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Wirtinger-type inequalities for Caputo fractional derivatives via Taylor's formula

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Abstract

In this study, we firstly derive a Wirtinger-type result, which gives the connection in between the integral of square of a function and the integral of square of its Caputo fractional derivatives with the help of left-sided and right-sided fractional Taylor's Formulas. Afterward, we provide a more general inequality involving Caputo fractional derivatives for L_r norm with $r > 1$ via Hölder's inequality. Also, similar inequalities for Riemann–Liouville fractional derivatives are presented by means of a relation between Caputo fractional derivatives and Riemann–Liouville fractional derivatives.

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Keywords: Wirtinger inequality; Caputo fractional derivatives; Taylor's formula

1 Introduction

Fractional calculus is one of the most important topics of mathematical analysis and its history is as old as that of differential calculus. In the 20th century, the theory of fractional integrals and derivatives swiftly developed by interested researchers, due to the abundance of application areas in different disciplines. During the last few decades, different types of these fractional integrals and derivatives have been defined and applied in many fields, such as viscoelasticity, optics, optimization, atmospheric and statistical physics, electrical and mechanical engineering, control theory, bioengineering, etc. Moreover, with the development of the theory of fractional integral and derivative of arbitrary order, the fractional integral inequalities that are usually used in optimization problems are improved by a lot of researchers in recent years. Also, a fractional generalization of Taylor's formula, which gives an approximation of a higher order differentiable function is provided by Anastassiou in [5] and it is named as right-sided Caputo fractional Taylor formula (CFTF). Afterward, some authors examined Ostrowski and Hermite–Hadamard-type inequalities involving fractional derivatives via right-sided CFTF. For instance, Sarikaya provided some Ostrowski type integral inequalities for functions whose Caputo fractional derivatives belong to L_p with $1 \leq p \leq \infty$ by means of right-sided CFTF in [29]. In addition, Anastassiou gave some results of Ostrowski type for functions whose Caputo fractional derivatives are bounded in [6]. In [15], Hermite–Hadamard-type results for higher-order differentiable mappings were obtained via Caputo fractional derivatives.

On the other side, theory of inequality plays a key role in many application areas of mathematics such as optimization, special means, estimations of integrals, etc. Wirtinger's inequality, which was first used to establish the isoperimetric inequality in 1904, is one of the most important inequalities in literature. What makes this result so important is that it compares integrals involving a function and its derivative. The classical Wirtinger inequality compares the integral of square of a function with the integral of square of its derivative. In other words, this inequality [18] expresses that if $\psi \in C^1([\lambda,\mu])$ satisfies $\psi(\lambda) = \psi(\mu) = 0$, then

$$\int_{\lambda}^{\mu} (\psi(t))^2 \, dt \leq \int_{\lambda}^{\mu} \left(\psi'(t)\right)^2 dt. \tag{1}$$

A large number of researchers focus on Wirtinger-type inequalities such as Bessel, Blaschke, Beesack, Poincare and Sobolev owing to importance of Wirtinger inequality. For example, Beesack extended the inequality (1) as follows:

Theorem 1 [8] *For any $\psi \in C^2([\lambda,\mu])$ providing $\psi(\lambda) = \psi(\mu) = 0$, following inequality holds*:

$$\int_{\lambda}^{\mu} \psi^4(t) \, dt \leq \frac{4}{3} \int_{\lambda}^{\mu} \left(\psi'(t)\right)^4 dt. \tag{2}$$

Beesack and other results are used in many issues such as the convergence of series, estimations of integrals and determination of the minimal eigenvalues of differential operators. To illustrate, the best constant in the Poincare inequality, which is a more general form of classical Wirtinger inequality, is known as the first eigenvalue of the Laplace operator and this result has been the motivation of diverse geometric works (see, e.g. [16, 20, 25, 33]). Furthermore, Böttcher and Widom [11] examined a sequence of constants which appear in some problems by considering the best constant Wirtinger–Sobolev inequality given the relation between the integral of the square of a function and the integral of the square of its higher-order derivative. Also, you can look over the references [4], [3], [9, 10, 12, 14] [23], [31, 32] to learn more about Wirtinger type inequalities and their application areas.

Sarikaya proved some generalized versions of Wirtinger-type inequalities in [30].

Theorem 2 [30] *Let $\psi \in C^1([\lambda,\mu])$ satisfy $\psi(\lambda) = \psi(\mu) = 0$ and $\psi' \in L_2[\lambda,\mu]$, then we have the following inequality*

$$\int_{\lambda}^{\mu} |\psi(t)|^2 \, dt \leq \frac{(\mu-\lambda)^2}{6} \int_{\lambda}^{\mu} \left|\psi'(t)\right|^2 dt. \tag{3}$$

The main purpose of this paper is to establish Wirtinger-type results, which give the connection between the integral of a function and the integral of its derivatives of arbitrary order via right-sided and left sided CFTF. We firstly deal with a Wirtinger-type inequality for functions whose Caputo fractional derivatives belong to L_2. Latter, we prove a more general version of this result for L_r norm with $r > 1$ via Hölder's inequality. Also, similar inequalities for Riemann-Liouville fractional derivatives are given with the help of a relation between Caputo and Riemann–Liouville fractional derivatives. Results presented in this work may be inspiration for further studies on inequalities involving Caputo fractional derivatives.

2 Preliminaries

In this part, we recall some definitions and properties of Riemann-Liouville and Caputo fractional derivatives by utilizing the references [13, 17, 19, 21, 22, 24] and [28]. Additionally, it should be noted that many mathematicians have studies on inequalities involving fractional integrals. The interested reader is able to look over the recent references [1, 2, 7, 26] and [27] for fractional theory, and the references included there.

Definition 1 Supposing the function ϕ is Lebesgue integrable on $[\lambda,\mu]$. The Riemann–Liouville fractional integrals (RLFI) $J^{\alpha}_{\lambda+}\phi$ and $J^{\alpha}_{\mu-}\phi$ of order $\alpha>0$ are defined by

$$J^{\alpha}_{\lambda+}\phi(\varkappa)=\frac{1}{\Gamma(\alpha)}\int_{\lambda}^{\varkappa}(\varkappa-\zeta)^{\alpha-1}\phi(\zeta)d\zeta,\ \varkappa>\lambda$$

and

$$J^{\alpha}_{\mu-}\phi(\varkappa)=\frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\mu}(\zeta-\varkappa)^{\alpha-1}\phi(\zeta)d\zeta,\ \ \varkappa<\mu,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function, i.e., $\Gamma(\alpha)=\int\limits_{0}^{\infty}e^{-u}u^{\alpha-1}du$. These integrals are named as the left-sided and the right-sided RLFI, respectively. Also, we note that $J^{0}_{\lambda+}\phi(\varkappa)=J^{0}_{\mu-}\phi(\varkappa)=\phi(\varkappa)$ for $\varkappa\in(\lambda,\mu)$.

Definition 2 Assume that $\alpha>0$, $m\in\mathbb{N}$ and $m=\lceil\alpha\rceil$, where $\lceil\cdot\rceil$ is the ceiling of the number. The left-sided and right-sided Riemann–Liouville fractional derivatives (RLFD) of order α are defined by

$$\begin{aligned}{}^{RL}D^{\alpha}_{\lambda+}\phi(\varkappa)&=\frac{d^m}{dx^m}J^{m-\alpha}_{\lambda+}\phi(\varkappa)\\&=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{d\varkappa^m}\int_{\lambda}^{\varkappa}\frac{\phi(\zeta)}{(\varkappa-\zeta)^{\alpha-m+1}}d\zeta,\ \ \varkappa>\lambda\end{aligned}$$

and

$$\begin{aligned}{}^{RL}D^{\alpha}_{\mu-}\phi(\varkappa)&=(-1)^m\frac{d^m}{d\varkappa^m}J^{m-\alpha}_{\mu-}\phi(\varkappa)\\&=\frac{(-1)^m}{\Gamma(m-\alpha)}\frac{d^m}{d\varkappa^m}\int_{\varkappa}^{\mu}\frac{\phi(\zeta)}{(\zeta-\varkappa)^{\alpha-m+1}}d\zeta,\ \ \varkappa<\mu,\end{aligned}$$

respectively. If $\alpha=m$ and usual derivatives of ϕ of order m exist, then $\left({}^{RL}D^{\alpha}_{\lambda+}\phi\right)(\varkappa)=\phi^{(m)}(\varkappa)$ and ${}^{RL}D^{\alpha}_{\mu-}\phi(\varkappa)=(-1)^m\phi^{(m)}(\varkappa)$. Specifically, if we choose $m=1$ and $\alpha=0$, then we possess $\left({}^{RL}D^{0}_{\lambda+}\phi\right)(\varkappa)=\left({}^{RL}D^{0}_{\mu-}\phi\right)(\varkappa)=\phi(\varkappa)$.

We indicate by $AC^m[\lambda,\mu]$ the space of real-valued mappings $\phi(\zeta)$ that possess derivatives up to $m-1$ order for $m\in\mathbb{N}$ on $[\lambda,\mu]$ such that $\phi^{(m-1)}(\zeta)$ is the element of $AC[\lambda,\mu]$

which is the space of absolutely continuous functions. That is,

$$AC^{m}[\lambda,\mu]=\left\{\phi:[\lambda,\mu]\to\mathbb{R}:\phi^{(m-1)}(\zeta)\in AC[\lambda,\mu]\right\}.$$

Definition 3 Assume that $\phi\in AC^{m}[\lambda,\mu]$ and $m=\lceil\alpha\rceil$ with $m\in\mathbb{N}$ and $\alpha>0$. The Caputo fractional derivatives (CFD) of order α are defined by

$$\begin{aligned}{}^{c}D_{\lambda+}^{\alpha}\phi(\varkappa)&=J_{\lambda+}^{m-\alpha}\phi^{(m)}(\varkappa)\\&=\frac{1}{\Gamma(m-\alpha)}\int_{\lambda}^{\varkappa}\frac{\phi^{(m)}(\zeta)}{(\varkappa-\zeta)^{\alpha-m+1}}d\zeta,\quad \varkappa>\lambda\end{aligned}$$

and

$$\begin{aligned}{}^{c}D_{\mu-}^{\alpha}\phi(\varkappa)&=(-1)^{m}J_{\mu-}^{m-\alpha}\phi^{(m)}(\varkappa)\\&=\frac{(-1)^{m}}{\Gamma(m-\alpha)}\int_{\varkappa}^{\mu}\frac{\phi^{(m)}(\zeta)}{(\zeta-\varkappa)^{\alpha-m+1}}d\zeta,\quad \varkappa<\mu,\end{aligned}$$

which are named as the left-sided and right-sided CFD, respectively. If $\alpha=m$ and usual derivatives of ϕ of order m exist, then $\left({}^{c}D_{\lambda+}^{\alpha}\phi\right)(\varkappa)=\phi^{(m)}(\varkappa)$ and ${}^{c}D_{\mu-}^{\alpha}\phi(\varkappa)=(-1)^{m}\phi^{(m)}(\varkappa)$. Especially, one has

$$\left({}^{c}D_{\lambda+}^{0}\phi\right)(\varkappa)=\left({}^{c}D_{\mu-}^{0}\phi\right)(\varkappa)=\phi(\varkappa),$$

where $m=1$ and $\alpha=0$.

Theorem 3 ([28]) *Suppose $\phi\in AC^{m}[\lambda,\mu]$ such that CFD ${}^{c}D_{\lambda+}^{\alpha}\phi(\varkappa)$ and ${}^{c}D_{\mu-}^{\alpha}\phi(\varkappa)$ exist together with RLFD ${}^{RL}D_{\lambda+}^{\alpha}\phi(\varkappa)$ and ${}^{RL}D_{\mu-}^{\alpha}\phi(\varkappa)$ for $\varkappa\in[\lambda,\mu]$. Then, for $\alpha>0$ and $m=\lceil\alpha\rceil\in\mathbb{N}$, we have*

$${}^{c}D_{\lambda+}^{\alpha}\phi(\varkappa)={}^{RL}D_{\lambda+}^{\alpha}\phi(\varkappa)-\sum_{i=0}^{m-1}\frac{\phi^{(i)}(\lambda)}{\Gamma(k-\alpha+1)}(\varkappa-\lambda)^{k-\alpha}$$

and

$${}^{c}D_{\mu-}^{\alpha}\phi(\varkappa)={}^{RL}D_{\mu-}^{\alpha}\phi(\varkappa)-\sum_{i=0}^{m-1}\frac{\phi^{(i)}(\mu)}{\Gamma(k-\alpha+1)}(\mu-\varkappa)^{k-\alpha}.$$

Also, it is clear that ${}^{c}D_{\lambda+}^{\alpha}\phi(\varkappa)={}^{RL}D_{\lambda+}^{\alpha}\phi(\varkappa)$ if $\phi^{(i)}(\lambda)=0$ for $i=0,1,\ldots,m-1$ and ${}^{c}D_{\mu-}^{\alpha}\phi(\varkappa)={}^{RL}D_{\mu-}^{\alpha}\phi(\varkappa)$ if $\phi^{(i)}(\mu)=0$ for $i=0,1,\ldots,m-1$.

Theorem 4 ([22]) *Let $\phi\in AC^{m}[\lambda,\mu]$, $x\in[\lambda,\mu]$, $\alpha>0$ and $m=\lceil\alpha\rceil\in\mathbb{N}$. Then, one has*

$$J_{\lambda+}^{\alpha}\,{}^{c}D_{\lambda+}^{\alpha}\phi(\varkappa)=J_{\lambda+}^{\alpha}J_{\lambda+}^{m-\alpha}D^{m}\phi(\varkappa)=J_{\lambda+}^{m}D_{\lambda+}^{m}\phi(\varkappa).$$

The right and left sided Caputo fractional Taylor formulas are introduced by Kilbas et al., as follows [19].

Definition 4 Supposing that ψ belongs to $AC^m[\lambda,\mu]$. Also, let $\varkappa\in[\lambda,\mu]$, $\alpha>0$ and $m=\lceil\alpha\rceil\in\mathbb{N}$. Then one has the right-sided Caputo fractional Taylor formula (CFTF)

$$\psi(\varkappa)=\sum_{i=0}^{m-1}\frac{\psi^{(i)}(\mu)}{i!}(\varkappa-\mu)^i+\frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\mu}(\tau-\varkappa)^{\alpha-1}\left[{}^cD_{\mu-}^{\alpha}\psi(\tau)\right]d\tau. \tag{4}$$

Definition 5 Assume the function ψ belongs to $AC^m[\lambda,\mu]$. Also, let $\varkappa\in[\lambda,\mu]$, $\alpha>0$ and $m=\lceil\alpha\rceil\in\mathbb{N}$. Then we have the left-sided Caputo fractional Taylor formula (CFTF)

$$\psi(\varkappa)=\sum_{i=0}^{m-1}\frac{\psi^{(i)}(\lambda)}{i!}(\varkappa-\lambda)^i+\frac{1}{\Gamma(\alpha)}\int_{\lambda}^{\varkappa}(\varkappa-\tau)^{\alpha-1}\left[{}^cD_{\lambda+}^{\alpha}\psi(\tau)\right]d\tau. \tag{5}$$

3 Some inequalities for Caputo and Riemann–Liouville fractional derivatives

In this section, it is observed Wirtinger-type inequalities involving CFD. Similar results including RLFD are presented by using connections in between CFD and RLFD. We first give the Taylor's formula, which forms the basis of the two identities we will use throughout the article. Let $\psi\in C^m([\lambda,\mu])$, $m\in\mathbb{N}\setminus\{0\}$. Then, from Taylor's theorem, we have

$$\psi(\varkappa)=\sum_{i=0}^{m-1}\frac{\psi^{(i)}(c)}{i!}(\varkappa-c)^i+\frac{1}{(m-1)!}\int_{c}^{\varkappa}(\varkappa-\tau)^{m-1}\psi^{(m)}(\tau)d\tau.$$

Theorem 5 *Supposing that $\psi\in C^m([\lambda,\mu])$ with $m\in\mathbb{N}\setminus\{0\}$ and ${}^cD_{\lambda+}^{\alpha}\psi(\tau)$, ${}^cD_{\mu-}^{\alpha}\psi(\tau)\in L_2[\lambda,\mu]$ with $\psi^{(i)}(\lambda)=\psi^{(i)}(\mu)=0$, $i=0,1,2,\dots,m-1$, $\alpha>0$, $m=\lceil\alpha\rceil$. Then one has the result*

$$\begin{aligned}\int_{\lambda}^{\mu}|\psi(\varkappa)|^2d\varkappa\le{}&\frac{(\mu-\lambda)^{2\alpha}}{[\Gamma(\alpha)]^2(2\alpha-1)(2\alpha)(2\alpha+1)}\\&\times\int_{\lambda}^{\mu}\left[\left|{}^cD_{\lambda+}^{\alpha}\psi(\tau)\right|^2+\left|{}^cD_{\mu-}^{\alpha}\psi(\tau)\right|^2\right]d\tau.\end{aligned} \tag{6}$$

Proof Applying Cauchy–Schwarz inequality to the resulting identities after taking absolute value of both sides of (4) and (5), owing to $\psi^{(i)}(\lambda)=\psi^{(i)}(\mu)=0$, for $i=0,1,2,\dots,m-1$, we see that

$$\begin{aligned}|\psi(\varkappa)|^2&=\left[\frac{1}{\Gamma(\alpha)}\int_{\lambda}^{\varkappa}(\varkappa-\varepsilon)^{\alpha-1}\left|{}^cD_{\lambda+}^{\alpha}\psi(\varepsilon)\right|d\varepsilon\right]^2\\&\le\frac{1}{[\Gamma(\alpha)]^2}\left(\int_{\lambda}^{\varkappa}(\varkappa-\varepsilon)^{2\alpha-2}d\varepsilon\right)\left(\int_{\lambda}^{\varkappa}\left|{}^cD_{\lambda+}^{\alpha}\psi(\varepsilon)\right|^2d\varepsilon\right)\\&=\frac{1}{[\Gamma(\alpha)]^2}\frac{(\varkappa-\lambda)^{2\alpha-1}}{2\alpha-1}\int_{\lambda}^{\varkappa}\left|{}^cD_{\lambda+}^{\alpha}\psi(\varepsilon)\right|^2d\varepsilon\end{aligned} \tag{7}$$

and

$$|\psi(\varkappa)|^2 = \left[\frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\mu}(\varepsilon-\varkappa)^{\alpha-1}\left|{}^{c}D_{\mu-}^{\alpha}\psi(\varepsilon)\right|d\varepsilon\right]^2 \tag{8}$$
$$\leq \frac{1}{[\Gamma(\alpha)]^2}\left(\int_{\varkappa}^{\mu}(\varepsilon-\varkappa)^{2\alpha-2}\,d\varepsilon\right)\left(\int_{\varkappa}^{\mu}\left|{}^{c}D_{\mu-}^{\alpha}\psi(\varepsilon)\right|^2 d\varepsilon\right)$$
$$= \frac{1}{[\Gamma(\alpha)]^2}\frac{(\mu-\varkappa)^{2\alpha-1}}{2\alpha-1}\int_{\varkappa}^{\mu}\left|{}^{c}D_{\mu-}^{\alpha}\psi(\varepsilon)\right|^2 d\varepsilon.$$

Integrating both sides of (7) with respect to $\varkappa$ from λ to $\rho\lambda+(1-\rho)\,\mu$ for $\rho\in[0,1]$ and then applying Dirichlet's formula to the double integral on the right side of the resulting statement, we find that

$$\int_{\lambda}^{\rho\lambda+(1-\rho)\mu}|\psi(\varkappa)|^2\,d\varkappa \tag{9}$$
$$\leq \frac{1}{[\Gamma(\alpha)]^2}\int_{\lambda}^{\rho\lambda+(1-\rho)\mu}\frac{(\varkappa-\lambda)^{2\alpha-1}}{2\alpha-1}\int_{\lambda}^{\varkappa}\left|{}^{c}D_{\lambda+}^{\alpha}\psi(\varepsilon)\right|^2 d\varepsilon d\varkappa$$
$$= \frac{1}{[\Gamma(\alpha)]^2}\int_{\lambda}^{\rho\lambda+(1-\rho)\mu}\left|{}^{c}D_{\lambda+}^{\alpha}\psi(\varepsilon)\right|^2\int_{\lambda}^{\rho\lambda+(1-\rho)\mu}\frac{(\varkappa-\lambda)^{2\alpha-1}}{2\alpha-1}d\varkappa d\varepsilon$$
$$= \frac{1}{[\Gamma(\alpha)]^2(2\alpha-1)}\int_{\lambda}^{\rho\lambda+(1-\rho)\mu}\frac{(1-\rho)^{2\alpha}(\mu-\lambda)^{2\alpha}-(\varepsilon-\lambda)^{2\alpha}}{2\alpha}\left|{}^{c}D_{\lambda+}^{\alpha}\psi(\varepsilon)\right|^2 d\varepsilon.$$

If similar processes are applied for the inequality (8), then one possesses

$$\int_{\rho\lambda+(1-\rho)\mu}^{\mu}|\psi(\varkappa)|^2\,d\varkappa \tag{10}$$
$$\leq \frac{1}{[\Gamma(\alpha)]^2}\int_{\rho\lambda+(1-\rho)\mu}^{\mu}\frac{(\mu-x)^{2\alpha-1}}{2\alpha-1}\int_{\varkappa}^{\mu}\left|{}^{c}D_{\mu-}^{\alpha}\psi(\varepsilon)\right|^2 d\varepsilon d\varkappa$$
$$= \frac{1}{[\Gamma(\alpha)]^2(2\alpha-1)}\int_{\rho\lambda+(1-\rho)\mu}^{\mu}\frac{\rho^{2\alpha}(\mu-\lambda)^{2\alpha}-(\mu-\varepsilon)^{2\alpha}}{2\alpha}\left|{}^{c}D_{\mu-}^{\alpha}\psi(\varepsilon)\right|^2 d\varepsilon.$$

If we apply the change of the variable $\varepsilon=\lambda\sigma+(1-\sigma)\,\mu$ to the right side of the results (9) and (10), then we possess

$$\int_{\lambda}^{\rho\lambda+(1-\rho)\mu}|\psi(\varkappa)|^2\,d\varkappa$$

$$\leq \frac{(\mu-\lambda)^{2\alpha+1}}{[\Gamma(\alpha)]^2(2\alpha-1)2\alpha} \times \int_{\rho}^{1}\left[(1-\rho)^{2\alpha}-(1-\sigma)^{2\alpha}\right]\left|{}^{c}D_{\lambda+}^{\alpha}\psi(\sigma\lambda+(1-\sigma)\mu)\right|^2 d\sigma$$

and

$$\int_{\rho\lambda+(1-\rho)\mu}^{\mu}|\psi(x)|^2\,dx \leq \frac{(\mu-\lambda)^{2\alpha+1}}{[\Gamma(\alpha)]^2(2\alpha-1)2\alpha}\int_{0}^{\rho}\left[\rho^{2\alpha}-\sigma^{2\alpha}\right]\left|{}^{c}D_{\mu-}^{\alpha}\psi(\sigma\lambda+(1-\sigma)\mu)\right|^2 d\sigma.$$

Integrating both sides of the resulting expression with respect to ρ from 0 to 1, placing the above inequalities side by side, it is seen that

$$\begin{aligned}\int_{\lambda}^{\mu}|\psi(\varkappa)|^2\,d\varkappa &\leq \frac{(\mu-\lambda)^{2\alpha+1}}{[\Gamma(\alpha)]^2(2\alpha-1)2\alpha}\\ &\times\left\{\int_0^1\int_{\rho}^{1}\left[(1-\rho)^{2\alpha}-(1-\sigma)^{2\alpha}\right]\left|{}^{c}D_{\lambda+}^{\alpha}\psi(\sigma\lambda+(1-\sigma)\mu)\right|^2 d\sigma\,d\rho\right.\\ &\left.+\int_0^1\int_0^{\rho}\left[\rho^{2\alpha}-\sigma^{2\alpha}\right]\left|{}^{c}D_{\mu-}^{\alpha}\psi(\sigma\lambda+(1-\sigma)\mu)\right|^2 d\sigma\,d\rho\right\}.\end{aligned}$$

We apply the change of order of integration method to double integrals in the right side of the above inequality, then we have

$$\begin{aligned}\int_{\lambda}^{\mu}|\psi(\varkappa)|^2\,d\varkappa &\leq \frac{(\mu-\lambda)^{2\alpha+1}}{[\Gamma(\alpha)]^2(2\alpha-1)2\alpha}\times\left\{\int_0^1 g(\sigma)\left|{}^{c}D_{\lambda+}^{\alpha}\psi(\sigma\lambda+(1-\sigma)\mu)\right|^2 d\sigma\right.\\ &\left.+\int_0^1 h(\sigma)\left|{}^{c}D_{\mu-}^{\alpha}\psi(\sigma\lambda+(1-\sigma)\mu)\right|^2 d\sigma\right\},\end{aligned}\tag{11}$$

where

$$g(\sigma)=\frac{1}{2\alpha+1}-\frac{(1-\sigma)^{2\alpha+1}}{2\alpha+1}-\sigma(1-\sigma)^{2\alpha}$$

and

$$h(\sigma) = \frac{1}{2\alpha+1} - \frac{\sigma^{2\alpha+1}}{2\alpha+1} - \sigma^{2\alpha}(1-\sigma).$$

Lastly, using the change of the variable $\tau = \sigma\lambda + (1-\sigma)\mu$ and from $d\tau = (\lambda-\mu)d\sigma$, because the maximum value of the functions $g(\sigma)$ and $h(\sigma)$ for $\sigma \in [0,1]$ is $\frac{1}{2\alpha+1}$, the desired inequality (6) can be easily obtained. □

Example 1 If we consider the function $\psi(\varkappa) = x^m(1-x)^m$ on $[0,1]$, then we have

$$\int_0^1 |\psi(\varkappa)|^2 \, d\varkappa = \frac{[\Gamma(1+2m)]^2}{\Gamma(2+4m)}$$

Later, choosing $m = 2$ and $\alpha = 1.5$ in the left side of the inequality (6), $\psi(0) = \psi(1) = \psi'(0) = \psi'(1) = 0$, which is necessary under the conditions of the theorem, and it is found that

$$\int_0^1 |\psi(\varkappa)|^2 \, d\varkappa = 0.0015873.$$

Thus, the integral value on the left side of the inequality (6) is 0.0015873. For the right side of the inequality (6), we have

$$\begin{aligned} \psi''(\varkappa) &= 2(1-x)^2 - 8(1-x)x + 2x^2, \\ {}^cD_{0+}^{1.5}\psi(x) &= 0.56419\left(4x^{0.5} - 16x^{1.5} + 12.8x^{2.5}\right), \end{aligned}$$

and

$$\begin{aligned} &{}^cD_{1-}^{1.5}\psi(x) \\ &= \frac{0.56419}{x^{0.5}}\left[4\left(x-x^2\right)^{0.5} + \left(x-x^2\right)^{0.5}((-3.2) + x((-9.6) + 12.8x))\right]. \end{aligned}$$

In this case, it follows that

$$\begin{aligned} &\int_0^1 \left[\left|{}^cD_{0+}^{1.5}\psi(x)\right|^2 + \left|{}^cD_{1-}^{1.5}\psi(x)\right|^2\right] dx \\ &= 0.203718. \end{aligned}$$

Also, we have the result

$$\frac{(\mu-\lambda)^{2\alpha}}{[\Gamma(\alpha)]^2(2\alpha-1)(2\alpha)(2\alpha+1)} = 0.0530516,$$

for $\alpha = 1.5$, $\mu = 1$ and $\lambda = 0$. Thus, for the right side of the inequality (6), we have

$$\begin{aligned}&\frac{(\mu-\lambda)^{2\alpha}}{[\Gamma(\alpha)]^2(2\alpha-1)(2\alpha)(2\alpha+1)}\int_0^1\left[\left|{}^cD_{0+}^{1.5}\psi(x)\right|^2+\left|{}^cD_{1-}^{1.5}\psi(x)\right|^2\right]dx\\&=0.203718\cdot 0.0530516\\&=0.0108076.\end{aligned}$$

So, the inequality (6) gives the numerical result

$$0.0015873\leq 0.0108076,$$

which shows that inequality is valid.

Since the left and right sides of the fractional derivatives cannot be written in common brackets, the functions whose maximum values need to be calculated are different. Therefore, special cases of inequalities obtained in this work must be evaluated by considering this situation.

Proposition 6 *Under the same assumptions of Theorem 5 with $\alpha = m = 1$, the inequality (3) is recaptured.*

Proof If we reconsider the inequality (11), because of ${}^cD_{\lambda+}^{\alpha}\psi(\varkappa)=\psi'(\varkappa)$ and ${}^cD_{\mu-}^{\alpha}\psi(\varkappa)=(-1)\psi'(\varkappa)$ in the case when $\alpha = m = 1$, we can write

$$\begin{aligned}&\int_\lambda^\mu|\psi(\varkappa)|^2\,d\varkappa\\&\leq\frac{(\mu-\lambda)^3}{2}\times\left\{\int_0^1 g(\sigma)\left|\psi'(\sigma\lambda+(1-\sigma)\mu)\right|^2d\sigma\right.\\&\left.\quad+\int_0^1 h(\sigma)\left|(-1)\psi'(\sigma\lambda+(1-\sigma)\mu)\right|^2d\sigma\right\}\\&=\frac{(\mu-\lambda)^3}{2}\int_0^1\left[g(\sigma)+h(\sigma)\right]\left|\psi'(\sigma\lambda+(1-\sigma)\mu)\right|^2d\sigma,\end{aligned}$$

where

$$g(\sigma)+h(\sigma)=\frac{2}{2\alpha+1}-\frac{(1-\sigma)^{2\alpha+1}}{2\alpha+1}-\sigma(1-\sigma)^{2\alpha}-\frac{\sigma^{2\alpha+1}}{2\alpha+1}-\sigma^{2\alpha}(1-\sigma).$$

Finally, because the maximum value of the function $g(\sigma)+h(\sigma)$ for $\sigma\in[0,1]$ is $\frac{1}{3}$, and so the inequality (3) is recaptured. □

Also, we have the following Wirtinger-type inequality involving RLFD.

Theorem 7 *Supposing that* $\psi \in C^m([\lambda,\mu])$ *with* $m \in \mathbb{N}\setminus\{0\}$ *and* ${}^{c}D^{\alpha}_{\lambda+}\psi(\tau), {}^{c}D^{\alpha}_{\mu-}\psi(\tau) \in L_2[\lambda,\mu]$ *with* $\psi^{(i)}(\lambda) = \psi^{(i)}(\mu) = 0$, $i = 0,1,2,\ldots,m-1$, $\alpha > 0$, $m = \lceil\alpha\rceil$. *Then, one has the result*

$$\int_{\lambda}^{\mu} |\psi(\varkappa)|^2 \, d\varkappa \leq \frac{(\mu-\lambda)^{2\alpha}}{[\Gamma(\alpha)]^2 (2\alpha-1)(2\alpha)(2\alpha+1)} \times \int_{\lambda}^{\mu} \left[\left|{}^{RL}D^{\alpha}_{\lambda+}\psi(\tau)\right|^2 + \left|{}^{RL}D^{\alpha}_{\mu-}\psi(\tau)\right|^2 \right] d\tau. \tag{12}$$

Proof If we use the Theorem 3, due to the acceptance of the equality given in the theorem $\psi^{(i)}(\lambda) = \psi^{(i)}(\mu) = 0$, $i = 0,1,2,\ldots,m-1$, we can write the identities

$${}^{c}D^{\alpha}_{\lambda+}\phi(\varkappa) = {}^{RL}D^{\alpha}_{\lambda+}\phi(\varkappa)$$

and

$${}^{c}D^{\alpha}_{\mu-}\phi(\varkappa) = {}^{RL}D^{\alpha}_{\mu-}\phi(\varkappa).$$

Then, subtituting ${}^{RL}D^{\alpha}_{\lambda+}\phi(\varkappa)$ and ${}^{RL}D^{\alpha}_{\mu-}\phi(\varkappa)$ instead of ${}^{c}D^{\alpha}_{\lambda+}\phi(\varkappa)$ and ${}^{c}D^{\alpha}_{\mu-}\phi(\varkappa)$ in (6), respectively, the inequality (12) can be readily deduced. □

Remark 1 Under the same assumptions of Theorem 5 with $\alpha = m = 1$, the inequality (3) is recaptured.

Proof The proof of this remark follows the same lines as the proof of Remark 5, because of ${}^{RL}D^{\alpha}_{\lambda+}\psi(\varkappa) = \psi'(\varkappa)$ and ${}^{RL}D^{\alpha}_{\mu-}\psi(\varkappa) = (-1)\psi'(\varkappa)$ in the case when $\alpha = m = 1$, the inequality (3) can be recaptured. □

Now, we derive a more general inequality by considering Taylor's formula in the following result.

Theorem 8 *Assume that* $\psi \in C^m([\lambda,\mu])$ *with* $m \in \mathbb{N}\setminus\{0\}$ *and* ${}^{c}D^{\alpha}_{\lambda+}\psi(\tau), {}^{c}D^{\alpha}_{\mu-}\psi(\tau) \in L_r[\lambda,\mu]$ *with* $r > 1$, $\alpha > 0$. *If* $\psi^{(i)}(\lambda) = \psi^{(i)}(\mu) = 0$ *for* $i = 0,1,2,\ldots,m-1$, $m = \lceil\alpha\rceil$, *then we have the inequality*

$$\int_{\lambda}^{\mu} |\psi(\varkappa)|^r \, d\varkappa \leq \frac{1}{[\Gamma(\alpha)]^r} \frac{(\mu-\lambda)^{\alpha r}}{(\alpha r)(\alpha r+1)} \left(\frac{r-1}{\alpha r-1}\right)^{r-1} \times \int_{\lambda}^{\mu} \left[\left|{}^{c}D^{\alpha}_{\lambda+}\psi(\tau)\right|^r + \left|{}^{c}D^{\alpha}_{\mu-}\psi(\tau)\right|^r \right] d\tau. \tag{13}$$

Proof Taking absolute value of both sides of (4) and (5), and later using Hölder's inequality with the indices r and $\frac{r}{r-1}$, due to $\psi^{(i)}(\lambda) = \psi^{(i)}(\mu) = 0$, for $i = 0,1,2,\ldots,m-1$, we find that

$$|\psi(\varkappa)|^r = \left[\frac{1}{\Gamma(\alpha)}\int_{\lambda}^{\varkappa}(\varkappa-\varepsilon)^{\alpha-1}\left|{}^cD^{\alpha}_{\lambda+}\psi(\varepsilon)\right|d\varepsilon\right]^r \tag{14}$$

$$\leq \frac{1}{[\Gamma(\alpha)]^r}\left(\int_{\lambda}^{\varkappa}(\varkappa-\varepsilon)^{\frac{\alpha r-r}{r-1}}d\varepsilon\right)^{r-1}\int_{\lambda}^{\varkappa}\left|{}^cD^{\alpha}_{\lambda+}\psi(\varepsilon)\right|^r d\varepsilon$$

$$= \frac{1}{[\Gamma(\alpha)]^r}\left(\frac{r-1}{\alpha r-1}\right)^{r-1}(\varkappa-\lambda)^{\alpha r-1}\int_{\lambda}^{\varkappa}\left|{}^cD^{\alpha}_{\lambda+}\psi(\varepsilon)\right|^r d\varepsilon$$

and

$$|\psi(\varkappa)|^r = \left[\frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\mu}(\varepsilon-\varkappa)^{\alpha-1}\left|{}^cD^{\alpha}_{\mu-}\psi(\varepsilon)\right|d\varepsilon\right]^r \tag{15}$$

$$\leq \frac{1}{[\Gamma(\alpha)]^r}\left(\int_{x}^{\mu}(\varepsilon-\varkappa)^{\frac{\alpha r-r}{r-1}}d\varepsilon\right)^{r-1}\int_{\varkappa}^{\mu}\left|{}^cD^{\alpha}_{\mu-}\psi(\varepsilon)\right|^r d\varepsilon$$

$$= \frac{1}{[\Gamma(\alpha)]^r}\left(\frac{r-1}{\alpha r-1}\right)^{r-1}(\mu-\varkappa)^{\alpha r-1}\int_{\varkappa}^{\mu}\left|{}^cD^{\alpha}_{\mu-}\psi(\varepsilon)\right|^r d\varepsilon.$$

Integrating both sides of (14) with respect to $\varkappa$ from λ to $\rho\lambda+(1-\rho)\mu$ for $\rho\in[0,1]$ and then applying Dirichlet's formula to the double integral in the right side of the resulting statement, we find that

$$\int_{\lambda}^{\rho\lambda+(1-\rho)\mu}|\psi(\varkappa)|^r d\varkappa \tag{16}$$

$$\leq \frac{1}{[\Gamma(\alpha)]^r}\left(\frac{r-1}{\alpha r-1}\right)^{r-1}\int_{\lambda}^{\rho\lambda+(1-\rho)\mu}(\varkappa-\lambda)^{\alpha r-1}\int_{\lambda}^{\varkappa}\left|{}^cD^{\alpha}_{\lambda+}\psi(\varepsilon)\right|^r d\varepsilon d\varkappa$$

$$= \frac{1}{[\Gamma(\alpha)]^r}\left(\frac{r-1}{\alpha r-1}\right)^{r-1}\int_{\lambda}^{\rho\lambda+(1-\rho)\mu}\frac{(1-\rho)^{\alpha r}(\mu-\lambda)^{\alpha r}-(\varepsilon-\lambda)^{\alpha r}}{\alpha r}\left|{}^cD^{\alpha}_{\lambda+}\psi(\varepsilon)\right|^r d\varepsilon.$$

Integrating both sides of (15) with respect to $\varkappa$ from $\rho\lambda+(1-\rho)\mu$ to μ for $\rho\in[0,1]$, and similar processes are applied for the inequality (15), then one has

$$\int_{\rho\lambda+(1-\rho)\mu}^{\mu}|\psi(\varkappa)|^r d\varkappa \tag{17}$$

$$\leq \frac{1}{[\Gamma(\alpha)]^r}\left(\frac{r-1}{\alpha r-1}\right)^{r-1}\int_{\rho\lambda+(1-\rho)\mu}^{\mu}(\mu-\varkappa)^{\alpha r-1}\int_{\varkappa}^{\mu}\left|{}^cD^{\alpha}_{\mu-}\psi(\varepsilon)\right|^r d\varepsilon d\varkappa$$

$$= \frac{1}{[\Gamma(\alpha)]^r}\left(\frac{r-1}{\alpha r-1}\right)^{r-1}\int_{\rho\lambda+(1-\rho)\mu}^{\mu}\frac{\rho^{\alpha r}(\mu-\lambda)^{\alpha r}-(\mu-\varepsilon)^{\alpha r}}{\alpha r}\left|{}^cD^{\alpha}_{\mu-}\psi(\varepsilon)\right|^r d\varepsilon.$$

If we apply the change of the variable $\varepsilon = \lambda\sigma + (1-\sigma)\mu$ to the right sides of the results (16) and (17), then we have the integrals

$$\begin{aligned}&\int_{\lambda}^{\rho\lambda+(1-\rho)\mu}|\psi(\varkappa)|^r d\varkappa\\&\leq \frac{(\mu-\lambda)^{\alpha r+1}}{[\Gamma(\alpha)]^r}\left(\frac{r-1}{\alpha r-1}\right)^{r-1}\frac{1}{\alpha r}\\&\times\int_{\rho}^{1}\left[(1-\rho)^{\alpha r}-(1-\sigma)^{\alpha r}\right]\left|{}^cD^{\alpha}_{\lambda+}\psi(\sigma\lambda+(1-\sigma)\mu)\right|^r d\sigma\end{aligned}$$

and

$$\begin{aligned}&\int_{\rho\lambda+(1-\rho)\mu}^{\mu}|\psi(x)|^r dx\\&\leq \frac{(\mu-\lambda)^{\alpha r+1}}{[\Gamma(\alpha)]^r}\left(\frac{r-1}{\alpha r-1}\right)^{r-1}\frac{1}{\alpha r}\int_{0}^{\rho}\left[\rho^{\alpha r}-\sigma^{\alpha r}\right]\left|{}^cD^{\alpha}_{\mu-}\psi(\sigma\lambda+(1-\sigma)\mu)\right|^r d\sigma.\end{aligned}$$

Integrating both sides of the resulting expression with respect to ρ from 0 to 1 after placing the above inequalities side by side, it is seen that

$$\begin{aligned}&\int_{\lambda}^{\mu}|\psi(\varkappa)|^r d\varkappa\\&\leq \frac{(\mu-\lambda)^{\alpha r+1}}{[\Gamma(\alpha)]^r}\left(\frac{r-1}{\alpha r-1}\right)^{r-1}\frac{1}{\alpha r}\\&\times\left\{\int_{0}^{1}\int_{\rho}^{1}\left[(1-\rho)^{\alpha r}-(1-\sigma)^{\alpha r}\right]\left|{}^cD^{\alpha}_{\lambda+}\psi(\sigma\lambda+(1-\sigma)\mu)\right|^r d\sigma d\rho\right.\\&\left.+\int_{0}^{1}\int_{0}^{\rho}\left[\rho^{\alpha r}-\sigma^{\alpha r}\right]\left|{}^cD^{\alpha}_{\mu-}\psi(\sigma\lambda+(1-\sigma)\mu)\right|^r d\sigma d\rho\right\}.\end{aligned}$$

Then, by using fundamental integral operations, it is found that

$$\begin{aligned}&\int_{\lambda}^{\mu}|\psi(\varkappa)|^r d\varkappa\\&\leq \frac{(\mu-\lambda)^{\alpha r+1}}{[\Gamma(\alpha)]^r}\left(\frac{r-1}{\alpha r-1}\right)^{r-1}\frac{1}{\alpha r}\end{aligned}\tag{18}$$

$$\times \left\{ \int_0^1 (\nu(\sigma)) \left|{}^c D^{\alpha}_{\lambda+} \psi(\sigma\lambda + (1-\sigma)\mu)\right|^r d\sigma \right.$$
$$\left. + \int_0^1 w(\sigma) \left|{}^c D^{\alpha}_{\mu-} \psi(\sigma\lambda + (1-\sigma)\mu)\right|^r d\sigma \right\},$$

where

$$\nu(\sigma) = \frac{1}{\alpha r + 1} - \frac{(1-\sigma)^{\alpha r + 1}}{\alpha r + 1} - \sigma (1-\sigma)^{\alpha r}$$

and

$$w(\sigma) = \frac{1}{\alpha r + 1} - \frac{\sigma^{\alpha r + 1}}{\alpha r + 1} - \sigma^{\alpha r} (1-\sigma).$$

Finally, using the change of the variable $\tau = \sigma\lambda + (1-\sigma)\mu$ and from $d\tau = (\lambda - \mu)d\sigma$, because the maximum value of the functions $\nu(\sigma)$ and $w(\sigma)$ for $\sigma \in [0,1]$ is $\frac{1}{\alpha r+1}$, the required inequality (13) can be easily obtained. □

Example 2 If we consider the function $\psi(\varkappa) = x^m (1-x)^m$ on $[0,1]$, then we have

$$\int_0^1 |\psi(\varkappa)|^r \, d\varkappa = \frac{[\Gamma(1+mr)]^2}{\Gamma(2+2mr)}.$$

Later, choosing $r = 4$, $m = 2$ and $\alpha = 1.5$ in the inequality (13), $\psi(0) = \psi(1) = \psi'(0) = \psi'(1) = 0$, which is necessary under the conditions of the theorem, it is found that

$$\int_0^1 |\psi(\varkappa)|^4 \, d\varkappa = 0.0000205677.$$

Thus, the integral value on the left side of the inequality (13) is 0.0000205677. For the right side of the inequality (13), we have

$$\psi''(\varkappa) = 2(1-x)^2 - 8(1-x)x + 2x^2,$$
$${}^c D^{1.5}_{0+} \psi(x) = 0.56419 \left(4x^{0.5} - 16x^{1.5} + 12.8x^{2.5}\right),$$

and

$${}^c D^{1.5}_{1-} \psi(x)$$
$$= \frac{0.56419}{x^{0.5}} \left[4\left(x - x^2\right)^{0.5} + \left(x - x^2\right)^{0.5} ((-3.2) + x((-9.6) + 12.8x))\right].$$

In this case, it follows that

$$\int_0^1 \left[\left|{}^c D^{1.5}_{0+} \psi(x)\right|^4 + \left|{}^c D^{1.5}_{1-} \psi(x)\right|^4 \right] dx$$
$$= 0.0310211.$$

Also, we have the result

$$\frac{1}{[\Gamma(\alpha)]^r}\frac{(\mu-\lambda)^{\alpha r}}{(\alpha r)(\alpha r+1)}\left(\frac{r-1}{\alpha r-1}\right)^{r-1}=0.00833729,$$

for $\alpha = 1.5$, $\mu = 1$ and $\lambda = 0$. Thus, for the right side of the inequality (13), we have

$$\begin{aligned}&\frac{1}{[\Gamma(\alpha)]^r}\frac{(\mu-\lambda)^{\alpha r}}{(\alpha r)(\alpha r+1)}\left(\frac{r-1}{\alpha r-1}\right)^{r-1}\int_0^1\left[\left|{}^cD_{0+}^{1.5}\psi(x)\right|^4+\left|{}^cD_{1-}^{1.5}\psi(x)\right|^4\right]dx\\&=0.0310211\cdot 0.00833729\\&=0.0002586321.\end{aligned}$$

So, the inequality (13) gives the numerical result

$$0.0000205677\le 0.0002586321,$$

which shows that inequality is valid.

Remark 2 Under the same assumptions of Theorem 8 with $m = \alpha = 1$, the following result holds:

$$\int_a^b|\psi(x)|^r\,dx\le\frac{(\mu-\lambda)^r}{r(r+1)}\int_a^b\left|\psi'(x)\right|^r dx, \tag{19}$$

which is given by Erden in [14].

Proof If we reconsider the inequality (18), because of ${}^cD^{\alpha}_{\lambda+}\psi(\varkappa)=\psi'(\varkappa)$ and ${}^cD^{\alpha}_{\mu-}\psi(\varkappa)=(-1)\psi'(\varkappa)$ in the case when $\alpha = m = 1$, we can write

$$\begin{aligned}&\int_\lambda^\mu|\psi(\varkappa)|^r\,d\varkappa\\&\le\frac{(\mu-\lambda)^r}{r}\times\left\{\int_0^1 v(\sigma)\left|\psi'(\sigma\lambda+(1-\sigma)\mu)\right|^r d\sigma\right.\\&\left.+\int_0^1 w(\sigma)\left|\psi'(\sigma\lambda+(1-\sigma)\mu)\right|^r d\sigma\right\},\end{aligned}$$

where

$$v(\sigma)+w(\sigma)=\frac{2}{r+1}-\frac{(1-\sigma)^{r+1}}{r+1}-\sigma(1-\sigma)^r-\frac{\sigma^{r+1}}{r+1}-\sigma^r(1-\sigma).$$

Finally, because the maximum value of the function $v(\sigma)+w(\sigma)$ for $\sigma\in[0,1]$ is $\frac{1}{r+1}$, and so the inequality (19) is recaptured. □

Remark 3 If $r = 4$ is chosen in (13), then one has

$$\int_{\lambda}^{\mu} |\psi(\varkappa)|^4 \, d\varkappa \leq \frac{1}{[\Gamma(\alpha)]^4} \frac{(\mu-\lambda)^{4\alpha}}{(4\alpha)(4\alpha+1)} \left(\frac{3}{4\alpha-1}\right)^3 \times \int_{\lambda}^{\mu} \left[\left|{}^{c}D^{\alpha}_{\lambda+}\psi(\tau)\right|^4 + \left|{}^{c}D^{\alpha}_{\mu-}\psi(\tau)\right|^4 \right] d\tau.$$

Also, if we take $r = 2$ in (13), then we get the inequality (6).

Remark 4 Under the same assumptions of Theorem 8 with $r = 4$ and $m = \alpha = 1$, the following result holds:

$$\int_{a}^{b} |\psi(x)|^4 \, dx \leq \frac{(\mu-\lambda)^4}{20} \int_{a}^{b} \left|\psi'(x)\right|^4 dx,$$

which is given by Erden in [14].

Furthermore, we have the following inequality including RLFD.

Theorem 9 *Assume that* $\psi \in C^m([\lambda,\mu])$ *with* $m \in \mathbb{N}\setminus\{0\}$ *and* ${}^{c}D^{\alpha}_{\lambda+}\psi(\tau), {}^{c}D^{\alpha}_{\mu-}\psi(\tau) \in L_r[\lambda,\mu]$ *with* $r > 1$, $\alpha > 0$. *If* $\psi^{(i)}(\lambda) = \psi^{(i)}(\mu) = 0$ *for* $i = 0,1,2,\ldots,m-1$, $m = \lceil \alpha \rceil$, *then we have*

$$\int_{\lambda}^{\mu} |\psi(\varkappa)|^r \, d\varkappa \leq \frac{1}{[\Gamma(\alpha)]^r} \frac{(\mu-\lambda)^{\alpha r}}{(\alpha r)(\alpha r+1)} \left(\frac{r-1}{\alpha r-1}\right)^{r-1} \times \int_{\lambda}^{\mu} \left[\left|{}^{RL}D^{\alpha}_{\lambda+}\psi(\tau)\right|^r + \left|{}^{RL}D^{\alpha}_{\mu-}\psi(\tau)\right|^r \right] d\tau. \tag{20}$$

Proof If we use the Theorem 3, owing to the acceptance of the equality given in the theorem $\psi^{(i)}(\lambda) = \psi^{(i)}(\mu) = 0$, $i = 0,1,2,\ldots,m-1$, we can write the identities

$${}^{c}D^{\alpha}_{\lambda+}\phi(\varkappa) = {}^{RL}D^{\alpha}_{\lambda+}\phi(\varkappa)$$

and

$${}^{c}D^{\alpha}_{\mu-}\phi(\varkappa) = {}^{RL}D^{\alpha}_{\mu-}\phi(\varkappa).$$

Then, substituting ${}^{RL}D^{\alpha}_{\lambda+}\phi(\varkappa)$ and ${}^{RL}D^{\alpha}_{\mu-}\phi(\varkappa)$ instead of ${}^{c}D^{\alpha}_{\lambda+}\phi(\varkappa)$ and ${}^{c}D^{\alpha}_{\mu-}\phi(\varkappa)$, respectively in (13), the inequality (20) can be readily deduced. □

Remark 5 If we choose $r = 2$ in (20), then the result (20) reduces to the inequality (12).

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Author contributions
The main idea of this paper was proposed by SE. SE and MZS prepared the manuscript initially and all authors performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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Declarations

Competing interests
The authors declare no competing interests.

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