u^r - \frac{1}{2}(b - \chi\theta_1 - \xi\theta_2)\int_\Omega u^{r+d}\\ &+ \chi\theta_1\int_\Omega |\Delta v|^{\sigma_7} + \xi\theta_2\int_\Omega |\Delta w|^{\sigma_5} + c_{34}, \quad t \in (t_0, T_{\max}),\end{aligned}$$

Applying the variation-of-constants formula, we get

$$\begin{aligned}\frac{1}{r}\int_\Omega u^r \le& e^{-\beta\sigma_7 t}\chi\theta_1\int_{t_0}^t\int_\Omega e^{\beta\sigma_7 s}|\Delta v|^{\sigma_7} - \frac{1}{4}e^{-\beta\sigma_7 t}(b - \chi\theta_1 - \xi\theta_2)\int_{t_0}^t\int_\Omega e^{\beta\sigma_7 s}u^{r+d}\\ &+ e^{-\delta\sigma_5 t}\xi\theta_2\int_{t_0}^t\int_\Omega e^{\delta\sigma_5 s}|\Delta w|^{\sigma_5} - \frac{1}{4}e^{-\delta\sigma_5 t}(b - \chi\theta_1 - \xi\theta_2)\int_{t_0}^t\int_\Omega e^{\delta\sigma_5 s}u^{r+d}\\ &+ c_{35}, \quad t \in (t_0, T_{\max}),\end{aligned}$$

where $c_{35} = e^{(\beta\sigma_7+\delta\sigma_5)t_0}\frac{1}{r}\int_\Omega u^r(\cdot,t_0) + \frac{c_{34}}{\beta\sigma_7+\delta\sigma_5}(1 + e^{(\beta\sigma_7+\delta\sigma_5)t_0})$. According to (2.2), we obtain

$$\begin{aligned}\frac{1}{r}\int_\Omega u^r \le& -\left(\frac{1}{4}(b - \chi\theta_1 - \xi\theta_2) - \chi\theta_1 C_{\sigma_7}\alpha^{\sigma_7}\right)e^{-\beta\sigma_7 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_7 s}u^{r+d}\\ &- \left(\frac{1}{4}(b - \chi\theta_1 - \xi\theta_2) - \xi\theta_2 C_{\sigma_5}\gamma^{\sigma_5}\right)e^{-\delta\sigma_5 t}\int_{t_0}^t\int_\Omega e^{\delta\sigma_5 s}u^{r+d}\\ &+ \chi\theta_1 e^{-\beta\sigma_7(t-t_0)}C_{\sigma_7}\left\|v(\cdot,t_0)\right\|_{W^{2,\sigma_7}}^{\sigma_7} + \xi\theta_2 e^{-\delta\sigma_5(t-t_0)}C_{\sigma_5}\left\|w(\cdot,t_0)\right\|_{W^{2,\sigma_5}}^{\sigma_5}\\ &+ c_{35}, \quad t \in (t_0, T_{\max}),\end{aligned}$$

where

$$b_1 = 4C_{\sigma_7}\alpha^{\sigma_7}, \qquad b_2 = 4C_{\sigma_5}\gamma^{\sigma_5}.$$

Choosing b large enough such that $\frac{1}{4}(b - \chi\theta_1 - \xi\theta_2) - \chi\theta_1 C_{\sigma_7}\alpha^{\sigma_7} > 0$ and $\frac{1}{4}(b - \chi\theta_1 - \xi\theta_2) - \xi\theta_2 C_{\sigma_5}\gamma^{\sigma_5} > 0$, under the conditions $\frac{b-\xi\theta_2}{\chi\theta_1} > 1 + b_1$ and $\frac{b-\chi\theta_1}{\xi\theta_2} > 1 + b_2$, we can derive (3.1).

(b) Let $q + k = d + 1 > g + l$. By Young's inequality we have

$$\begin{aligned}-\frac{\chi(r-1)}{r+q-1}\int_\Omega u^{r+q-1}\Delta v &\le \frac{\chi(r-1)}{r+q-1}\left(\int_\Omega u^{r+d} + \int_\Omega |\Delta v|^{\sigma_7}\right), \quad t \in (t_0, T_{\max}),\\ \frac{\xi(r-1)}{r+g-1}\int_\Omega u^{r+g-1}\Delta w &\le \frac{\xi(r-1)}{r+g-1}\left(\int_\Omega u^{r+g+l-1} + \int_\Omega |\Delta w|^{\sigma_5}\right), \quad t \in (t_0, T_{\max}).\end{aligned}$$

By Young's inequality and (3.2), for any $\eta_6 > 0$, we have

$$\begin{aligned}\frac{\xi(r-1)}{r+g-1}\left(\int_\Omega u^{r+g+l-1} + \int_\Omega |\Delta w|^{\frac{r+g+l-1}{l}}\right) \le& \frac{b}{2}\int_\Omega u^{r+d} + \eta_6\int_\Omega |\Delta w|^{\sigma_5}\\ &+ c_{36}, \quad t \in (t_0, T_{\max}),\end{aligned}$$

where $c_{36} = c_{36}(r) > 0$. Thus

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u^r &\le -\frac{1}{2}(b-2\chi\theta_1-2\eta_6)\int_\Omega u^{r+d}+\chi\theta_1\int_\Omega|\Delta v|^{\sigma_7}+\eta_6\int_\Omega|\Delta w|^{\sigma_5}+a\int_\Omega u^r\\ &= -(\beta\sigma_7+\delta\sigma_5)\frac{1}{r}\int_\Omega u^r-\frac{1}{2}(b-2\chi\theta_1-2\eta_6)\int_\Omega u^{r+d}+\chi\theta_1\int_\Omega|\Delta v|^{\sigma_7}\\ &\quad+\eta_6\int_\Omega|\Delta w|^{\sigma_5}+\left(a+(\beta\sigma_7+\delta\sigma_5)\frac{1}{r}\right)\int_\Omega u^r+c_{36},\quad t\in(t_0,T_{\max}).\end{aligned}$$

By Young's inequality we have

$$\left(a+(\beta\sigma_7+\delta\sigma_5)\frac{1}{r}\right)\int_\Omega u^r\le\frac{1}{4}(b-2\chi\theta_1)\int_\Omega u^{r+d}+c_{37},\quad t\in(t_0,T_{\max}),$$

where $c_{37} = c_{37}(r) > 0$. We have

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\int_\Omega u^r &\le -(\beta\sigma_7+\delta\sigma_5)\frac{1}{r}\int_\Omega u^r-\frac{1}{4}(b-2\chi\theta_1-4\eta_6)\int_\Omega u^{r+d}+\chi\theta_1\int_\Omega|\Delta v|^{\sigma_7}\\ &\quad\times\eta_6\int_\Omega|\Delta w|^{\sigma_5}+c_{38},\quad t\in(t_0,T_{\max}),\end{aligned}$$

where $c_{38} = c_{36} + c_{37} > 0$. Applying the variation-of-constants formula, we get

$$\begin{aligned}\frac{1}{r}\int_\Omega u^r &\le e^{-\beta\sigma_7 t}\chi\theta_1\int_{t_0}^t\int_\Omega e^{\beta\sigma_7 s}|\Delta v|^{\sigma_7}-\frac{1}{8}e^{-\beta\sigma_7 t}(b-2\chi\theta_1-4\eta_6)\int_{t_0}^t\int_\Omega e^{\beta\sigma_7 s}u^{r+d}\\ &\quad+e^{-\delta\sigma_5 t}\eta_6\int_{t_0}^t\int_\Omega e^{\delta\sigma_5 s}|\Delta w|^{\sigma_5}-\frac{1}{8}e^{-\delta\sigma_5 t}(b-2\chi\theta_1-4\eta_6)\int_{t_0}^t\int_\Omega e^{\delta\sigma_5 s}u^{r+d}\\ &\quad+c_{39},\quad t\in(t_0,T_{\max}),\end{aligned}$$

where $c_{39} = e^{(\beta\sigma_7+\delta\sigma_5)t_0}\frac{1}{r}\int_\Omega u^r(\cdot,t_0)+\frac{c_{38}}{\beta\sigma_7+\delta\sigma_5}(1+e^{(\beta\sigma_7+\delta\sigma_5)t_0})$. Combining this with (2.2), we get

$$\begin{aligned}\frac{1}{r}\int_\Omega u^r &\le -\left(\frac{1}{8}(b-2\chi\theta_1-4\eta_6)-\chi\theta_1 C_{\sigma_7}\alpha^{\sigma_7}\right)e^{-\beta\sigma_7 t}\int_{t_0}^t\int_\Omega e^{\beta\sigma_7 s}u^{r+d}\\ &\quad-\left(\frac{1}{8}(b-2\chi\theta_1-4\eta_6)-\eta_6 C_{\sigma_5}\gamma^{\sigma_5}\right)e^{-\delta\sigma_5 t}\int_{t_0}^t\int_\Omega e^{\delta\sigma_5 s}u^{r+d}\\ &\quad+\chi\theta_1 e^{-\beta\sigma_7(t-t_0)}C_{\sigma_7}\left\|v(\cdot,t_0)\right\|_{W^{2,\sigma_7}}^{\sigma_7}+\eta_6 e^{-\delta\sigma_5(t-t_0)}C_{\sigma_5}\left\|w(\cdot,t_0)\right\|_{W^{2,\sigma_5}}^{\sigma_5}\\ &\quad+c_{39},\quad t\in(t_0,T_{\max}),\end{aligned}$$

where

$$b_3 = 8C_{\sigma_7}\alpha^{\sigma_7}.$$

Choosing η_6 sufficiently small and b sufficiently large, we can ensure that $\frac{b}{\chi\theta_1} > 2 + b_3$ and $b > 2\chi\theta_1$. This guarantees the derivation of (3.1).

We have established the L^r-boundedness (3.1) for certain cases. From this, employing a standard Alikakos–Moser iteration [16] and (2.4), we deduce

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max})$$

with some $C > 0$. The boundedness can be derived using the standard parabolic regularity theory. This, in conjunction with Lemma 2, establishes Theorem 2. □

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